ALGORITHMS FOR GEOMETRIC CONTROL OF SYSTEMS OVER RINGS

Anna Maria Perdon * Giovanna Guidone Peroli * Massimo Caboara **

* *Dipartimento di Matematica, Università di Ancona via Brecce Bianche, 60131 Ancona, Italy*
** *Università di Genova via Dodecaneso 35, 16146 Genova, Italy*

Abstract: In this paper we describe some algorithms for implementing geometric procedures related to analysis and synthesis of control systems over rings. Software performing symbolic computations allows practical use of the algorithms in particular dealing with delay differential systems.

Keywords: Systems over rings, Geometric approach, Algorithms

1. INTRODUCTION

In recent time, many efforts have been made to extend the so called geometric approach to the class of systems with coefficients in a ring or, simply, systems over rings. The study of this kind of systems is receiving an increasing attention, as they prove to be an efficient tool for investigating and solving design problems concerning continuous-time, delay-differential systems (see (Conte G., 1997) and the references therein).
The relevance of the geometric approach as design methodology in controlling systems with coefficients in a field relies on the existence of numeric algorithmic procedures to check set theoretic relations and to construct feedbacks. It is more difficult to provide similar results for systems over rings. We have developed algorithms relying on Gröbner basis computation; for other approaches see (Bareiss, 1968), (Malashonok, 1991).

These new procedures, usually acting on vectors and matrices with elements in a ring of polynomials in one ore several indeterminates, cannot be implemented efficiently using numeric algorithms, but they require the use of symbolic computations tools, such as MapleV, Mathematica or CoCoA, see (Capani A., 1995).

In this paper we first present a complete account of the results which, within the geometric approach, allow the solution of many design problems such as that of the Disturbance Decoupling. Then we propose new algorithms to compute feedback solutions to such problems.

In this draft, due to the page limit, only some of the algorithms are explicitly described.

2. PRELIMINARIES

Assume that a delay-differential system Σ_d is described by the equations

$$\mathbf{\Sigma_d} = \begin{cases} \dot{x}(t) = A_0 x(t) + \sum_{i=1}^{a} A_i \delta^i x(t) + \\ \qquad + B_0 u(t) + \sum_{i=1}^{b} + B_i \delta^i u(t) \\ y(t) = C_0 x(t) + \sum_{i=1}^{c} C_i \delta^i x(t) \end{cases} \quad (1)$$

where, denoting by $\mathbf{R}$ the field of real numbers, x belongs to the vector space $\mathbf{R}^n$, u belongs to the input space $\mathbf{R}^m$, y belongs to the output space $\mathbf{R}^p$, A_i, $i = 0, 1, ..., a$, B_i, $i = 0, 1, ..., b$, C_i, $i =$

$0, 1, ..., c$ are matrices of suitable dimensions with entries in $\mathbf{R}$ and δ is the delay operator defined, for any time function $f(t)$, by $\delta f(t) = f(t-h)$, for a given $h \in \mathbf{R}^+$. Replacing formally the delay operator δ by the algebraic indeterminate Δ, it is possible to associate to Σ_d the system Σ over the ring $R = \mathbf{R}[\Delta]$ of real polynomials in one indeterminate defined by the equations

$$\begin{cases} x(t+1) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \end{cases} \qquad (2)$$

where x belongs to the free state module $\mathcal{X} = R^n$, u belongs to the free input module $\mathcal{U} = R^m$, y belongs to the free output module $\mathcal{Y} = R^p$ and A, B, C are matrices with entries in $R = \mathbf{R}[\Delta]$ given by $A = \sum_{i=0}^{a} A_i \Delta^i$, $B = \sum_{i=0}^{b} B_i \Delta^i$ and $C = \sum_{i=0}^{c} C_i \Delta^i$. Note that, although they have different meaning, we used the same letters to represent corresponding variables in equations (1) and (2). Actually the systems Σ_d and Σ are quite different objects from a dynamical point of view. The key point is, however, that Σ_d and Σ have the same signal flow graph, so that control problems concerning the input/output behaviour of Σ_d can naturally be formulated in terms of the input/output behaviour of Σ and can possibly be solved in the framework of systems over rings. In turn, solutions found in that framework can be brought back to the original delay-differential framework.
For delay-differential systems with non commensurable delays, the above procedure gives rise to systems over the ring $R = \mathbf{R}[\Delta_1, \Delta_2, ..., \Delta_k]$ of polynomials in several indeterminates with real coefficients. Therefore, in the systems which we are mainly interested in, the ring R can be a Principal Ideal Domain, as in the case of $\mathbf{R}[\Delta]$, or more generally a Noetherian ring, as in the case of $R = \mathbf{R}[\Delta_1, \Delta_2, ..., \Delta_k]$.

