First Order Perturbation Analysis of Groebner Basis Representations

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1 Introduction

2 Groebner Basis Representations

3 Syzygies

4 Representation Singularities

A severe obstacle to the computational use of Groebner basis representations is the fact that they may "jump", i.e. *change discontinuously*, upon small changes of the ideal without the presence of a situation where such a behavior would appear natural.

Consider $\mathcal{I} = \langle p_1, \dots, p_n \rangle$; for each $p_\nu = \sum_{j \in J_\nu} a_{\nu j} x^j, \nu = 1(1)n$, we admit potential *perturbations* $e_\nu = \sum_{j \in \bar{J}_\nu} \epsilon_{\nu j} x^j$ where $\bar{J}_\nu := \{ j \in \mathbb{N}^s : j \leq \bar{j}_\nu := max_{j \in J_\nu} j \}$ and the order in $\mathbb{N}^s$ is induced by the term order in $\mathcal{T}_s$. More intuitively, the perturbation may affect the coefficients of all terms not greater than the leading term $\mathrm{LT}(p_\nu) = x_\nu^{\bar{j}}$ of p_ν; this may include terms whose coefficients vanish in p_ν. For some specified $\epsilon > 0, \epsilon$ small, we consider the set of *near-by* ideals

$$\overline{\mathcal{I}}_\epsilon := \{ \langle \tilde{p}_1, \dots, \tilde{p}_n \rangle = \langle p_\nu + \sum_{j \in \bar{J}_\nu} \epsilon_{\nu j} x^j, \ \nu = 1(1)n \rangle, \text{ with } \max_\nu \sum_j |\epsilon_{\nu j}| \leq \epsilon \} \ .$$

Note that $\overline{\mathcal{I}}_\epsilon$ depends on the generating set of $\mathcal{I}$. This is meaningful since we assume that $\mathcal{I}$ has been specified by a specific generating set whose elements have (some) coefficients of limited accuracy. Then $\overline{\mathcal{I}}_\epsilon$ contains those ideals which cannot be distinguished from $\mathcal{I}$.

Definition (near-singular, singular ideal): For specified $\epsilon > 0$, $\mathcal{I} = \langle p_1, \dots, p_n \rangle$ is called *near-singular* if $\overline{\mathcal{I}}_\epsilon$ contains ideals which differ in the dimension(s) of (some of) their zero component(s), or – for zero-dimensional ideals – in the number of their isolated zeros, counting multiplicities. $\mathcal{I}$ is called *singular* if it is near-singular for arbitrary small $\epsilon > 0$. $\square$

Example: The ideal $\mathcal{I}$ generated by s polynomials of total degree 1 in s variables is near-singular if and only if the associated quadratic matrix of the coefficients of the linear terms is near-singular in the sense of numerical linear algebra. In the near-singular case, the zero sets of ideals in $\overline{\mathcal{I}}_\epsilon$ are either one point, empty, or a one-dimensional variety; possibly, there are even higher-dimensional varieties.

Accordingly, there appear at least 3 types of Groebner bases for ideals in $\overline{\mathcal{I}}_\epsilon$:

$$\{x_1 - z_1, \ldots, x_s - z_s\},\ \{1\},\ \{x_1 - a_1 x_s - b_1, \ldots, x_{s-1} - a_{s-1}x_s - b_{s-1}\}\ .$$

Obviously, there cannot be a continuous transition from one type to another. □

The discontinuous behavior observed in the example is natural if $\mathcal{I}$ is near-singular. Unfortunately, such a behavior may also occur at manifolds in the data space where the zeros of the ideals *behave perfectly smoothly.*

Example: $n = s = 2$,[1] $\mathcal{I} = \langle x^2 + y^2 - 1,\ 3x^2y - y^3\rangle$. Obvioulsy, the zero set of $\mathcal{I}$ consists of the six corners of the regular hexagon inscribed into the unit circle. With the term order $x > y$, we have

$$\begin{aligned} \mathcal{G}_0 &= \{x^2 + y^2 - 1,\ y^3 - \tfrac{3}{4}y\}\ , \\ \mathcal{N}_0 &= \{\,1,\ y,\ x,\ y^2,\ xy,\ xy^2\}\ . \end{aligned}$$

