Approximately Singular Multivariate Polynomials ⋆

Tateaki Sasaki[1]

Institute of Mathematics, University of Tsukuba
Tsukuba-shi, Ibaraki 305-8577, Japan
sasaki@math.tsukuba.ac.jp

Abstract. Approximate singularity is defined for bivariate and multivariate polynomials. For bivariate polynomials, approximately singular points in weak and strong senses are defined, and a method to determine an approximately singular point in weak sense is given. For bivariate and multivariate polynomials, a method to expand the roots into (fractional-)power series at a singular point is presented, in which the small (erroneous) terms are treated as perturbations. Furthermore, a concept of approximate unconjugacy of power-series roots is introduced.

Key words: algebraic-numeric computation, approximate algebra, approximate algebraic computation, approximately singular point, approximate singularity, singular point, singularity.

1 Introduction

¿From the late 1980's, the author has been studying approximate algebraic computation [11], and he and his coworkers introduced so far concepts of approximate greatest common divisor (GCD) of multivariate as well as univariate polynomials [15, 9, 10], approximate square-free decomposition [15] and approximate multivariate factorization [17, 16, 12]. Corless et al. introduced a concept of approximate decomposition of multivariate as well as univariate polynomials [2]. We call the algebra based on such approximate operations *approximate algebra*.

The main purpose of approximate algebra is not to handle polynomials with coefficients of floating-point numbers, as in [18], although it is an important purpose. The main purpose is, given a polynomial or a set of polynomials not having a desirable property, to find another polynomial or a set of polynomials which has a desirable property, by changing the coefficients slightly. For example, factorizability is a desirable property because we can separate a system into simpler subsystems. Thus, given a multivariate polynomial which is nearly factorizable, we want to obtain a factorizable polynomial by changing the coefficients slightly. Then, as a natural development of approximate algebra, we are led to concepts of approximate singularity.

We have already several such concepts; the nearest polynomial [25, 4, 5, 20] and the pseudo-variety [8, 22, 23, 21, 6, 3]. If a univariate polynomial has multiple roots then it is called singular. Then, we can say that a polynomial is approximately singular if it has no multiple root but close roots. Given a non-singular univariate polynomial, the nearest polynomial is a singular polynomial obtained by changing the coefficients minimally in the sense of some norm. The nearest polynomial plays an important role in control theory; see [1], for example. The approximate square-free decomposition with the minimum tolerance is then a generalization of the nearest polynomial, although the theory is not developed yet. Note that the approximate square-free decomposition is applicable to polynomials with inexact coefficients, while the nearest polynomial can be defined only for exact polynomials. The pseudo-variety is a set of algebraic varieties of a polynomial system and its nearby systems, where the nearby systems are obtained by changing the coefficients of the given system slightly. Stetter and Thallinger [19] called a multivariate polynomial system singular if it has an infinite number of solutions.

In this paper, we consider bivariate and multivariate polynomials, with small perturbations. The coefficients may be exact or inexact. Singularity of a polynomial may be changed even by a small perturbation, as we will see in **2** for bivariate polynomials. Furthermore, a small perturbation often

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makes the algebraic curve pretty complicated in structure. Therefore, given a polynomial with a small perturbation, we want to determine a polynomial which gives an algebraic curve of a simple structure and which represents a set of nearby polynomials. This desire leads us to a concept of approximately singular point, in weak and strong senses. Converting approximately singular polynomials to singular ones, we will be able to see and treat characteristic behaviors of algebraic curves (surfaces) defined by the original polynomials. In **2**, we will give a method to determine approximately singular points (in weak sense only) and corresponding perturbations.

Next, we investigate the power-series expansion of the roots of bivariate and multivariate polynomials with perturbations. As is well-known, the roots of bivariate polynomial can be expanded at a singular point into Puiseux series (fractional-power series). As we will show by an example in **2**, if a given polynomial contains small (erroneous) terms, the resulting series becomes sometimes strange in that the convergence radius is very small. In **3**, we will present an expansion method, both for bivariate and multivariate polynomials, in which the small (erroneous) terms are treated as perturbations. The resulting series reproduces the global structure of the curve, and converges to the power series with no perturbation; it, however, diverges at the singular point. Then, we will be led to a concept of approximate unconjugacy of power-series roots.

This paper is of such a kind that presents new basic concepts and imposes related problems, hence we explain them by many examples. The concepts presented will become important in approximate algebra.