3. BASIC GEOMETRIC CONCEPTS

The geometric approach to linear dynamical systems relies upon the notion of controlled invariance and its equivalence to feedback invariance, which is crucial in the applications. For systems over a field, (Basile G., 1992) and (Wonham, 1985), controlled invariance is equivalent to invariance with respect to a static state feedback, while for systems over a ring R the situation is more complex.

Definition *Given the system (2) over a ring R, a submodule $\mathcal{V} \subset \mathcal{X}$ is called*

i) (A,B)-invariant or controlled invariant if $A\mathcal{V} \subset \mathcal{V} + ImB$;

ii) feedback invariant or $(A+BF)$-invariant if there exists a static R-linear state feedback $F{:}X \longrightarrow U$ such that $(A+BF)\mathcal{V} \subset \mathcal{V}$.

A feedback law satisfying *ii*) is called a *friend* of $\mathcal{V}$.

If R is a field, i) and ii) are equivalent, but over a ring controlled invariance does not necessarily imply feedback invariance. The crucial property which allows to pass from the geometric property i) to the existence of a feedback is related to the possibility of extending a map defined on $\mathcal{V}$ to a map defined on the whole state module. In particular, the following propositions hold.

Proposition 1 *(see (Conte G., 1995*b*)) Assume that the (A, B)-invariant submodule $\mathcal{V}$ is a direct summand of the state module $\mathcal{X}$ (i.e. there exists a submodule $\mathcal{W}$ such that $\mathcal{X} = \mathcal{W} \oplus \mathcal{V}$). Then, $\mathcal{V}$ is of feedback type.*

The above result motivates our interest in the following property.

Proposition 2 *(see (Conte G., 1995*b*)) Over a PID ring R a submodule $\mathcal{V} \subseteq \mathcal{X}$ is a direct summand if and only if the Smith form of any matrix whose columns generate $\mathcal{V}$ is the identity.*

In general, if R is not a PID and Proposition 2 does not apply , it is difficult to prove that a submodule is a direct summand but the following result helps us in checking the feedback invariance property. See also (Perdon A. M. and Peroli, 2003) for another approach.

Proposition 3 *(see (Assan J. and Perdon, 2002)) Let Σ be a system defined by (2) over a commutative ring R, let $\mathcal{V}$ be a submodule of R^n and let v be an element of $\mathcal{V}$. Denote by $\mathcal{F}_V(v)$ the set of all friends of v, i.e. all $F \in R^{m\times n}$ such that $(A+BF)v \in \mathcal{V}$ and let $\{v_i,\ i \in \mathcal{I}\}$ be a set of generators for a submodule $\mathcal{V}$ of R^n.*
Then $\mathcal{V}$ is feedback invariant if and only if

$$\bigcap_{i\in\mathcal{I}} \mathcal{F}_V(v_i) \neq \emptyset. \qquad (3)$$

The above condition is practically computable in many cases which are important for applications as, for instance, when R is the rings of polynomials in one or several indeterminates over the reals. If a controlled invariant submodule is not feedback invariant, a more general property of the same kind is still valid (see (Conte G., 1995*b*)). In order to describe the situation, assume that $\mathcal{V} \subset \mathcal{X} \subset$

R^n is an (A,B)-invariant submodule generated by the q columns of the matrix V, i.e. there exist two matrices $L \in R^{q\times q}$ and $M \in R^{m\times q}$ such that

$$AV = VL + BM,$$

and consider the extension Σ_e of the system (2) given by the equations

$$\Sigma_e = \begin{cases} x_e(t+1) = A_e x_e(t) + B_e u_e(t) \\ y(t) = C_e x_e(t) \end{cases}, \quad (4)$$

where

$$x_e = \begin{bmatrix} x(t) \\ x_a(t) \end{bmatrix},\ u_e = \begin{bmatrix} u(t) \\ u_a(t) \end{bmatrix},$$

$$A_e = \begin{bmatrix} A & 0 \\ 0 & 0_{q\times q} \end{bmatrix},\ B_e = \begin{bmatrix} B & 0 \\ 0 & I_{q\times q} \end{bmatrix}$$

$$\textit{and } C_e = [C\ 0_{q\times q}].$$

The module $\mathcal{V}_e$ generated in $\mathcal{X}_e = R^{n+q}$ by the columns of $V_e = \begin{bmatrix} V \\ I_{q\times q} \end{bmatrix}$ is a direct summand of $\mathcal{X}_e = R^n \bigoplus R^q$ and a controlled invariant submodule for Σ_e, since

$$A_e V_e = \begin{bmatrix} AV \\ 0_{k\times k} \end{bmatrix} = V_e L + B_e \begin{bmatrix} M \\ -L \end{bmatrix}. \quad (5)$$

