PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 141, Number 7, July 2013, Pages 2485–2499
S 0002-9939(2013)11682-5
Article electronically published on March 12, 2013

HOMOGENEOUS EINSTEIN METRICS ON G_2/T

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(Communicated by Lei Ni)

Abstract. We construct the Einstein equation for an invariant Riemannian metric on the exceptional full flag manifold $M = G_2/T$. By computing a Gröbner basis for a system of polynomials on six variables we prove that this manifold admits exactly two non-Kähler invariant Einstein metrics. Thus G_2/T turns out to be the first known example of an exceptional full flag manifold which admits a non-Kähler and not normal homogeneous Einstein metric.

Introduction

A Riemannian manifold (M, g) is called Einstein if the metric g has constant Ricci curvature, that is, $\mathrm{Ric}_g = \lambda g$ for some $\lambda \in \mathbb{R}$, where Ric_g is the Ricci tensor corresponding to g. The question whether M carries an Einstein metric, and if so, how many, is a fundamental one in Riemannian geometry. A number of interesting results in geometry have been motivated and inspired by this hard problem. The Einstein equation is a nonlinear second order PDE, and a good understanding of its solutions in the general case seems far from being attained. It becomes more manageable in the homogeneous setting. Most known examples of compact simply connected Einstein manifolds are homogeneous. In the homogeneous case the Einstein equation reduces to a system of algebraic equations for which we are looking for positive solutions. For some cases such solutions have been obtained explicitly. We refer to [14] and the references therein for more details on compact homogeneous Einstein manifolds. The low-dimensional cases were also examined in [5].

Let K be a compact, connected and semisimple Lie group. A full flag manifold is a compact homogeneous space of the form K/T, where T is a maximal torus in K. It is known that K/T admits a unique (up to isometry) K-invariant Kähler-Einstein metric (cf. [13]).

Non-Kähler homogeneous Einstein metrics on full flag manifolds corresponding to classical Lie groups have been studied by several authors (cf. [2], [16], [10]). Although various existence results of homogeneous Einstein metrics on these spaces have been obtained, the classification of such metrics is a demanding task which remains widely open. In the present paper we study the classification problem of homogeneous Einstein metrics on the full flag manifold G_2/T. The isotropy representation of this space decomposes into six inequivalent irreducible submodules.

Received by the editors October 14, 2011.

2010 *Mathematics Subject Classification.* Primary 53C25; Secondary 53C30.

Key words and phrases. Homogeneous Einstein metric, full flag manifold, exceptional Lie group G_2.

The third author was supported by Grant-in-Aid for Scientific Research (C) 21540080.

There are three (nonisomorphic) flag manifolds corresponding to the exceptional Lie group G_2, since there are exactly three different ways to paint black the simple roots in the Dynkin diagram of $\mathfrak{g}_2$, as shown in Figure 1 (for the classification of generalized flag manifolds in terms of painted Dynkin diagrams, we refer to [1] or [6]).

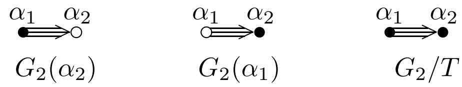

FIGURE 1. The painted Dynkin diagrams corresponding to G_2

If we paint black one simple root in the Dynkin diagram of $\mathfrak{g}_2$, then we obtain a flag manifold of the form $G_2/U(2)$ with two or three isotropy summands, depending on the height of this simple root. Recall that for $\mathfrak{g}_2$ we can choose a set of simple roots by $\Pi_M = \{\alpha_1, \alpha_2\}$ with $(\alpha_1, \alpha_1) = 3(\alpha_2, \alpha_2)$, so the highest root has the form $\widetilde{\alpha} = 2\alpha_1 + 3\alpha_2$ (see Section 5). Thus, the flag manifold $G_2(\alpha_2)$ in Figure 1 has two isotropy summands, and $U(2)$ is represented by the short root of $\mathfrak{g}_2$. For this space, all G_2-invariant Einstein metrics have been obtained explicitly in [3]. The second flag manifold $G_2(\alpha_1)$ in Figure 1 has three isotropy summands, and the isotropy group $U(2)$ is represented by the long root of $\mathfrak{g}_2$. For this space, the G_2-invariant Einstein metrics were studied in [12], [2].

The full flag manifold G_2/T, where $T = U(1) \times U(1)$ is a maximal torus in G_2, is obtained by painting black both simple roots in the Dynkin diagram of $\mathfrak{g}_2$. According to [19] a full flag manifold K/T is a normal homogeneous Einstein manifold if and only if all roots of K have the same length, and in this case the normal metric of K/T is never Kähler. Therefore, if K is an exceptional Lie group, then K/T is a normal homogeneous Einstein manifold if and only if $K \in \{E_6, E_7, E_8\}$, so G_2/T is not normal. Our main result is the following:

Theorem A. *The full flag manifold G_2/T admits exactly three G_2-invariant Einstein metrics* (*up to isometry*). *There is a unique Kähler-Einstein metric given* (*up to a scalar*) *by $g = (3, 1, 4, 5, 6, 9)$, and the other two are non-Kähler. The approximate values of these invariant metrics are given in Theorem* 4.1.

As a consequence of Theorem A, G_2/T is the first known example of an exceptional full flag manifold which admits a non-Kähler and not normal homogeneous Einstein metric. Also, the present work on G_2/T is the first attempt towards the classification of homogeneous Einstein metrics on generalized flag manifolds with six isotropy summands.