2 Approximately Singular Bivariate Polynomial

Let $F(y,x)$ be a bivariate polynomial over $\mathbf{C}$. Singularity is one of the most important concept in algebraic geometry, see [24] for example. A point $(\hat{x},\hat{y}) \in \mathbf{C}^2$ satisfying

$$F(\hat{x},\hat{y}) = \frac{\partial F}{\partial y}(\hat{x},\hat{y}) = \frac{\partial F}{\partial x}(\hat{x},\hat{y}) = 0 \tag{2.1}$$

is called a *singular point* of $F(y,x)$, and $F(y,x)$ is called *singular* at $(\hat{x},\hat{y})$. For the later use, we define a point $(\tilde{x},\tilde{y}) \in \mathbf{C}^2$ to be *semi-singular* if it is a solution of one of the above three systems, i.e. if $(\tilde{x},\tilde{y}) \in V_1 \cup V_2 \cup V_3$, where

$$\begin{aligned} V_1 &= \text{variety}(\,\{F=0,\ \partial F/\partial y=0\}\,), \\ V_2 &= \text{variety}(\,\{\partial F/\partial y=0,\ \partial F/\partial x=0\}\,), \\ V_3 &= \text{variety}(\,\{F=0,\ \partial F/\partial x=0\}\,). \end{aligned} \tag{2.2}$$

Throughout this paper, by $\|\cdot\|$ we denote a norm (the infinity norm in examples).

Let us consider what happens on the singularity if $F(y,x)$ is perturbed slightly.

Example 1 $F(y,x) = y^2 - x^3 - \delta_2 x^2 - \delta_1 x, \quad 0 \le |\delta_1|, |\delta_2| \le \varepsilon \ll 1.$

Let $F_0(y,x) = y^2 - x^3$, $F_1(y,x) = y^2 - x^3 - \delta_1 x$, and $F_2(y,x) = y^2 - x^3 - \delta_2 x^2$. Figures 1-1, 1-2 and 1-3 show the curves determined by $F_0(y,x) = 0$, $F_1(y,x) = 0$ and $F_2(y,x) = 0$, respectively.

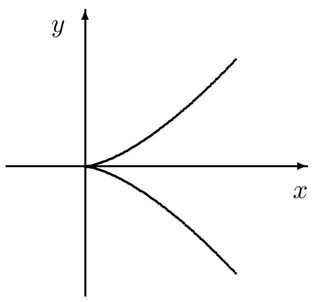


Fig.1-1 $\delta_1 = \delta_2 = 0$

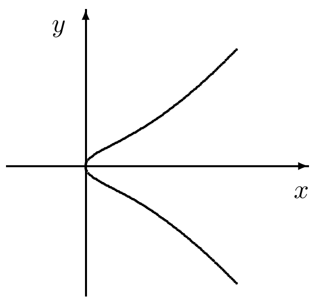


Fig.1-2 $\delta_1 = 0.05$

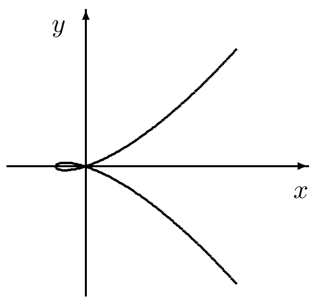


Fig.1-3 $\delta_2 = 0.10$

$F_0(y,x)$ has only one singular point at $(0,0)$, $F_1(y,x)$ has no singular point, and $F_2(y,x)$ has only one singular point at $(0,0)$. $F(y,x)$ has no singular point if $\delta_2^2-3\delta_1 = 0$, and it has a singular point at $(x,y) = (-\delta_2/2,\, 0)$ if $\delta_2^2-4\delta_1 = 0$. In fact, if $\delta_2^2-4\delta_1 = 0$, we can rewrite $F(y,x)$ as $F(y,x) \to y^2-x\,(x+\delta_2/2)^2 \stackrel{\text{def}}{=} F_3(y,x)$. $F_3(y,x)$ is obtained from F by the minimum perturbation $\Delta = (\delta_1 - \delta_2^2/4)x : F_3(y,x) = F(y,x) + \Delta(y,x)$. ♢

As Example 1 shows, the singular point may move, appear or disappear if $F(y,x)$ is perturbed even slightly. However, the global structure of the curves determined by $F_0(y,x) = 0$, $F_1(y,x) = 0$ and $F_2(y,x) = 0$ are approximately the same. Furthermore, among the polynomials which give curves of a globally similar structure, one polynomial ($F_0(y,x)$ in the above example) gives a curve of extremely simple structure. So, we want to treat the curves determined by $F(y,x) = 0$, with $0 \le |\delta_1|, |\delta_2| \le \varepsilon \ll 1$, as a set of curves which are represented by a curve determined by $F_0(y,x) = 0$, for example. Thus, we are led to a concept of approximate singularity, in weak and strong senses, as follows.

Definition 1 (approximate singularity in weak sense) *Let $F(y,x)$ and $\Delta(y,x)$ be*

$$\begin{cases} F(y,x) = \sum_{i,j} c_{ij} y^i x^j, \\ \Delta(y,x) = \sum_{i,j} \delta_{ij} y^i x^j, \end{cases} \quad \|\Delta\|/\|F\| < \varepsilon \ll 1. \tag{2.3}$$

(The support of Δ, $\mathrm{supp}(\Delta) = \{\, x^i y^j \mid \delta_{ij} \ne 0 \,\}$, may be chosen rather arbitrarily such as $\mathrm{supp}(\Delta) \subseteq \mathrm{supp}(F)$ or $\mathrm{supp}(\Delta) \not\subseteq \mathrm{supp}(F)$). If we can choose $\Delta(y,x)$ such that $F{+}\Delta$ is singular at $(\hat{x},\hat{y}) \in \mathbf{C}^2$ then we say that $F(y,x)$ is approximately singular at $(\hat{x},\hat{y})$ in weak sense, with tolerance ε and that $(\hat{x},\hat{y})$ is an approximately singular point of $F(y,x)$.