Then, by Proposition 1, $\mathcal{V}_e$ is feedback invariant and

$$\begin{bmatrix} u(t) \\ u_a(t) \end{bmatrix} = F_e \begin{bmatrix} x(t) \\ x_a(t) \end{bmatrix} + G_e v(t), \quad (6)$$

where

$$F_e = \begin{bmatrix} -FV - M & F \\ -L & 0 \end{bmatrix} \text{ and } G_e = \begin{bmatrix} G \\ G_a \end{bmatrix} \quad (7)$$

is a friend of $\mathcal{V}_e$ for any F.

Remark that the action of F_e can be seen as the action on the original system Σ of a dynamic feedback of the form

$$\begin{cases} x_a(t+1) = A_1 x(t) + A_2 x_a(t) + G_1 v(t) \\ u(t) = Fx(t) + Hx_a(t) + G_2 v(t) \end{cases} \quad (8)$$

where $x_a \in \mathcal{X}_a = R^q$, $v \in R^m$, A_1, A_2, F, H, G_1 and G_2 are matrices of suitable dimensions with entries in the ring R.

Another geometric notion, crucial in dealing with non interacting control problems, is the notion of controllability submodule.

Definition *A submodule $\mathcal{R} \subseteq \mathcal{X}$ is said to be a controllability submodule if and only if there exist R-linear maps $F : \mathcal{X} \to \mathcal{U}$, $G : \mathcal{U} \to \mathcal{U}$ such that*

$$\mathcal{R} = < A + BF | Im(BG) > .$$

A controllability submodule is (controlled invariant and) feedback invariant. In particular, for every friend F of $\mathcal{R}$ one has

$$\mathcal{R} = < A + BF | Im B \cap \mathcal{R} > .$$

The fact that the one given above is not a purely geometric characterization, but one which relies on the existence of a feedback, makes its use difficult for system with coefficients in a ring. In order to avoid this, a weaker notion was introduced.

Definition *(see (Conte G., 1995*a*)) A submodule $\mathcal{R} \subseteq \mathcal{X}$ is said to be a pre-controllability submodule if the following holds:*
i) $\mathcal{R}$ is (A,B)-invariant;
ii) $\mathcal{R}$ coincides with the minimum element $\mathcal{S}_*$ of the family $\mathcal{S}_R$, defined by

$$\mathcal{S}_{\mathcal{R}} = \{\mathcal{S} \subseteq \mathcal{X} \ s.\ t.\ \ \mathcal{S} = \mathcal{R} \cap (A\mathcal{S} + ImB)\}.$$

The relationship between this notion and that of controllability submodule, is described in (Conte G., 1995*a*). Here, it is sufficient to recall that a submodule $\mathcal{R} \subseteq \mathcal{X}$ is a controllability submodule if and only if it is a pre-controllability submodule and feedback invariant.

4. THE DISTURBANCE DECOUPLING PROBLEM

Conditions for the solvability of many control problems are formulated in terms of set theoretic relations involving controlled invariant submodules or pre-controllability submodules. For example, let us recall that, for a system Σ described over a Noetherian ring R by equations of the form

$$\begin{cases} x(t+1) = Ax(t) + Bu(t) + Dq(t) \\ y(t) = Cx(t) \end{cases} \quad (9)$$

where $q \in R^k$ denotes a disturbance, the Disturbance Decoupling Problem for Σ consists in finding a feedback law of the form (8) such that, in the compensated system, the disturbance does not appear on the output. Denoting by $\mathcal{V}^*$ the maximum (A,B)-invariant submodule contained in $Ker\ C$, we have

Proposition 4 *(see (Conte G., 1995*b*)) Let Σ be the system defined by equations (9) over a Noetherian ring R. Then, the Disturbance Decoupling Problem for Σ is solvable if and only if*

$$Im\ D \subseteq \mathcal{V}^* + Im\ B, \quad (10)$$

where $\mathcal{V}^*$ is the maximum (A,B)-invariant submodule contained in $Ker\ C$.