The paper is organised as follows: In Section 2 we recall the Lie-theoretic description of a full flag manifold K/T of a compact and connected semisimple Lie group K, and we study its isotropy representation. Next, following [16], we describe the structure constants of K/T relative to the associated isotropy decomposition, and we give the expression of the Ricci tensor of a K-invariant metric on K/T. In Section 3 we consider the exceptional full flag manifold G_2/T and we give its Lie-theoretic description. Then we construct the Einstein equation for a G_2-invariant Riemannian metric. In the last section, we give the corresponding polynomial system, and by computing a Gröbner basis for this system, we prove Theorem A and obtain the full classification of homogeneous Einstein metrics on G_2/T.

2. Full flag manifolds

Let K/T be a full flag manifold where T is a maximal torus of a compact semisimple Lie group K. We will give a characterization of K/T in terms of root system theory, and we will describe some topics of the associated Kähler geometry. Then, we study the isotropy representation of K/T and we give the expression of the Ricci tensor for a K-invariant metric on K/T.

2.1. **Lie-theoretic description of K/T.** Assume that $\dim_{\mathbb{R}} T = \operatorname{rank} G = \ell$. We denote by $\mathfrak{k}$, $\mathfrak{t}$ the Lie algebras of K and T respectively, and by $\mathfrak{k}_{\mathbb{C}} = \mathfrak{k} \oplus i\mathfrak{k}$, $\mathfrak{t}_{\mathbb{C}} = \mathfrak{t} \oplus i\mathfrak{t}$, the corresponding complexifications. Let $\mathfrak{t}^*$ and $\mathfrak{t}^*_{\mathbb{C}}$ be the dual spaces of $\mathfrak{t}$ and $\mathfrak{t}_{\mathbb{C}}$, respectively. The subalgebra $\mathfrak{t}_{\mathbb{C}}$ is a Cartan subalgebra of the complex semisimple Lie algebra $\mathfrak{k}_{\mathbb{C}}$, and thus we obtain the root space decomposition $\mathfrak{k}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \oplus \sum_{\alpha \in R} \mathfrak{k}^{\alpha}_{\mathbb{C}}$, where R is the root system of $\mathfrak{k}_{\mathbb{C}}$ relative to $\mathfrak{t}_{\mathbb{C}}$ and $\mathfrak{k}^{\alpha}_{\mathbb{C}}$ are the corresponding root spaces. Recall that by $\mathbb{C}$-linearity, a root $\alpha \in R$ is completely determined by its restriction to either $\mathfrak{t}$ or $i\mathfrak{t}$. Since the Killing form B of $\mathfrak{k}_{\mathbb{C}}$ is nondegenerate, for any $\lambda \in \mathfrak{t}^*_{\mathbb{C}}$ we define $H_\lambda \in i\mathfrak{t}$ by the equation $B(H_\lambda, H) = \lambda(H)$ for all $H \in \mathfrak{t}_{\mathbb{C}}$. Let $i\mathfrak{t}^*$ denote the real linear subspace of $\mathfrak{t}^*_{\mathbb{C}}$ which consists of all $\lambda \in \mathfrak{t}^*_{\mathbb{C}}$ such that the restriction $\lambda|_{\mathfrak{t}}$ has values in $i\mathbb{R}$. Note that the restriction map $\lambda \mapsto \lambda|_{i\mathfrak{t}}$ defines an isomorphism from $i\mathfrak{t}^*$ onto the real linear dual space $i\mathfrak{t}^*$, which allows us to identify these spaces. Then, it is well known that R spans $i\mathfrak{t}^*$ and that R is a finite subset of $i\mathfrak{t}^* \backslash \{0\}$. Thus, if $\alpha \in R$, then $\alpha \in i\mathfrak{t}^*$.

Let $(\ , \ )$ be the bilinear form on $\mathfrak{t}^*_{\mathbb{C}}$ induced by the Killing form, that is, $(\lambda, \mu) = B(H_\lambda, H_\mu)$, for any $\lambda, \mu \in \mathfrak{t}^*_{\mathbb{C}}$. Then, since B is negative definite on $\mathfrak{t}$ and positive definite on $i\mathfrak{t}$, the restriction of $(\ , \ )$ on $i\mathfrak{t}^*$ is a positive definite inner product. The weight lattice of $\mathfrak{k}_{\mathbb{C}}$ with respect to $\mathfrak{t}_{\mathbb{C}}$ is given by $\Lambda = \{\lambda \in i\mathfrak{t}^* \mid \dfrac{2(\lambda, \alpha)}{(\alpha, \alpha)} \in \mathbb{Z} \ \text{ for all } \ \alpha \in R\}$. Let $\Pi = \{\alpha_1, \ldots, \alpha_\ell\}$ be a simple root system of R, and let R^+ be the set of all positive roots with respect to Π. Consider the fundamental weights corresponding to Π, that is, $\Lambda_1, \ldots, \Lambda_\ell \in \Lambda$ such that

$$\frac{2(\Lambda_i, \alpha_j)}{(\alpha_j, \alpha_j)} = \delta_{ij} \qquad (1 \leq i, j \leq \ell). \tag{2.1}$$

Then $\{\Lambda_1, \ldots, \Lambda_\ell\}$ forms a $\mathbb{Z}$-basis for the weight lattice Λ, and since $i\mathfrak{t} \cong i\mathfrak{t}^*$, it is $i\mathfrak{t} = \sum_{i=1}^{\ell} \mathbb{R}\Lambda_i$. In the weight lattice Λ there is a distinguished subset Λ^+ given by $\Lambda^+ = \{\lambda \in \Lambda \mid (\lambda, \alpha) > 0 \ \text{ for any } \ \alpha \in R^+\}$. One can see that $\Lambda^+ = \Lambda \cap C(\Pi)$, where $C(\Pi)$ is the fundamental Weyl chamber corresponding to Π, given by $C(\Pi) = \{\lambda \in i\mathfrak{t}^* \mid (\lambda, \alpha_i) > 0 \ \text{ for all } \ \alpha_i \in \Pi\}$. The elements of Λ^+ are usually called dominant weights relative to R^+, and any dominant weight can be expressed as a linear combination of the fundamental weights with nonnegative coefficients. For example, set $\delta = \frac{1}{2}\sum_{\alpha \in R^+} \alpha \in i\mathfrak{t}^*$. Then, $\delta = \sum_{i=1}^{\ell} \Lambda_i$ and thus $\delta \in \Lambda^+$ (cf. [8, p. 168]).