Definition 2 (approximate singularity in strong sense) *Let $F(y,x)$ and $\Delta(y,x)$ be the same as in Def. 1. Suppose that, for a suitably chosen $\Delta(y,x)$, $(\hat{x},\hat{y})$ is a singular point of $\tilde{F}(y,x) \stackrel{\text{def}}{=} F(y,x){+}\Delta(y,x)$. Furthermore, suppose that we can choose a small domain $D(\hat{x},\hat{y})$ such that, for every $\Delta(y,x)$ satisfying (2.3), $D(\hat{x},\hat{y})$ contains only one or no singular point around $(\hat{x},\hat{y})$ and it contains all the semi-singular points around $(\hat{x},\hat{y})$. If we can choose $\Delta(y,x)$ so that $\tilde{F}(y,x)$ has a singular point at $(\hat{x},\hat{y})$ and no other semi-singular point in $D(\hat{x},\hat{y})$, then we say that $F(y,x)$ is approximately singular at $(\hat{x},\hat{y})$ in strong sense, with tolerance ε.*

Example 2 Approximately singular point of $F(y,x)$ in Example 1.
Semi-singular points $(\tilde{x}_i,\tilde{y}_i) \in V_i$ $(i=1,2,3)$ of $F(y,x)$ are computed as follows.

$$\begin{aligned}
&(\tilde{x}_1,\tilde{y}_1) : \Big(\, \frac{1}{2}\Big(-\delta_2 \pm \sqrt{\delta_2^2-4\delta_1}\,\Big),\ 0\,\Big), \quad (0,\,0),\\
&(\tilde{x}_2,\tilde{y}_2) : \Big(\, \frac{1}{3}\Big(-\delta_2 \pm \sqrt{\delta_2^2-3\delta_1}\,\Big),\ 0\,\Big),\\
&(\tilde{x}_3,\tilde{y}_3) : \Big(\, \frac{1}{3}\Big(-\delta_2 + \sqrt{\delta_2^2-3\delta_1}\,\Big),\ \pm\frac{1}{9}\sqrt{6\delta_2^3-27\delta_2\delta_1-6\sqrt{(\delta_2^2-3\delta_1)^3}}\ \Big),\\
&\qquad\qquad\ \Big(\, \frac{1}{3}\Big(-\delta_2 - \sqrt{\delta_2^2-3\delta_1}\,\Big),\ \pm\frac{1}{9}\sqrt{6\delta_2^3-27\delta_2\delta_1+6\sqrt{(\delta_2^2-3\delta_1)^3}}\ \Big).
\end{aligned}$$

Points $(\tilde{x}_i,\tilde{y}_i)$ $(i=1,2,3)$ become identical only when $\tilde{y}_3 = 0$, or $(6\delta_2^3-27\delta_2\delta_1)^2 = 36(\delta_2^2-3\delta_1)^3$ $\Longrightarrow 27\delta_1^2(\delta_2^2-4\delta_1) = 0$. Setting $\delta_1 = 0$, we obtain $F(y,x) = F_2(y,x)$ and we know that $F_2(y,x)$ has a singular point at $(0,0)$. This approximately singular point is, unless $\delta_2 = 0$, not in strong sense because there are several semi-singular points around $(0,0)$. Setting $\delta_2^2-4\delta_1 = 0$, we obtain $F(y,x) = y^2-x\,(x+\delta_2/2)^2 = F_3(y,x)$ and we know that $F_3(y,x)$ has a singular point at $(-\delta_2/2,\,0)$. This point is also not approximately singular in strong sense. The approximately singular point in strong sense is obtained only by setting $\delta_1 = \delta_2 = 0$. ♢

The above definitions impose us the following problems immediately.

Problem 1 Given a bivariate polynomial $F(y,x)$ which has an approximately singular point at $(\hat{x},\hat{y})$ with tolerance ε, determine a polynomial $\tilde{F}(y,x)$ which has a singular point at $(\hat{x},\hat{y})$.

Problem 2 Determine a polynomial $\tilde{F}(y,x)$ which shows approximate singularity in strong sense and a corresponding approximately singular point $(\hat{x},\hat{y})$.

Problem 3 Determine the minimum value of the tolerance ε and the corresponding perturbation $\Delta(y,x)$, for which $F(y,x)$ has an approximately singular point.

Problem 1 is not difficult to solve. Here, we describe a numerical method.