When we perturb the first generating polynomial into $x^2 + \epsilon xy + y^2 - 1$, the unit circle is slightly distorted into an ellipse and the zeros move continuously and by $0(\epsilon)$ as ϵ increases from 0. However, the Groebner basis of the associated ideal becomes

$$\begin{aligned} \mathcal{G}_\epsilon &= \{x^2 + \epsilon xy + y^2 - 1,\ xy^2 + \tfrac{4}{3\epsilon}y^3 - \tfrac{1}{\epsilon}y,\ y^4 + \tfrac{9\epsilon}{16-3\epsilon^2}xy - \tfrac{12}{16-3\epsilon^2}y^2\}\ , \\ \mathcal{N}_\epsilon &= \{\,1,\ y,\ x,\ y^2,\ xy,\ y^3\}, \end{aligned}$$

and there is no continuous transition from $\mathcal{G}_\epsilon$ to $\mathcal{G}_0$ as they have different numbers of elements and different normal sets.

Naturally, the above perturbation of the generating polynomials is just one of many that must be considered to establish that $\mathcal{I}$ is *not* near-singular. A concise analysis is lengthy; intuitively, it is clear that the zeros must remain near the real unit circle (in $\mathbb{C}^2$) and near that curve an ϵ-perturbation of $3x^2y - y^3$ can move the zero set only by $O(\epsilon)$. Thus, there are 6 isolated zeros for each $\mathcal{I} \in \overline{\mathcal{I}}_\epsilon$. □

We will call this phenomenon of a discontinuous change in the structure of the Groebner basis caused by a small perturbation of the original generating set *without* the presence of a (genuine) singularity a *representation singularity.*

Definition (representation near-singular, representation singular ideal): Assume that $\mathcal{I} = \langle p_1, \ldots, p_n\rangle$ is *not* near-singular for a specified small $\epsilon > 0$. $\mathcal{I}$ is called *representation near-singular* if $\overline{\mathcal{I}}_\epsilon$ contain ideals whose Groebner bases have different structures (number of elements, leading terms). $\mathcal{I}$ is called *representation singular* if it is representation near-singular for arbitrary small $\epsilon > 0$. □

[1] For a more intuitive notation, we use $(x, y) \in \mathbb{C}^2$ in place of (x_1, x_2).

Obviously, $\mathcal{I} = \langle x^2 + y^2 - 1,\ 3xy^2 - y^3 \rangle$ is representation singular. Its zero set is distinguished by a special symmetry from most of the zero sets which arise for a slightly perturbed generating set. Quite generally, representation singularities appear when the zeros assume some special "degenerate" constellation. Therefore we will call the Groebner bases of representation singular ideals *degenerate*; they are commonly "simpler" than the structurally different Groebner bases for nearby perturbed generating sets.

Typically, some coefficients in the Groebner basis of a representation near-singular ideal have large moduli which diverge to infinity as the perturbation takes the ideal into the representation singular situation. In particular, the multiplication tables A_σ associated with such Groebner bases also contain some large elements. Thus the computational value of these matrices, e.g. for the determination of their eigenvectors which yield the zeros of the ideal, is dubious. The fact that these matrices become arbitrarily ill-conditioned with respect to their eigenproblem is revealed by the *jump in the normal set* which occurs at the ideal with the degenerate Groebner basis: The normal set used with the neighboring ideals does not span the residue class ring of the limiting degenerate ideal as two or more of its members are linearly dependent on the zero set of the degenerate ideal.

Example, continued: The multiplication tables of $\langle x^2 + \epsilon xy + y^2 - 1,\ 3x^2y - y^3 \rangle$ w.r.t. $\mathcal{N}_\epsilon$ are

$$A_x = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & -1 & -\epsilon & 0 \\ 0 & \frac{1}{\epsilon} & 0 & 0 & 0 & -\frac{4}{3\epsilon} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{3} \\ 0 & 0 & 0 & -\frac{3}{16-3\epsilon^2} & \frac{12}{16-3\epsilon^2} & 0 \end{pmatrix},$$

$$A_y = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & \frac{1}{\epsilon} & 0 & 0 & 0 & -\frac{4}{3\epsilon} \\ 0 & 0 & 0 & \frac{12}{16-3\epsilon^2} & -\frac{9}{16-3\epsilon^2} & 0 \end{pmatrix}.$$

The matrix X of the common eigenvectors of A_x and A_y (normalized for a first component 1) has the columns $\mathbf{t}(z_\mu)$ for the zeros $z_\mu, \mu = 1(1)6$, with $\mathbf{t}$ from $\mathcal{N}_\epsilon$. The max-norm condition number of X is approxiately $5/\varepsilon$ which characterizes the deteriorating condition of the eigenvalue problem of A_x and A_y for $\epsilon \to 0$. Obviously, $t_6 = y^3 = \frac{3}{4}y = \frac{3}{4}t_2$ in $\mathcal{N}_\varepsilon$ at $\varepsilon = 0$.