We now define the complex Lie subalgebras of $\mathfrak{k}_{\mathbb{C}}$ by $\mathfrak{n} = \sum_{\alpha \in R^+} \mathfrak{k}^{\alpha}_{\mathbb{C}}$ and $\mathfrak{b} = \mathfrak{t}_{\mathbb{C}} \oplus \mathfrak{n}$. One can easily show that $\mathfrak{n}$ is a nilpotent ideal of $\mathfrak{k}_{\mathbb{C}}$ and that $\mathfrak{b}$ is a maximal solvable Lie subalgebra of $\mathfrak{k}_{\mathbb{C}}$; i.e. it is a Borel subalgebra. Let $K_{\mathbb{C}}$ denote the complex simply connected semisimple Lie group whose Lie algebra is $\mathfrak{k}_{\mathbb{C}}$. Then, the connected subgroup $B \subset K_{\mathbb{C}}$ with Lie algebra $\mathfrak{b}$ is a Borel subgroup of $K_{\mathbb{C}}$ and $K_{\mathbb{C}}/B \cong K/T$ as C^∞-manifolds.

Since $K_{\mathbb{C}}$ is a complex Lie group and B a closed complex subgroup, the quotient $K_{\mathbb{C}}/B$ admits a K-invariant complex structure. Furthermore, the K-invariant complex structures on $K_{\mathbb{C}}/B = K/T$ are in 1-1 correspondence with different choices of positive roots for $\mathfrak{k}_{\mathbb{C}}$ (cf. [7]). Since the Weyl group $W(R)$ of the root system of $\mathfrak{k}_{\mathbb{C}}$ acts transitively on the sets of systems of positive roots, all these complex structures are equivalent. Moreover, the following holds:

Theorem 2.1 ([7], [17]). *There is a* 1-1 *correspondence between K-invariant Kähler metrics on $K_{\mathbb{C}}/B$ and dominant weights in Λ^+. In particular, the K-invariant Kähler metric on $K_{\mathbb{C}}/B$ corresponding to 2δ is a Kähler-Einstein metric.*

According to [7, p. 504] a full flag manifold admits a unique (up to equivalence) invariant complex structure, hence a unique (up to scale) Kähler–Einstein metric (cf. also [13]). This Kähler–Einstein metric will be computed in Section 2.

2.2. **The isotropy representation of K/T.** We will now examine the isotropy representation of a full flag manifold $K_{\mathbb{C}}/B = K/T$. Consider the reductive decomposition $\mathfrak{k} = \mathfrak{t} \oplus \mathfrak{m}$ of $\mathfrak{k}$ with respect to the negative of the Killing form $Q = -B(\ , \ )$, that is, $\mathfrak{m} = \mathfrak{t}^{\perp}$ and $\mathrm{Ad}(T)\mathfrak{m} \subset \mathfrak{m}$. As usual, we identify $\mathfrak{m} = T_o(K/T)$ (where $o = eT$ is the identity coset of K/T).

Choose a Weyl basis $\{H_{\alpha_1}, \ldots, H_{\alpha_\ell}\} \cup \{E_\alpha \in \mathfrak{k}_{\mathbb{C}}^{\alpha} \mid \alpha \in R\}$ with $B(E_\alpha, E_{-\alpha}) = -1$, $[E_\alpha, E_{-\alpha}] = -H_\alpha$. Recall for later use that the root vectors satisfy $[E_\alpha, E_\beta] = N_{\alpha,\beta} E_{\alpha+\beta}$ if $\alpha, \beta, \alpha+\beta \in R$ and $[E_\alpha, E_\beta] = 0$ otherwise. The numbers $N_{\alpha,\beta} \in \mathbb{R}$ satisfy $N_{\alpha,\beta} = -N_{\beta,\alpha}$, $N_{\alpha,\beta} = N_{-\alpha,-\beta} \in \mathbb{R}$ if $\alpha, \beta, \alpha+\beta \in R$, and $N_{\alpha,\beta} = 0$ if $\alpha, \beta \in R$, $\alpha + \beta \notin R$. They can also be chosen so that $N_{\alpha,-\beta} = N_{-\alpha,\beta}$. Then the real subalgebra $\mathfrak{k}$ is given by

$$\mathfrak{k} = \sum_{j=1}^{\ell} \mathbb{R} i H_{\alpha_j} \oplus \sum_{\alpha \in R^+} (\mathbb{R} A_\alpha + \mathbb{R} B_\alpha) = \mathfrak{t} \oplus \sum_{\alpha \in R^+} (\mathbb{R} A_\alpha + \mathbb{R} B_\alpha), \tag{2.2}$$

where $A_\alpha = E_\alpha + E_{-\alpha}$ and $B_\alpha = i(E_\alpha - E_{-\alpha})$ $(\alpha \in R^+)$.