Procedure ApproxSPweak($F(y,x)$, ε) ==

Step 1 : Compute the semi-singular points numerically ;
If V_1, V_2 and V_3 in (2.2) contain no mutually close solution then there is no possibility that $F(y,x)$ has an approximately singular point ;

Step 2 : Suppose that V_1, V_2 and V_3 contain solutions $(\tilde{x}_i,\tilde{y}_i) \in V_i$ $(i=1,2,3)$ such that $(\tilde{x}_1,\tilde{y}_1) \simeq (\tilde{x}_2,\tilde{y}_2) \simeq (\tilde{x}_3,\tilde{y}_3)$, then put $G(y,x) = F(y+\tilde{y}, x+\tilde{x})$, where

$$\tilde{x} = (\tilde{x}_1+\tilde{x}_2+\tilde{x}_3)/3, \quad \tilde{y} = (\tilde{y}_1+\tilde{y}_2+\tilde{y}_3)/3 \ ;$$

Step 3 : Put $\Delta = d_0 + d_1 y + d_2 x + \cdots$, and determine $d_0, d_1, d_2, \ldots$ so as to satisfy (not only the followings but also higher derivatives, if Δ contains higher degree terms)

$$(G+\Delta)(0,0) = 0, \quad \partial(G+\Delta)/\partial y(0,0) = 0, \quad \partial(G+\Delta)/\partial x(0,0) = 0 \ ;$$

Step 4 : Check the tolerance, and repeat the above procedure if necessary. (In the repetition, choose a semi-singular point which is not equal to but nearest to $(\tilde{x},\tilde{y})$ as a candidate of an approximately singular point.) ♢

The merit of utilizing semi-singular points is not only easiness of computation but also that, even if $F(y,x)$ is approximately singular, the system (2.1) may have no solution because it is an over-determined one. The reason for moving the origin is that, if the singular point is near the origin, higher total-degree terms are ineffective for determining the singular point, and we may determine perturbations only for lower total-degree terms.

Example 3 Let $F(y,x)$ be as follows (F has a singular point at $(-0.51,\ 0.49)$).

$$\begin{aligned} F(y,x) = {} & y^3 - 2y^2x^2 + 1.96y^2x + 1.0498y^2 + yx^3 - 3.51yx^2 + 2.7195yx \\ & + 0.172847y + 0.51x^3 - 2.2699x^2 + 1.857149x - 0.29235001 \ . \end{aligned}$$

Suppose that, by computing semi-singular points numerically, we find $(-0.5,\ 0.5)$ as a candidate of an approximately singular point of tolerance $O(0.01)$. Moving the origin to $(-0.5,\ 0.5)$, we put $G(y,x) = F(y-0.5,\ x+0.5)$. Then, putting $\Delta = d_0 + d_1 y + d_2 x + d_3 y^2 + d_4 yx + d_5 x^2 + \cdots$, we determine $d_0, d_1, d_2, \ldots$ as follows.

$$\begin{aligned} (G+\Delta)(0,0) = 0 \ &\Rightarrow d_0 = 0.000099\cdots, \\ \partial(G+\Delta)/\partial y(0,0) = 0 \ &\Rightarrow d_1 = 0.00029\cdots, \\ \partial(G+\Delta)/\partial y(0,0) = 0 \ &\Rightarrow d_2 = 0.020001, \\ \partial^2(G+\Delta)/\partial y^2(0,0) = 0 \ &\Rightarrow d_3 = -0.0298, \\ \partial^2(G+\Delta)/\partial y \partial x(0,0) = 0 \ &\Rightarrow d_4 = 0.00049\cdots, \\ \partial^2(G+\Delta)/\partial x^2(0,0) = 0 \ &\Rightarrow d_5 = 0.9999 \ . \end{aligned}$$

Among these, d_5 is too large, so should be abandoned. ♢

In the above Example 3, we obtain the following polynomial

$$G+\Delta = y^3 - 2y^2x^2 - 0.04y^2x + yx^3 - 0.01yx^2 + 0.01x^3 - 0.9999x^2.$$

This polynomial has only one singular point at $(0, 0)$; we can check this by computing the Gröbner basis w.r.t. the lexicographic order. However, the origin is a singular point in weak sense and not in strong sense; each of V_1, V_2, V_3 for $G+\Delta$ contain several semi-singular points around the origin. As this example shows, for a given polynomial, there are usually infinitely many nearby polynomials which have approximately singular points in weak sense. Hence, the important and interesting problem is to find a polynomial having approximately singular points in strong sense. We will be able to obtain required polynomials by such an algebraic method that was described in Example 2, but the method seems to be time-consuming and Problems 2 and 3 are open now.

The next important problem is on the solution $y = \phi(x)$ of the equation $F(y, x) = 0$. Once an approximately singular point in strong sense and the corresponding polynomial $\tilde{F}(y, x)$ are determined, the solution $y = \tilde{\phi}(x)$ of $\tilde{F}(y, x) = 0$ will show a simple analytic behavior around the singular point. However, if $F(y, x)$ has several semi-singular points around a point, $\phi(x)$ may show a complicated behavior around the point. The function $\phi(x)$ can be expanded into Taylor series if the expansion point is not a semi-singular point. If the expansion point is a semi-singular point in V_1, $\phi(x)$ is expanded into fractional-power series or Puiseux series in general. A well-known classical method for computing the Puiseux-series solution is Newton-Puiseux's method. Being applied to approximately singular polynomials, Newton-Puiseux's method sometimes gives unexpected results, as the following example shows.