Computation of $\mathcal{V}^*$ is necessary in order to check condition (10), but the I.S.A. algorithm for computing $\mathcal{V}^*$, see (Wonham, 1985), does not converge in a finite number of steps in the case of systems over rings. Although a new, different algorithm has been found for systems over a PID in (Assan J. and Loiseau, 1999), the computation of $\mathcal{V}^*$ over more general rings is still an open problem.

However, the solution of the DDP problem can be studied by means of the following alternative result, which makes use of $\mathcal{R}^*_{(BD)}$, the maximum pre-controllability submodule in $Ker\ C$ with respect to the input matrix $(B\ \ D)$.

Proposition 5 *(see (Assan J. and Perdon, 1999)) Given a system Σ, defined over the Noetherian ring R by equations of the form (9), denote by $\mathcal{R}^*_{(BD)}$ the maximum pre-controllability submodule in $Ker\ C$ with respect to the input matrix $(B\ \ D)$. Then, the DDP for Σ is solvable if and only if*

$$Im\ D \subseteq \mathcal{R}^*_{(BD)}. \tag{11}$$

Efficient algorithms based on Gröbner basis theory for checking if a given submodule is contained in another one are available over rings of polynomials in several indeterminate over a field. They will be discussed in the following section, where we explicitly present an algorithm to compute the maximum pre-controllability submodule. Other algorithms are available and, due to the pages limit, will be presented in the full paper.
A novel algorithm to compute $\mathcal{R}^*$ by a converging sequence of submodules is proposed in (Assan J. and Perdon, 1999). In (Assan J., 1999) we also find a first attempt of implementing such algorithm, relying on the computation of counterimages and intersections of modules over polynomial rings. Since such computations can be very expensive, it crucial to optimize them. We here propose new and optimized algorithms which compute the substeps of such implementation.

5. ALGORITHMS

In the following we only consider the general case of a noetherian ring: the PID case is much simpler and all the procedures proposed below work without modifications also in that case. Given a system described by equations (2) over a ring R and a submodule $\mathcal{V} \subseteq \mathcal{X}$, spanned by the columns of a matrix V, the controlled invariance property check consists practically in finding matrices L and M satisfying

$$AV = VL + BM, \tag{12}$$

i. e. in solving with respect to the matrices L ed M, the linear system of equations over R

$$[V \mid B]\left[\frac{L}{M}\right] = AV \tag{13}$$

When the coefficients of the system belong, for instance, to a ring of polynomials in one or several indeterminates over the rationals, this can be done by syzygy computations (see any Computer Algebra text, e.g. (Kreuzer M., 2000)) or by another technique called *module components elimination*, see (Caboara M., 2003) and applied to geometric approach in (Perdon A. M. and Peroli, 2003). Both techniques are supported by modern Computer Algebra Systems. See also (Malashonok, 1991) for a survey of other techniques.
In the following, we use the sub procedure `ModImage` and the check of inclusion (11) which are trivial, and the sub procedures `Ker` and `ModCImage`, which can be realized by syzygy computations or by module components elimination. The procedure `SpecialInverse_One` returns, if it exists, one inverse of the argument solving the associate linear system.

<ALGORITHM 1:>

```
Define Is_V_AB_Invariant(A,B,V)
  L:=ColumnVectors(BlockMatrix([[-A*V,V,B]]));
  S:=Gens(Syz(L));
  X0:=Transposed(
       Mat([First(Vi,NumCol(B))|Vi In S]));
  X1:=Transposed(
       Mat([GetFromIToJ(Vi,NumCol(B)+1,
           NumCol(B)+NumCol(V))|Vi In S]));
  X2:=Transposed(
        Mat([Last(Vi,NumCol(V))|Vi In S]));
  TT:=SpecialInverse_One(X0);
  If Not TT[1] Then Return [False]; End;
  K0:=TT[2][1];//K0 is the inverse of X0
  OK:=A*V=V*X1*K0+B*X2*K0;
  Return [True,X1*K0,X2*K0];
End;
```

Assume now that $\mathcal{V}$ is (A,B)-invariant. The existence of the feedback can then be checked by Proposition 3.