Since $\mathfrak{t} = \mathrm{span}_{\mathbb{R}}\{iH_{\alpha_j} \mid 1 \le j \le \ell\}$, the reductive decomposition $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{m}$ implies that

$$\mathfrak{m} = T_o(K/T) = \sum_{\alpha \in R^+} (\mathbb{R} A_\alpha + \mathbb{R} B_\alpha). \tag{2.3}$$

Set $\mathfrak{m}_\alpha = \mathbb{R} A_\alpha + \mathbb{R} B_\alpha$ for any $\alpha \in R^+$. The linear space $\mathfrak{m}_\alpha$ is an irreducible $\mathrm{Ad}(T)$-module which does not depend on the choice of an ordering in R. Furthermore, since the roots of $\mathfrak{k}_{\mathbb{C}}$ with respect to $\mathfrak{t}_{\mathbb{C}}$ are distinct and the root spaces are one-dimensional, it is obvious that $\mathfrak{m}_\alpha \ncong \mathfrak{m}_\beta$ as $\mathrm{Ad}(T)$-representations, for any two roots $\alpha, \beta \in R^+$. Thus, by using (2.3) we obtain the following:

Proposition 2.2. *Let $M = K/T$ be a full flag manifold of a compact simple Lie group K. Then the isotropy representation of M decomposes into a direct sum of* 2*-dimensional pairwise inequivalent irreducible T-submodules $\mathfrak{m}_\alpha$ as follows:*

$$\mathfrak{m} = \sum_{\alpha \in R^+} \mathfrak{m}_\alpha. \tag{2.4}$$

The number of these submodules is equal to the cardinality $|R^+|$.

2.3. **The Ricci tensor for a K-invariant metric on K/T.** Since K/T is a reductive homogeneous space, there is a natural 1-1 correspondence between K-invariant symmetric covariant 2-tensors on K/T and $\mathrm{Ad}(T)$-invariant symmetric bilinear forms on $\mathfrak{m}$. For example, in this correspondence a K-invariant Riemannian metric g on K/T corresponds to an $\mathrm{Ad}(T)$-invariant inner product $\langle\ ,\ \rangle$ on $\mathfrak{m}$. In particular, since $\mathfrak{m}$ admits the decomposition (2.4) and the $\mathrm{Ad}(T)$-submodules are mutually inequivalent, the space of K-invariant Riemannian metrics on K/T is given by

$$\Big\{g = \langle\ ,\ \rangle = \sum_{\alpha\in R^+} x_\alpha \cdot Q|_{\mathfrak{m}_\alpha} \mid x_\alpha \in \mathbb{R}^+\Big\}. \tag{2.5}$$

The K-invariant Kähler-Einstein metric on $K_{\mathbb{C}}/B = K/T$ corresponding to $2\delta = 2\sum_{i=1}^{\ell}\Lambda_i$ is given by

$$g_{2\delta} = \sum_{\alpha\in R^+} 2(\Lambda_1 + \cdots + \Lambda_\ell, \alpha)\cdot Q|_{\mathfrak{m}_\alpha}. \tag{2.6}$$

Similarly, the Ricci tensor Ric_g of a K-invariant metric g on K/T, as a K-invariant covariant 2-tensor, will be described by an $\mathrm{Ad}(T)$-invariant symmetric bilinear form on $\mathfrak{m}$ given by

$$\mathrm{Ric}_g = \sum_{\alpha\in R^+} r_\alpha x_\alpha \cdot Q|_{\mathfrak{m}_\alpha},$$

where r_α ($\alpha \in R^+$) are the components of the Ricci tensor on each submodule $\mathfrak{m}_\alpha$. Since $\mathfrak{m}_\alpha \not\cong \mathfrak{m}_\beta$ for any $\alpha, \beta, \in R^+$, it is $\mathrm{Ric}_g(\mathfrak{m}_\alpha, \mathfrak{m}_\beta) = 0$ (cf. [20]).

There is a useful description of the components r_α associated to the isotropy decomposition (2.4). Let K/L be a compact homogeneous space of a compact semisimple Lie group K whose isotropy representation $\mathfrak{m}$ decomposes into s pairwise inequivalent irreducible $\mathrm{Ad}(L)$-submodules $\mathfrak{m}_i$ as $\mathfrak{m} = \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_s$. Following [20] and [15] we choose a Q-orthonormal basis $\{e_p\}$ adapted to $\mathfrak{m} = \bigoplus_{i=1}^s \mathfrak{m}_i$. Let $A^r_{pq} = Q([e_p, e_q], e_r)$ so that $[e_p, e_q]_{\mathfrak{m}} = \sum_\gamma A^r_{pq} e_r$, and set

$$\begin{bmatrix} k \\ ij \end{bmatrix} = \sum (A^r_{pq})^2 = \sum \big(Q([e_p, e_q], e_r)\big)^2, \tag{2.7}$$

where the sum is taken over all indices p, q, r with $e_p \in \mathfrak{m}_i, e_q \in \mathfrak{m}_j$, and $e_r \in \mathfrak{m}_k$. The triples $\begin{bmatrix} k \\ ij \end{bmatrix}$ are called the *structure constants* of K/L with respect to the decomposition $\mathfrak{m} = \bigoplus_{i=1}^s \mathfrak{m}_i$ and are symmetric to all three indices.

For the case of a full flag manifold K/T we study its structure constants with respect to the Q-orthogonal decomposition $\mathfrak{m} = \sum_{\alpha\in R^+}\mathfrak{m}_\alpha$, where $Q = -B(\ ,\ )$ and $\mathfrak{m}_\alpha = \mathbb{R}A_\alpha + \mathbb{R}B_\alpha$. Note that we can rewrite the previous splitting as $\mathfrak{m} = \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_s$, where $s = |R^+|$. Since $B(E_\alpha, E_{-\beta}) = -\delta_{\alpha,\beta}$ one can verify that the vectors A_α and B_α are such that $B(A_\alpha, A_\beta) = B(B_\alpha, B_\beta) = -2\delta_{\alpha,\beta}$ and $B(A_\alpha, B_\beta) = 0$. Therefore, the set $\{X_\alpha = A_\alpha/\sqrt{2},\ Y_\alpha = B_\alpha/\sqrt{2} \mid \alpha \in R^+\}$ is a Q-orthonormal basis of each $\mathfrak{m}_\alpha$.