Example 4 Power-series solutions by Newton-Puiseux's method.

$$
\begin{aligned}
F_0 &= y^2 - x^3 &&: \ y_0 = \pm\sqrt{x^3}, \\
F_1 &= y^2 - x^3 - \delta x &&: \ y_1 = \pm\sqrt{\delta}\sqrt{x}\,(1 + \frac{x^2}{2\delta} - \frac{x^4}{8\delta^2} + \cdots), \\
F_2 &= y^2 - x^3 - \delta x^2 &&: \ y_2 = \pm\sqrt{\delta}\,x\,(1 + \frac{x}{2\delta} - \frac{x^2}{8\delta^2} + \cdots).
\end{aligned}
$$

The convergence radii of y_0, y_1 and y_2 are ∞, $\sqrt{\delta}$ and δ, respectively. ♢

The above solutions y_1 and y_2 may be useful if δ is a definite number, but they are completely nonsense if δ is an error. Even if δ is not an error, the solutions will be very erroneous if δ is an erroneous number. On the other hand, we want to treat all the solutions of $F(y, x) = 0$ with small δ_1 and δ_2 as a set of solutions which are approximately the same to one representative solution (y_0 in the above example). Therefore, we impose the following problem.

Problem 4 Determine (fractional-)power series solutions of $F(y, x) = 0$ (and solutions of $F(y, x_1, \ldots, x_\ell) = 0$ w.r.t. y in the multivariate case), which converge to those of the equation with no perturbation.

We can give a definite answer to this problem. Required solutions can be obtained by the extended Hensel construction proposed independently by Kuo [7] for bivariate polynomial and Sasaki-Kako [14] for multivariate polynomials; we will explain the extended Hensel construction and give the answer to Problem 4 in the next section. Here, we give one example.

Example 5 $F(y, x) = y^2 + yx^2 - x^3 - \delta x^2$ (δ is a small number).

We determine fractional-power series $\phi_1^{(k)}(x)$ and $\phi_2^{(k)}(x)$, with $k \in \mathbf{N}$, such that

$$F(y, x) \equiv [y - \phi_1^{(k)}(x)] \cdot [y - \phi_2^{(k)}(x)] \pmod{x^{k+1+3/2}}.$$

For $k = 2$, for example, the following series satisfy this equality.

$$
\begin{cases}
\phi_1^{(2)}(x) = +x^{3/2} - x^2/2 + x^{5/2}/8 - x^{7/2}/128 + O(x^{9/2}) \\
\qquad\qquad + \delta x^{1/2}/2 - \delta x^{3/2}/16 + \delta^2 x^{-1/2}/8 + O(\delta x^{5/2}, \delta^2 x^{1/2}, \delta^3 x^{-3/2})), \\
\phi_2^{(2)}(x) = -x^{3/2} - x^2/2 - x^{5/2}/8 + x^{7/2}/128 - O(x^{9/2}) \\
\qquad\qquad - \delta x^{1/2}/2 + \delta x^{3/2}/16 - \delta^2 x^{-1/2}/8 - O(\delta x^{5/2}, \delta^2 x^{1/2}, \delta^3 x^{-3/2})).
\end{cases}
$$

We note that these power series converge to the solution of $F_0(y, x) = y^2 + yx^2 - x^3 = 0$ as $\delta \to 0$. However, they diverge as $x \to 0$. ♢

3 Power-Series Roots with Error Terms

In this section, by "multivariate polynomials" we mean polynomials in three or more variables. We denote the main variable and sub-variables by y and $x_1, \ldots, x_\ell$, respectively, and abbreviate $x_1, \ldots, x_\ell$ to $\boldsymbol{x}$. Let $F(y, \boldsymbol{x})$ be a given multivariate polynomial over $\mathbf{C}$. In algebraic geometry, the point $(\hat{y}, \hat{\boldsymbol{x}}) \in \mathbf{C}^{\ell+1}$ that satisfies

$$F(\hat{y}, \hat{\boldsymbol{x}}) = \frac{\partial F}{\partial y}(\hat{y}, \hat{\boldsymbol{x}}) = \frac{\partial F}{\partial x_1}(\hat{y}, \hat{\boldsymbol{x}}) = \cdots = \frac{\partial F}{\partial x_\ell}(\hat{y}, \hat{\boldsymbol{x}}) = 0 \tag{3.1}$$

is called a *singular point* of $F(y, \boldsymbol{x})$. Note that singular points of $F(y, \boldsymbol{x})$ may form a curve or a surface (for example, the singular points of $F(z, y, x) = z^3 - (y - x)^2$ form a line $(x = y,\ z = 0)$). This fact makes the choice of the "small domain $D(\hat{x}, \hat{y})$" in Definition 2 complicated. Except for this complication, we can define approximate singularity for multivariate polynomials, just as we have done for bivariate polynomials.