It is thus obvious that genuine Groebner bases do not furnish a suitable reference system for the representation of polynomials in or near a representation (near-)singular ideal, at least not for computational purposes with data of limited accuracy. Naturally, this raises the question what should be used in their place. The remainder of this paper will be devoted to a proposed answer to this question.

Before we continue, we wish to point to the fact that in the case of $n > s$ polynomials in the generating set there are two possibilities: Either the set is redundant and s of the n polynomials define the same ideal, or the generated ideal is singular. Therefore, we will restrict our considerations in the following to the case $n = s$, the *complete intersection case*, without a serious loss of generality, at least not within the type of applications which we have in mind (solution of polynomial systems of equations).

5 Perturbed Groebner Bases

At a representation singularity, there is no chance to extend the non-degenerate Groebner bases for neighboring ideals into the degenerate Groebner basis for the representation singular ideal because they have different leading terms, i.e. different normal sets. More promising appears the opposite direction: The normal set $\mathcal{N}_0$ of the degenerate Groebner basis $\mathcal{G}_0$ is retained but the basis elements are modified by small perturbations so that they become generating sets of the neighboring ideals. In this way, a continuous change in the zero sets is reflected by a continuous change in the reference system.

Definition (perturbed Groebner basis): For a specified fixed term order, consider a Groebner basis $\mathcal{G}_0 = \{g_1, \ldots, g_k\} \subset \mathbb{P}^s$, with the associated normal set $\mathcal{N} = \{t_1, \ldots, t_m\} \subset \mathcal{T}_s$. A polynomial system $\tilde{\mathcal{G}}_c = \{\tilde{g}_1, \ldots, \tilde{g}_\kappa\}$, with (cf. (...) for the notation),

$$\tilde{g}_\kappa(x) \;=\; g_\kappa(x) + c_\kappa^T \mathbf{t}(x), \quad c_\kappa \in \mathbb{C}^m \;, \quad \|c_\kappa\| \text{ small},$$

will be called a *perturbed Groebner basis*, more specifically a perturbation of $\mathcal{G}_0$. □

Generally, $\tilde{\mathcal{G}}_c$ will *not* be a genuine Groebner basis because there will be some $t_\mu \in \mathcal{N}$ with $t_\mu > \mathrm{LT}[g_\kappa]$ for one or several $\kappa \in \{1, \ldots, k\}$. We could propose to omit these t_μ in the perturbation of the g_κ concerned; however, this would be counterproductive: Due to the uniqueness of Groebner bases, there *cannot* be a genuine Groebner basis which is a smooth perturbation of the degenerate basis in cases of a representation singularity.

Example, continued: The perturbed Groebner basis

$$\tilde{\mathcal{G}} = \{x^2 + y^2 - 1 + \epsilon \cdot xy,\; y^3 - \tfrac{3}{4}y + \tfrac{3}{4}\epsilon \cdot xy^2\}$$

generates the ideal $\langle x^2 + \epsilon xy + y^2 - 1,\; 3x^2y - y^3\rangle$ as is immediately seen. Due to the presence of the xy^2 term in the perturbation, $\tilde{\mathcal{G}}$ is not a Groebner basis; but it is a continuous extension of the degenerate Groebner basis $\{x^2+y^2-1,\ y^3-\frac{3}{4}y\}$. □

What advantage may we gain by the use of perturbed Groebner bases in place of genuine Groebner bases near representation singularities? A genuine Groebner basis of a representation near-singular ideal contains at least one element with coefficients of very large moduli; this cause for numerical instabilities has now been eliminated. More important, the *normal set* of the degenerate ideal spans the residue class ring of all neighboring ideals.

This follows from the fact that the matrix of the eigenvectors of the multiplication tables A_σ is the *Gram matrix* of the *interpolation problem* on the zeros of the ideal, with the span of the normal set as interpolation subspace; cf., e.g., [..]. The Gram matrix is regular at the degenerate situation; by continuity is remains regular at all neighboring situations. Thus the multiplication tables with respect to the "degenerate" normal set retain the good condition of their eigenproblem in a neighborhood of the representation singular ideal.