We use the notation $\left[{\gamma \atop \alpha\beta}\right]$ for $\alpha, \beta, \gamma \in R$ instead of $\left[{k \atop ij}\right]$ for submodules $\mathfrak{m}_\alpha = \mathfrak{m}_{-\alpha}, \mathfrak{m}_\beta = \mathfrak{m}_{-\beta}$, and $\mathfrak{m}_\gamma = \mathfrak{m}_{-\gamma}$. Recall that if $\alpha, \beta, \alpha+\beta \in R$, then $[\mathfrak{k}^\alpha_{\mathbb{C}}, \mathfrak{k}^\beta_{\mathbb{C}}] = \mathfrak{k}^{\alpha+\beta}_{\mathbb{C}}$ and $B(\mathfrak{k}^\alpha_{\mathbb{C}}, \mathfrak{k}^\beta_{\mathbb{C}}) = 0$ (cf. [11, p. 168]). Since $\left[{\gamma \atop \alpha\beta}\right] \neq 0$ if and only if $Q([\mathfrak{m}_\alpha, \mathfrak{m}_\beta], \mathfrak{m}_\gamma) \neq 0$, we can easily conclude that $\left[{\gamma \atop \alpha\beta}\right] \neq 0$ if and only if the roots α, β, γ satisfy one of the relations $\alpha+\beta-\gamma=0$, $\alpha-\beta+\gamma=0$, $-\alpha+\beta+\gamma=0$. Note that if $\left[{\gamma \atop \alpha\beta}\right]\neq 0$ then we can rewrite $\left[{\gamma \atop \alpha\beta}\right]$ as $\left[{\gamma' \atop \alpha'\beta'}\right]$ with $\alpha',\beta',\gamma' \in R^+$ by rearranging roots and changing the sign of roots.

By using the above notation, it can be shown ([16, p. 75]) that for each $\alpha\in R^+$, the Ricci component r_α corresponding to the isotropy summand $\mathfrak{m}_\alpha$ is given by

$$r_\alpha = \frac{1}{2x_\alpha} + \frac{1}{8}\sum_{\beta,\gamma\in R^+}\frac{x_\alpha}{x_\beta x_\gamma}\begin{bmatrix}\alpha\\ \beta\gamma\end{bmatrix} - \frac{1}{4}\sum_{\beta,\gamma\in R^+}\frac{x_\gamma}{x_\alpha x_\beta}\begin{bmatrix}\gamma\\ \alpha\beta\end{bmatrix}. \tag{2.8}$$

Hence, a K-invariant metric (2.5) on K/T is an Einstein metric with Einstein constant k if and only if it is a positive real solution of the system $\{r_\alpha = k \mid \alpha\in R^+\}$.

Proposition 2.3. *For a full flag manifold K/T the triples $\left[{\alpha+\beta \atop \alpha\ \beta}\right]$ are given by*

$$\begin{bmatrix}\alpha+\beta\\ \alpha\ \beta\end{bmatrix} = 2N^2_{\alpha,\beta}. \tag{2.9}$$

Proof. By definition (2.7) we see that

$$\begin{aligned}\begin{bmatrix}\alpha+\beta\\ \alpha\ \beta\end{bmatrix} &= (B([X_\alpha,X_\beta],X_{\alpha+\beta}))^2 + (B([X_\alpha,X_\beta],Y_{\alpha+\beta}))^2\\ &\quad + (B([Y_\alpha,X_\beta],X_{\alpha+\beta}))^2 + (B([Y_\alpha,X_\beta],Y_{\alpha+\beta}))^2 + (B([X_\alpha,Y_\beta],X_{\alpha+\beta}))^2\\ &\quad + (B([X_\alpha,Y_\beta],Y_{\alpha+\beta}))^2 + (B([Y_\alpha,Y_\beta],X_{\alpha+\beta}))^2 + (B([Y_\alpha,Y_\beta],Y_{\alpha+\beta}))^2.\end{aligned}$$

Since $B(B_{\alpha+\beta},A_{\alpha+\beta})=0$ and $B(A_{\alpha+\beta},A_{\alpha+\beta}) = B(B_{\alpha+\beta},B_{\alpha+\beta}) = -2$, a straightforward computation using the properties of the root vectors and the numbers $N_{\alpha,\beta}$ gives that

$$\begin{aligned}B([X_\alpha,X_\beta],X_{\alpha+\beta}) &= 1/(2\sqrt{2})B(N_{\alpha,\beta}A_{\alpha+\beta}+N_{\alpha,-\beta}A_{\alpha-\beta},A_{\alpha+\beta}) = -N_{\alpha,\beta}/\sqrt{2},\\ B([Y_\alpha,X_\beta],Y_{\alpha+\beta}) &= B([X_\alpha,Y_\beta],Y_{\alpha+\beta}) = -B([Y_\alpha,Y_\beta],X_{\alpha+\beta}) = -N_{\alpha,\beta}/\sqrt{2},\\ B([X_\alpha,X_\beta],Y_{\alpha+\beta}) &= B([Y_\alpha,X_\beta],X_{\alpha+\beta})\\ &= B([X_\alpha,Y_\beta],X_{\alpha+\beta}) = B([Y_\alpha,Y_\beta],Y_{\alpha+\beta}) = 0,\end{aligned}$$

and the result follows. ☐

Remark 2.4. Two roots $\alpha,\beta\in R$ have the same length with respect to the Killing form B if and only if there is an element w of the Weyl group $W(R)$ of the root system R such that $\beta = w(\alpha)$ (see for example [18, p. 242]). Thus, because of the invariance of the Killing form under $W(R)$, it is obvious that for any element $w\in W(R)$ we have that $\left[{w(\gamma) \atop w(\alpha)\ w(\beta)}\right] = \left[{\gamma \atop \alpha\ \beta}\right]$.