We do not consider the approximate singularity of multivariate polynomial any more, but we investigate how to compute the power-series solutions of $F(y, \boldsymbol{x}) = 0$, which converge to those of the equation with no perturbation. If such a method is developed, the resulting power series will be quite useful for visualizing pseudo-varieties.

We can expand the solutions of $F(y, \boldsymbol{x}) = 0$ w.r.t. y in fractional-power series in $x_1, \ldots, x_\ell$ (multivariate Puiseux series). Unfortunately, the fractional-power series expansion of multivariate polynomial at a singular point is not unique, and the analytic structure of the solutions in the neighborhood of the singular point is not easy to see from the expansion. We use a rather new expansion method called the *extended Hensel construction* which is a Hensel construction at a "singular point" (see below). With this method, we can obtain the solutions of fractional-power series in the total-degree variable, with coefficients of algebraic functions in $x_1, \ldots, x_\ell$; see [14, 13] for details. Furthermore, the solutions manifest the analytic structure in the neighborhood of the singular point, as we will see below.

Definition 3 (singular point for Hensel construction) *Let $F(y, \boldsymbol{x})$ be*

$$F(y, \boldsymbol{x}) = f_n(\boldsymbol{x}) y^n + f_{n-1}(\boldsymbol{x}) y^{n-1} + \cdots + f_0(\boldsymbol{x}), \quad f_n(\boldsymbol{x}) \neq 0. \tag{3.2}$$

A point $(\breve{\boldsymbol{x}}) = (\breve{x}_1, \ldots, \breve{x}_\ell) \in \mathbf{C}^\ell$ is called a singular point for Hensel construction if it satisfies one or both of the conditions **(C1)** *$f_n(\breve{\boldsymbol{x}}) = 0$ and* **(C2)** *$F(y, \breve{\boldsymbol{x}})$ is not square-free.*

Remark Condition (C2) means that $\breve{\boldsymbol{x}}$ satisfies $F(\alpha, \breve{\boldsymbol{x}}) = \frac{\partial F}{\partial y}(\alpha, \breve{\boldsymbol{x}}) = 0$ for some $\alpha \in \mathbf{C}$. Therefore, if $(\breve{y}, \breve{\boldsymbol{x}})$ is a singular point of $F(y, \boldsymbol{x})$ then $(\breve{\boldsymbol{x}})$ is a singular point for the Hensel construction. Condition (C2) implies further that $F(y, \breve{\boldsymbol{x}}) = (y - \alpha)^m \tilde{F}(y, \breve{\boldsymbol{x}})$, $m \geq 2$, and the generalized Hensel construction cannot be applied for the factor corresponding to $(y - \alpha)^m$. This is the reason why $(\breve{\boldsymbol{x}})$ is called a singular point for the Hensel construction. ♢

In this paper, for simplicity, we consider $F(y, \boldsymbol{x})$ satisfying

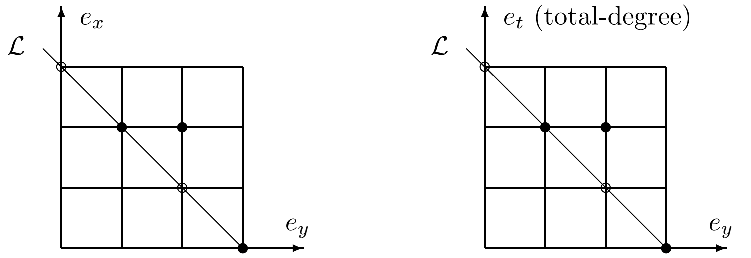
$$f_n(\mathbf{0}) \neq 0, \quad F(y, \mathbf{0}) = f_n(\mathbf{0})\, y^n. \tag{3.3}$$

Note that the origin $(\mathbf{0})$ is a singular point for Hensel construction of $F(y, \boldsymbol{x})$. Choosing the origin as the expansion point, we explain the extended Hensel construction of $F(y, \boldsymbol{x})$. (See [13] for a treatment of the case of $f_n(\mathbf{0}) = 0$). In the extended Hensel construction, the so-called "Newton polynomial" plays an essential role.