Example, continued: The multiplication tables for the perturbed Groebner basis $\tilde{\mathcal{G}}_\varepsilon = \langle x^2+y^2-1+\varepsilon xy, y^3-\frac{3}{4}y+\frac{3}{4}\varepsilon xy^2\rangle$ w.r.t. $\mathcal{N}_0$ are

$$A_x = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & -1 & -\varepsilon & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & \frac{1}{4} & 0 & 0 & 0 & -\frac{\varepsilon}{4} \\ 0 & 0 & 0 & \frac{4}{16-3\varepsilon^2} & -\frac{3\varepsilon}{16-3\varepsilon^2} & 0 \end{pmatrix}, A_y = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & \frac{3}{4} & 0 & 0 & 0 & -\frac{3\varepsilon}{4} \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -\frac{3\varepsilon}{16-3\varepsilon^2} & \frac{12}{16-3\varepsilon^2} & 0 \end{pmatrix};$$

these matrices converge to the multiplication tables of $\mathcal{G}_0$ for $\epsilon \to 0$. The matrix X of the common eigenvectors has a max-norm condition of approximately 8 independently of ε (for small $|\varepsilon|$).

This shows that the perturbed Groebner basis $\tilde{\mathcal{G}}_\epsilon$ which retains the degenerate normal set $\mathcal{N}_0$ provides the appropriate representation near the representation singularity. □

The preceding trivial example was chosen because it permits an explicit computation of interesting quantities in terms of ϵ. In a real-life situation, there arise two questions:

How do we realize the closeness of a representation singularity during the computation of the Groebner basis for a representation near-singular ideal, and how do we find the degenerate Groebner basis $\mathcal{G}_0$ of the nearby representation singular ideal?

Given $\mathcal{G}_0$, how do we find the perturbation coefficients c_κ and the multiplication tables w.r.t. $\mathcal{N}_0$ for our specified representation near-singular ideal?

The first problem is very non-trivial. An initial analysis, with some preliminary results, has been made in the Ph.D. thesis of V. Hribernig [..]. The investigations have been continued; further results will be published elsewhere. WE will not address this issue in the present paper.

The second question amounts to the following problem: Consider the generating set $\{p_1, \ldots, p_s\}$; by assumption, the p_ν have small residuals in a representation (..) w.r.t. the Groebner basis $\mathcal{G}_0$ of the nearby representation singular ideal ($\mathbf{t}$ refers to the associated $\mathcal{N}_0$):

$$p_\nu = e_\nu^T \mathbf{t} + \sum_{\kappa=1}^{\kappa} (d_{\nu\kappa}^T \mathbf{t}) g_\kappa + \sum_{\kappa_2 \le \kappa_1} (d_{\nu\kappa_1\kappa_2} \mathbf{t}) g_{\kappa_1} g_{\kappa_2} + \ldots$$

Now we replace the g_κ by $\tilde{g}_\kappa = g_\kappa + c_\kappa^T \mathbf{t}$ so that the coefficients e_ν, $d_{\nu\kappa}$, $d_{\nu\kappa_1\kappa_2}, \ldots$ become functions of the c_κ; this defines a mapping $(c_\kappa) \to (e_\nu)$. Of the possibly several branches of this mapping we choose the one which coincides with (..) for $(c_\kappa) = 0$. Since the e_ν are small for $(c_\kappa) = 0$, there will be a set of small c_κ which maps on $(e_\nu) = 0$, under suitable technical conditions. These are the c_κ which we have to compute to obtain $\langle p_1, \ldots, p_s \rangle = \langle \tilde{g}_1, \ldots, \tilde{g}_\kappa \rangle$. In general, this amounts to the solution of a polynomial system for the c_κ which may be more complicated then the system of the p_ν. Only in near-trivial cases, like the preceeding example, the c_κ may be determined without serious computation.

Therefore, we resort to a tool which is well-established in many parts of mathematics, viz. *first order perturbation analysis:* We compute c_κ such that the associated residuals for the p_ν w.r.t. $\tilde{\mathcal{G}}_c$ are $O(\|c\|^2)$. Similarly, we require the multiplication tables to represent the multiplicative structure of $\mathbb{P}^s / \langle p_1, \ldots, p_s \rangle$ correctly up to terms of $O(\|c\|^2)$. The remainder of the paper will establish that this first order perturbation analysis reduces to simple linear computations and that its results are meaningful under the assumption of this paper.

6 Manipulations in Perturbed Groebner Bases

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