3. The full flag manifold G_2/T

We now study the geometry of the full flag manifold G_2/T, where T is a maximal torus of G_2. We start by describing its isotropy representation.

3.1. The decomposition of the isotropy representation of G_2/T.

The root system of the exceptional complex simple Lie algebra $\mathfrak{g}_2$ can be chosen to be $R = \{\pm\alpha_1, \pm\alpha_2, \pm(\alpha_1+\alpha_2), \pm(\alpha_1+2\alpha_2), \pm(\alpha_1+3\alpha_2), \pm(2\alpha_1+3\alpha_2)\}$. We fix a system of simple roots to be $\Pi=\{\alpha_1,\alpha_2\}$. With respect to Π the positive roots are given by

$$R^+ = \{\alpha_1,\alpha_2,\alpha_1+\alpha_2,\alpha_1+2\alpha_2,\alpha_1+3\alpha_2,2\alpha_1+3\alpha_2\}. \tag{3.1}$$

The highest root is $\widetilde{\alpha} = 2\alpha_1 + 3\alpha_2$ (see Figure 2). Also, it is $\|\alpha_1\| = \sqrt{3}\,\|\alpha_2\|$, and the roots of $\mathfrak{g}_2$ make succesive angles of $\pi/6$. The Weyl group is generated by rotations of $\mathbb{R}^2$ about the origin through an angle $\pi/6$ and reflections about the vertical axis.

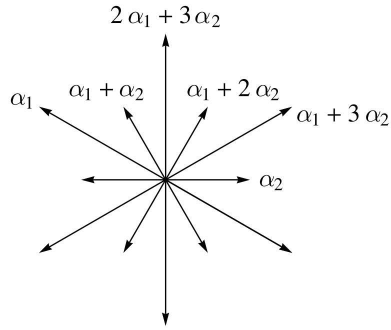

FIGURE 2. The root system of $\mathfrak{g}_2$

The full flag manifold G_2/T is obtained by painting black both simple roots in the Dynkin diagram of $\mathfrak{g}_2$. Proposition 2.2 implies that the isotropy representation $\mathfrak{m}$ of G_2/T decomposes into six inequivalent irreducible ad($\mathfrak{k}$)-submodules, i.e. $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3 \oplus \mathfrak{m}_4 \oplus \mathfrak{m}_5 \oplus \mathfrak{m}_6$, where the submodules $\mathfrak{m}_i$ $(1 \leq i \leq 6)$ are given by

$$\left.\begin{array}{lll} \mathfrak{m}_1 = \mathfrak{m}_{\alpha_1}, & \mathfrak{m}_2 = \mathfrak{m}_{\alpha_2}, & \mathfrak{m}_3 = \mathfrak{m}_{\alpha_1+\alpha_2}, \\ \mathfrak{m}_4 = \mathfrak{m}_{\alpha_1+2\alpha_2}, & \mathfrak{m}_5 = \mathfrak{m}_{\alpha_1+3\alpha_2}, & \mathfrak{m}_6 = \mathfrak{m}_{2\alpha_1+3\alpha_2}. \end{array}\right\} \tag{3.2}$$

3.2. **Kähler-Einstein metrics.** It is well known ([7, p. 504]) that a full flag manifold K/T admits $|W(K)|/2$ invariant complex structures (here $W(K)$ is the Weyl group of K), which are all equivalent under an automorphism of K. We now compute the unique Kähler–Einstein metric which is compatible with the natural complex structure J_{nat}, that is, the complex structure corresponding to the natural invariant ordering R^+ given by (3.1). From (2.5) a G_2-invariant Riemannian metric on G_2/T is given by

$$g = x_1 \cdot Q|_{\mathfrak{m}_1} + \cdots + x_6 \cdot Q|_{\mathfrak{m}_6}, \tag{3.3}$$

where we have set $x_1 = x_{\alpha_1}, x_2 = x_{\alpha_2}, x_3 = x_{\alpha_1+\alpha_2}$, $x_4 = x_{\alpha_1+2\alpha_2}$, $x_5 = x_{\alpha_1+3\alpha_2}$, $x_6 = x_{2\alpha_1+3\alpha_2}$, and the $\mathfrak{m}_k$ $(k = 1, \ldots, 6)$ are given by (3.2). We will denote such metrics by $g = (x_1, x_2, x_3, x_4, x_5, x_6) \in \mathbb{R}^6_+$.

Theorem 3.1. *The full flag manifold G_2/T admits six invariant Kähler–Einstein metrics which are isometric to each other. The Kähler-Einstein metric $g_{2\delta}$ which is compatible with the natural invariant ordering J_{nat} is given (up to a scale) by $g_{2\delta} = (3, 1, 4, 5, 6, 9)$.*

Proof. According to the notation of Section 2.1, the weight δ for G_2/T is given by $\delta = \frac{1}{2}\sum_{\alpha\in R^+}\alpha = \sum_{i=1}^{2}\Lambda_i$, where Λ_1 and Λ_2 are the fundamental weights corresponding to the simple roots α_1 and α_2, respectively. In Figure 2 one can easily distinguish the long roots

$$\mathcal{L}_1 = \alpha_1,\ \mathcal{L}_2 = \alpha_1 + 3\alpha_2,\ \mathcal{L}_3 = 2\alpha_1 + 3\alpha_2$$

from the short roots

$$\mathcal{S}_1 = \alpha_2, \ \mathcal{S}_2 = \alpha_1 + \alpha_2, \ \mathcal{S}_3 = \alpha_1 + 2\alpha_2.$$