Definition 4 (Newton line and Newton polynomial) *For each nonzero term $c y^{e_y} x_1^{e_1} \cdots x_\ell^{e_\ell}$ of $F(y, \boldsymbol{x})$, we plot a dot at the point (e_y, e_t), where $e_t = e_1 + \cdots + e_\ell$, in (e_y, e_t) plane. Let $\mathcal{L}$ be a straight line such that it passes the point $(n, 0)$ as well as another dot plotted and that no dot is plotted below $\mathcal{L}$. The line $\mathcal{L}$ is called* Newton line *for F. The sum of all the terms plotted on $\mathcal{L}$ is called* Newton polynomial *for F and abbreviated to F_{New}.*

Example 6 Newton line $\mathcal{L}$ for F_2 and F_3, where

$$F_2 = y^3 + 2y^2x^2 - yx^2 + \delta_2 y^2 x + \delta_0 x^3 \quad \text{(left figure)}$$
$$F_3 = y^3 + 2y^2(x_1^2 + x_2^2) - yx_1^2 + \delta_2 y^2(x_1 - x_2) + \delta_0 x_2^3 \quad \text{(right figure)}$$



Each dot shows a bivariate term (left figure) or multivariate terms (right figure). White dots indicate terms with small coefficients δ_0 and δ_2. ♢

As Example 6 suggests, we can perform the extended Hensel construction for both bivariate and multivariate polynomials. In the case of multivariate polynomials, it is convenient to introduce the total-degree variable t for sub-variables $x_1, \ldots, x_\ell$: $(x_1, \ldots, x_\ell) \mapsto (tx_1, \ldots, tx_\ell)$ (we may define t by weighting variables as $(x_1, \ldots, x_\ell) \mapsto (t^{w_1}x_1, \ldots, t^{w_\ell}x_\ell)$, where $w_1, \ldots, w_\ell$ are natural numbers). Observing only the main variable y and the total-degree variable t, we see a close similarity between the extended Hensel construction of bivariate and multivariate polynomials.

Let the Newton line $\mathcal{L}$ and a set of integers $(\hat{n}, \hat{e})$ be

$$\mathcal{L}: \ \frac{e_y}{n} + \frac{e_t}{e_0} = 1, \qquad \frac{\hat{n}}{n} = \frac{\hat{e}}{e_0}, \ \ \gcd(\hat{n}, \hat{e}) = 1. \tag{3.4}$$

The procedure of extended Hensel construction is as follows.

Procedure ExtHensel($F(y, \boldsymbol{x}), \kappa$) ==

% Construct $G_1^{(\kappa)}, \ldots, G_r^{(\kappa)}$ s.t. $F \equiv G_1^{(\kappa)} \cdots G_r^{(\kappa)} \pmod{J_{\kappa+1}}$. ($J_k$ is given below)

Step 1 : Determine the Newton polynomial F_{New} by introducing the total-degree variable t for $x_1, \ldots, x_\ell$, and set the ideal J_k as $J_k = \langle t^{k/\hat{n}} \rangle \cdot \langle y^n t^0, \ y^{n-1} t^{\hat{e}/\hat{n}}, \ \cdots, \ y^0 t^{e_0} \rangle$;

Step 2 : Factorize F_{New} into mutually prime factors: $F_{\mathrm{New}} = G_1^{(0)} \cdots G_r^{(0)}$;

Step 3 : Compute Moses-Yun's polynomials $W_1^{(l)}, \ldots, W_r^{(l)}$ satisfying

$$W_1^{(l)} \frac{F_{\mathrm{New}}}{G_1^{(0)}} + \cdots + W_r^{(l)} \frac{F_{\mathrm{New}}}{G_r^{(0)}} = y^l \quad (l = 0, 1, \ldots, n) \ ; \tag{3.5}$$

Step 4 : For $k = 1 \Rightarrow 2 \Rightarrow \cdots \Rightarrow \kappa$, do **4.1** and **4.2** iteratively:

4.1 Compute $\delta F^{(k)} \equiv F - G_1^{(k-1)} \cdots G_r^{(k-1)} \pmod{J_{k+1}} = \delta f_n^{(k)} y^n + \cdots + \delta f_0^{(k)}$;

4.2 Set $G_i^{(k)} = G_i^{(k-1)} + \delta G_i^{(k)}$, where $\delta G_i^{(k)} = \delta f_n^{(k)} W_i^{(n)} + \cdots + \delta f_0^{(k)} W_i^{(0)}$. ♢

Remark Consider a bivariate polynomial $F(y, x)$ and assume that $F_{\mathrm{New}} = y^n - x^{e_0}$, hence $F(y, x)$ is singular at the origin. Put $\hat{x} = x^{e_0/n}$, then the factorization of $F(y, x)$ in $\mathbf{C}[y, \hat{x}]$ is

$$F_{\mathrm{New}} = (y - \omega_1 \hat{x}) \cdots (y - \omega_n \hat{x}), \quad \omega_i^n = 1 \ (i = 1, \ldots, n).$$

Performing the extended Hensel construction with initial factors $(y - \omega_1 \hat{x}), \ldots, (y - \omega_n \hat{x})$, we obtain the Puiseux-series expansion of the solutions of $F(y, x) = 0$. Similarly, we can compute power-series solutions of $F(x, \boldsymbol{x}) = 0$; see [14] for details. ♢

Now, we consider how to compute the power-series roots of $F(y, \boldsymbol{x})$ with perturbation terms. We assume for simplicity that the perturbation terms are proportional to a parameter δ representing a small number. Procedure ExtHensel tells us that, if the Newton polynomial F_{New} is well-determined, the power-series roots of $F(y, \boldsymbol{x})$ do not show such pathological behaviors as in Example 4. Thus, our method consists of the following two operations.