If the controlled invariant submodule is not feedback invariant we can compute the extended system and the dynamic feedback (7), as it has been described in Section 3. We refer to (Perdon A. M. and Peroli, 2003) for the details and the implementation.

The condition in Proposition 5 needs, to be verified, the computation of $\mathcal{R}^*$. We here propose the following algorithm in CoCoA:

<ALGORITHM 2>

```
Define SStar(A,B,C)
  KerC:=Ker(C);
  NextS:=B;
  Repeat
    LastS:=NextS;
    NextS:=
      B+ModImage(A,Intersection(LastS,KerC));
  Until NextS=LastS;
  Return LastS;
End;


Define RStar(A,B,C)
  SBKerC:=
  Intersection(SStar(A,B,C),Ker(C));
  NextR:=
  Intersection(SBKerC,ModCImage(A,B));
  Repeat
    LastR:=NextR;
    NextR:=
    Intersection(SBKerC,
                 ModCImage(A,LastR+B));
  Until LastR=NextR;
  Return LastR;
End;
```

Remark Syzygy computations over a computable polynomial ring rely on Gröbner basis computations, hence, they present a quite high worst-case complexity. Practical complexity for given examples is quite difficult to be determined *a priori*. Moreover, Gröbner basis computations are notoriously not stable over the reals, requiring use of unlimited precision rational arithmetic. Nevertheless, modern systems, like CoCoA, see (Kreuzer M., 2000), or Singular, see (Greuel *et al.*, 2001), offer quite advanced capabilities in Gröbner basis computations and many cases of practical interest can in fact be worked out. See (Cox D. and O'Shea, 1993) for an easy introduction or (Kreuzer M., 2000) for more mathematical detail on Gröbner basis theory.

In (Perdon A. M. and Peroli, 2003) we will present several examples where all the computations have been performed with CoCoA 4.1 on a AMD 7 1GHz processor with 750MB RAM running Linux in less than one second CPU time. Here we only report a small example, for space reasons.

Example Let us consider the delay-differential system Σ_d described by the equations

$$\begin{cases} \dot{x}_1(t) = x_1(t) + x_2(t-h_1-h_2)+ \\ \qquad +u_1(t-h_1) \\ \dot{x}_2(t) = x_1(t) + x_2(t-h_1-h_2)+ \\ \dot{x}_3(t) = x_4(t) + x_2(t-h_1-h_2)+ \\ \qquad +q(t-h_1-h_2) \\ \dot{x}_4(t) = x_2(t-h_2) + x_4(t)+ \\ \qquad +u_2(t) + q(t) \\ \dot{x}_5(t) = x_4(t-h_2) + x_2(t-h_1-h_2)+ \\ \qquad +u_3(t-h_2) \\ y_1(t) = x_1(t-h_1) - x_2(t) \\ y_2(t) = x_4(t) \end{cases} \tag{14}$$

We can associate to Σ_d the system described over the Noetherian ring $R[\Delta_1, \Delta_2]$ by equations of the form (9), where

$$A = \begin{bmatrix} 1 & \Delta_1\Delta_2 & 0 & 0 & 0 \\ 1 & \Delta_1\Delta_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \Delta_1 & 0 \\ 0 & \Delta_2 & 0 & 1 & 0 \\ 0 & 0 & 0 & \Delta_2 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} \Delta_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \Delta_2 \end{bmatrix},$$

$$C = \begin{bmatrix} \Delta_1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and

$$D = \begin{bmatrix} 0 \\ 0 \\ \Delta_1\Delta_2 \\ 1 \\ 0 \end{bmatrix}.$$

Applying Algorithm 3 we have that that the columns of the matrix

$$V := \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \Delta_1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

generate $\mathcal{R}^*_{(BD)}$, the maximum pre-controllability submodule in $Ker\ C$ with respect to the input matrix $(B\,D)$ and condition (11) is satisfied. Although $\mathcal{R}^*_{(BD)}$ is not a direct summand, condition (3) is satisfied (the computations lasted Cpu times less than 0.5 seconds), therefore $\mathcal{R}^*_{(BD)}$ is feedback invariant. By using Algorithm 1, we obtain

$$L := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

and

$$M := \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Then, a possible feedback is given by (7). Going back to the original delay differential sysytem we have the following state feedback

$$F = \begin{cases} u_1(t) = 0 \\ u_2(t) = -x_3(t) - x_4(t) \\ u_3(t) = -x_4(t) \end{cases}$$