Then $\|\mathcal{L}_i\| = \sqrt{3}\,\|\mathcal{S}_j\|$, where $1 \leq i, j \leq 3$ and i, j are independent. We set $(\mathcal{L}_i, \mathcal{L}_i) = 3$ and $(\mathcal{S}_i, \mathcal{S}_i) = 1$, for any $1 \leq i \leq 3$. We denote the Kähler-Einstein metric $g_{2\delta}$ by

$$g_{\alpha_1} \cdot Q|_{\mathfrak{m}_1} + g_{\alpha_2} \cdot Q|_{\mathfrak{m}_2} + g_{\alpha_1+\alpha_2} \cdot Q|_{\mathfrak{m}_3} + g_{\alpha_1+2\alpha_2} \cdot Q|_{\mathfrak{m}_4} + g_{\alpha_1+3\alpha_2} \cdot Q|_{\mathfrak{m}_5} + g_{2\alpha_1+3\alpha_2} \cdot Q|_{\mathfrak{m}_6},$$

which is compatible with the natural invariant complex structure J_{nat} defined by the ordering R^+. By using (2.1) and applying relation (2.6) we obtain the following values for the components $g_\alpha = (2\delta, \alpha)$, $\alpha \in R^+$ of this metric:

$$\begin{aligned} g_{\alpha_1} &= 2(\Lambda_1, \alpha_1) = (\alpha_1, \alpha_1) = 3, \quad & g_{\alpha_2} &= 2(\Lambda_2, \alpha_2) = (\alpha_2, \alpha_2) = 1, \\ g_{\alpha_1+\alpha_2} &= 2(\Lambda_1, \alpha_1) + 2(\Lambda_2, \alpha_2) = 4, \quad & g_{\alpha_1+2\alpha_2} &= 2(\Lambda_1, \alpha_1) + 4(\Lambda_2, \alpha_2) = 5, \\ g_{\alpha_1+3\alpha_2} &= 2(\Lambda_1, \alpha_1) + 6(\Lambda_2, \alpha_2) = 6, \quad & g_{2\alpha_1+3\alpha_2} &= 4(\Lambda_1, \alpha_1) + 6(\Lambda_2, \alpha_2) = 9. \end{aligned}$$

□

3.3. **Homogeneous Einstein metrics.** We now proceed to the calculation of the Ricci tensor Ric_g corresponding to a G_2-invariant metric (3.3) on G_2/T. Following the notation of Section 2.3, the tensor Ric_g as a G_2-invariant symmetric covariant 2-tensor on G_2/T is given by $\mathrm{Ric}_g = r_1 x_1 \cdot Q|_{\mathfrak{m}_1} + \cdots + r_6 x_6 \cdot Q|_{\mathfrak{m}_6}$, where for simplicity we have set $r_1 = r_{\alpha_1}, r_2 = r_{\alpha_2}, r_3 = r_{\alpha_1+\alpha_2}$, $r_4 = r_{\alpha_1+2\alpha_2}$, $r_5 = r_{\alpha_1+3\alpha_2}$ and $r_6 = r_{2\alpha_1+3\alpha_2}$. In order to apply (2.8) we first need to find the nonzero structure constants $\left[{k \atop ij}\right]$ of G_2/T. By using (3.1) and (3.2) it follows that these are

$$c_{12}^3 = \begin{bmatrix} \alpha_1 + \alpha_2 \\ \alpha_1 \ \alpha_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 12 \end{bmatrix}, \ c_{23}^4 = \begin{bmatrix} \alpha_1 + 2\alpha_2 \\ \alpha_2 \ \alpha_1 + \alpha_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 23 \end{bmatrix}, \ c_{24}^5 = \begin{bmatrix} \alpha_1 + 3\alpha_2 \\ \alpha_2 \ \alpha_1 + 2\alpha_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 24 \end{bmatrix},$$

$$c_{15}^6 = \begin{bmatrix} 2\alpha_1 + 3\alpha_2 \\ \alpha_1 \ \alpha_1 + 3\alpha_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 15 \end{bmatrix}, \ c_{34}^6 = \begin{bmatrix} 2\alpha_1 + 3\alpha_2 \\ \alpha_1 + \alpha_2 \ \alpha_1 + 2\alpha_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 34 \end{bmatrix}.$$

By Remark 2.4 and the remarks on the notation for $\left[{\gamma \atop \alpha\beta}\right]$, we obtain the following:

Lemma 3.2. *The triples c_{12}^3, c_{24}^5 and c_{34}^6 are equal.*

Proof. The Weyl group $W(R)$ is generated by the reflections $\{s_{\alpha_1}, s_{\alpha_2}\}$, and we have that $s_{\alpha_1}(\alpha_1) = -\alpha_1, s_{\alpha_1}(\alpha_2) = \alpha_1 + \alpha_2, s_{\alpha_2}(\alpha_2) = -\alpha_2, s_{\alpha_2}(\alpha_1) = \alpha_1 + 3\alpha_2$. Now we see that $s_{\alpha_2}(\alpha_1 + \alpha_2) = \alpha_1 + 2\alpha_2$ and hence we have that

$$\begin{aligned} c_{24}^5 = \begin{bmatrix} \alpha_1 + 3\alpha_2 \\ \alpha_2 \ \alpha_1 + 2\alpha_2 \end{bmatrix} = \begin{bmatrix} s_{\alpha_2}(\alpha_1) \\ -s_{\alpha_2}(\alpha_2) \ s_{\alpha_2}(\alpha_1 + \alpha_2) \end{bmatrix} &= \begin{bmatrix} \alpha_1 \\ -\alpha_2 \ \alpha_1 + \alpha_2 \end{bmatrix} \\ &= \begin{bmatrix} \alpha_1 + \alpha_2 \\ \alpha_1 \ \alpha_2 \end{bmatrix} = c_{12}^3. \end{aligned}$$