(O1) Set a weight w_δ for δ so that no perturbation term is plotted below the Newton line.
(O2) Perform the extended Hensel construction with this weight.

For example, the solutions $\phi_1^{(2)}$ and $\phi_2^{(2)}$ in Example 5 were computed by setting the weights as $(w_x,\, w_y,\, w_\delta) = (1,\ 3/2,\ 2)$. We give another example of three variables.

Example 7 $F(y,u,v) = y^3 + 2y^2(u^2+v^2) - yu^2 + \delta v^3 = (y-\phi_1)\,(y-\phi_2)\,(y-\phi_3)$.
We set $w_\delta = 0$, then the Newton polynomial and its factorization are

$$F_{\mathrm{New}} = y^3 - yu^2 + \delta v^3 \stackrel{\mathrm{def}}{=} (y-\theta_1)\,(y-\theta_2)\,(y-\theta_3).$$

Here, θ_i $(i=1,2,3)$ are algebraic functions the minimal polynomial of which is F_{New}. Performing the extended Hensel construction, we obtain the power-series roots $\phi_i(u,v)$ $(i=1,2,3)$ of $F(y,u,v)$ as

$$\phi_i \ = \ \theta_i - \frac{2(u^2+v^2)}{4u^6-27\delta^2v^6}\,(2u^4\,\theta_i^2 + 3\delta u^2v^3\,\theta_i - 9\delta^2v^6) + \cdots.$$

Higher order terms of ϕ_i are also expressed by θ_i^2 and θ_i, and we see that ϕ_i $(i=1,2,3)$ manifest the conjugacy clearly. Furthermore, they show singularity, too: each ϕ_i diverges as $4u^6–27\delta^2v^6 \to 0$, showing that the singular points for the Hensel construction constitute hyper-surfaces determined by $4u^6-27\delta^2v^6 = 0$. Expression $4u^6-27\delta^2v^6$ is the discriminant of F_{New}. If $4u^6-27\delta^2v^6 = 0$ then we need another technique to obtain the power-series roots; see [14] for details. ♢

As this example shows, the Newton polynomial F_{New} determines the conjugacy of the roots; if F_{New} is square-free, it determines the conjugacy fully. In particular, if F_{New} is irreducible in $\mathbf{C}[y,\boldsymbol{x}]$ then the roots of $F(y,\boldsymbol{x})$ w.r.t. y are *conjugate* one another (see [14] for details). The viewpoint of approximate algebra then leads us to a concept of approximate unconjugacy, as follows.

Definition 5 (approximate unconjugacy) *Let the Newton polynomial F_{New} be irreducible in $\mathbf{C}[y,\boldsymbol{x}]$, but be approximately factolizable at tolerance ε as $F_{\mathrm{New}} = G_1^{(0)}\cdots G_r^{(0)} + \Delta^{(0)}$, where $G_i^{(0)}$ and $G_j^{(0)}$ $(\forall i \neq j)$ have no approximate GCD at tolerance ε and $\|\Delta^{(0)}\|/\|F_{\mathrm{New}}\| \le \varepsilon$. Then, we say that roots of $F(y,\boldsymbol{x})$ which correspond to the roots of $G_i^{(0)}$ are approximately unconjugate at tolerance ε, to others which correspond to the roots of $G_j^{(0)}$ $(j \neq i)$.*

Example 8 Consider polynomial $F(y,u,v)$ in Example 7.
Discarding the term δv^3, we have $F(y,u,v) \simeq \tilde{F}(y,u,v)$, where

$$\tilde{F}(y,u,v) = y^3 + 2y^2\,(u^2+v^2) - yu^2 \ = \ y\,(y-\tilde{\phi}_2)\,(y-\tilde{\phi}_3).$$

Since $\tilde{F}_{\mathrm{New}} = y^3 - yu^2 = y\,(y-u)\,(y+u)$, the roots of $F(y,u,v)$ are approximately unconjugate each other at tolerance δ. In fact, the power-series roots of $\tilde{F}$ are

$$\begin{cases} \tilde{\phi}_1 = \ \ 0, \\ \tilde{\phi}_2 = \ \ u - (u^2+v^2) + (u^2+v^2)^2/2u + (u^2+v^2)^4/8u^3 + \cdots, \\ \tilde{\phi}_3 = -u - (u^2+v^2) - (u^2+v^2)^2/2u - (u^2+v^2)^4/8u^3 + \cdots, \end{cases}$$

and we see that these approximate roots are not algebraic. ♢

Remark If we set w_δ as $w_\delta > 0$ in Example 7, we obtain mutually unconjugate roots in which the term δv^3 is treated as a perturbation. The resulting power-series roots are much simpler than those in Example 7. Thus, there is a freedom in setting the weights in our method. ♢

Although multivariate power-series roots computed by the extended Hensel construction are pretty complicated, we think that they play important roles in analyzing algebraic varieties of positive dimensions.

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