It is easy to verify that the disturbance is decoupled from the output of the compensated system

$$\begin{cases} \dot{x}_1(t) = x_1(t) + x_2(t - h_1 - h_2) \\ \dot{x}_2(t) = x_1(t) + x_2(t - h_1 - h_2) \\ \dot{x}_3(t) = x_4(t - h_1) + q(t - h_1 - h_2) \\ \dot{x}_4(t) = x_2(t - h_2) + x_4(t) \\ \dot{x}_5(t) = 0 \\ y_1(t) = x_1(t - h_1) - x_2(t) \\ y_2(t) = x_4(t) \end{cases} \tag{15}$$

6. CONCLUSIONS

Symbolic computation tools which perform efficiently provide the possibility of implementing basic algorithms for analyzing geometric properties of dynamical systems with coefficients in a ring and for constructing feedback solutions to control problems. Their use allows to develop practical design methodologies for delay differential systems based on the geometric approach.

As a matter of fact, though the geometric approach extended to the rings has been allowing for a couple of decades a deep insight in the properties of several kind of systems, the complete lack of computational techniques prevented such results to be of immediate use: throughout the bibliography concerning this subject it is not easy to find practical examples which are worked out by an application of the theoretical results coming from the geometrical approach. And this happens since, up to now, no computational technique has been developed within the ring context, as it requires a complex knowledge in computational commutative algebra. The results which we here propose represent a first step towards the filling of such a gap: we can now compute the solutions of several design problems, such as DDP, based on the effective computations of fundamental objects of the geometrical approach, and we could consider issues coming from practical design problems, such as robustness of solutions.

7. REFERENCES

Assan J., A.M. Perdon (1999). An efficient computation of the solution of the block decoupling problem with coefficient assignment over a ring. *Kybernetika* **35**(6), 765–776.

Assan J., J.F. Lafay, A.M. Perdon and J.J. Loiseau (1999). Effective computation of maximal controlled invariant submodules over a principal ring. *Proc. 38th IEEE Conference on Decision and Control.*

Assan J., J.F. Lafay and A.M. Perdon (1999). Computation of maximal pre-controllability submodules over a noetherian ring. *System and Control Letters* **37**(3), 153–161.

Assan J., J.F. Lafay and A.M. Perdon (2002). Feedback invariance and injection invariance for systems over rings. *Proc. 15th IFAC Wordl Congress.*

Bareiss, E.H. (1968). Sylvester identity and multistep integer-preserving gaussian elimination. *Math. Comp.* **22**(103), 565–578.

Basile G., G. Marro (1992). *Controlled and Conditioned Invariants in Linear System Theory.* Prentice Hall. Englewood Cliffs, New York.

Caboara M., C. Traverso (2003). Efficient algorithms for basic module operations. Submitted.

Capani A., G. Niesi, L. Robbiano (1995). CoCoA, a system for doing computations in commutative algebra. Available via anonymous ftp from `cocoa.dima.unige.it`.

Conte G., A.M. Perdon (1995*a*). The decoupling problem for systems over a ring. *Proc. 34th IEEE Conference on Decision and Control.*

Conte G., A.M. Perdon (1995*b*). The disturbance decoupling problem for systems over a ring. *SIAM Journal on Control and Optimization.*

Conte G., A.M. Perdon (1997). Noninteracting control problems for delay-differential systems via systems over a ring. *European Journal of Automation* **31**(6), 1059–1076.

Cox D., J. Little and D. O'Shea (1993). *Ideal, Varieties and Algorithms.* Springer Verlag. Berlin, New York.

Greuel, G.-M., G. Pfister and H. Schönemann (2001). SINGULAR 2.0. A Computer Algebra System for Polynomial Computations. Centre for Computer Algebra. University of Kaiserslautern. `http://www.singular.uni-kl.de`.

Kreuzer M., L. Robbiano (2000). *Computational Commutative Algebra 1.* Springer Verlag. Berlin, New York.

Malashonok, G.I. (1991). Algorithms for the solution of systems of linear equations in commutative rings. In: *Effective Methods in Algebraic Geometry* (T Mora and C. Traverso, Eds.). pp. 289–298. Birkhauser.

Perdon A. M., M. Caboara and G. Guidone Peroli (2003). Symbolic computations in the geometric approach to systems over rings. Submitted.

Wonham, M. (1985). *Linear Multivariable Control: a Geometric Approach.* 3rd Ed., Springer Verlag. Englewood Cliffs, New York.