We also see that $s_{\alpha_1}(\alpha_1 + 3\alpha_2) = 2\alpha_1 + 3\alpha_2$ and $s_{\alpha_1}(\alpha_1 + 2\alpha_2) = \alpha_1 + 2\alpha_2$, and hence we have that

$$c_{34}^6 = \begin{bmatrix} 2\alpha_1 + 3\alpha_2 \\ \alpha_1 + \alpha_2 \ \alpha_1 + 2\alpha_2 \end{bmatrix} = \begin{bmatrix} s_{\alpha_1}(\alpha_1 + 3\alpha_2) \\ s_{\alpha_1}(\alpha_2) \ s_{\alpha_1}(\alpha_1 + 2\alpha_2) \end{bmatrix} = \begin{bmatrix} \alpha_1 + 3\alpha_2 \\ \alpha_2 \ \alpha_1 + 2\alpha_2 \end{bmatrix} = c_{24}^5. \ \square$$

For the calculation of the above triples we use Proposition 2.3 and the fact that

$$N^2_{\alpha,\beta} = N_{\alpha,\beta}N_{-\alpha,-\beta} = \frac{q(p+1)}{2}Q(\alpha,\alpha), \tag{3.4}$$

where p, q are the largest nonnegative integers such that $\beta + k\alpha \in R$, with $-p \leq k \leq q$.

We first proceed to the calculation of $c^3_{12} = \left[\begin{smallmatrix}\alpha_1+\alpha_2\\ \alpha_1\ \alpha_2\end{smallmatrix}\right]$. By using the relation $(\alpha_1,\alpha_1) = 3(\alpha_2,\alpha_2)$ and equation (3.4) we obtain that $N^2_{\alpha_1,\alpha_2} = 3Q(\alpha_2,\alpha_2)/2$, so Proposition 2.3 implies that $c^3_{12} = 3Q(\alpha_2,\alpha_2)$. The normalizing value $Q(\alpha_2,\alpha_2)$ is given by $Q(\alpha_2,\alpha_2) = 1/12$ (cf. [8]); thus $c^3_{12} = 1/4$. Similarly, we obtain that

$$c^4_{23} = \begin{bmatrix}\alpha_1+2\alpha_2\\ \alpha_2\ \alpha_1+\alpha_2\end{bmatrix} = 2N^2_{\alpha_2,\alpha_1+\alpha_2} = \frac{1}{3}, \quad c^6_{15} = \begin{bmatrix}2\alpha_1+3\alpha_2\\ \alpha_1\ \alpha_1+3\alpha_2\end{bmatrix} = 2N^2_{\alpha_1,\alpha_1+3\alpha_2} = \frac{1}{4}.$$

The above computations combined with Lemma 3.2 give the following:

Proposition 3.3. *The nonzero triples $\left[\begin{smallmatrix}k\\ ij\end{smallmatrix}\right]$ of the full flag manifold G_2/T are given by*

$$\begin{bmatrix}3\\12\end{bmatrix} = \begin{bmatrix}5\\24\end{bmatrix} = \begin{bmatrix}6\\34\end{bmatrix} = \begin{bmatrix}6\\15\end{bmatrix} = \frac{1}{4} \quad \textit{and} \quad \begin{bmatrix}4\\23\end{bmatrix} = \frac{1}{3}.$$

Therefore, we obtain the following proposition for the Ricci tensor from (2.8):

Proposition 3.4. *The components r_i $(i = 1,\ldots,6)$ of the Ricci tensor associated to the G-invariant Riemannian metric g given by* (3.3) *are the following:*

$$\begin{aligned}
r_1 &= \frac{1}{2x_1} + \frac{1}{16}\Big(\frac{x_1}{x_2x_3} - \frac{x_2}{x_1x_3} - \frac{x_3}{x_1x_2}\Big) + \frac{1}{16}\Big(\frac{x_1}{x_5x_6} - \frac{x_5}{x_1x_6} - \frac{x_6}{x_1x_5}\Big),\\
r_2 &= \frac{1}{2x_2} + \frac{1}{16}\Big(\frac{x_2}{x_1x_3} - \frac{x_1}{x_2x_3} - \frac{x_3}{x_1x_2}\Big) + \frac{1}{12}\Big(\frac{x_2}{x_3x_4} - \frac{x_3}{x_2x_4} - \frac{x_4}{x_2x_3}\Big)\\
&\quad + \frac{1}{16}\Big(\frac{x_2}{x_4x_5} - \frac{x_4}{x_2x_5} - \frac{x_5}{x_2x_4}\Big),\\
r_3 &= \frac{1}{2x_3} + \frac{1}{16}\Big(\frac{x_3}{x_1x_2} - \frac{x_2}{x_1x_3} - \frac{x_1}{x_2x_3}\Big) + \frac{1}{12}\Big(\frac{x_3}{x_2x_4} - \frac{x_2}{x_3x_4} - \frac{x_4}{x_2x_3}\Big)\\
&\quad + + \frac{1}{16}\Big(\frac{x_3}{x_4x_6} - \frac{x_4}{x_3x_6} - \frac{x_6}{x_3x_4}\Big),\\
r_4 &= \frac{1}{2x_4} + \frac{1}{12}\Big(\frac{x_4}{x_2x_3} - \frac{x_2}{x_3x_4} - \frac{x_3}{x_2x_4}\Big) + \frac{1}{16}\Big(\frac{x_4}{x_2x_5} - \frac{x_2}{x_4x_5} - \frac{x_5}{x_2x_4}\Big)\\
&\quad + \frac{1}{16}\Big(\frac{x_4}{x_3x_6} - \frac{x_3}{x_4x_6} - \frac{x_6}{x_3x_4}\Big),\\
r_5 &= \frac{1}{2x_5} + \frac{1}{16}\Big(\frac{x_5}{x_1x_6} - \frac{x_1}{x_5x_6} - \frac{x_6}{x_1x_5}\Big) + \frac{1}{16}\Big(\frac{x_5}{x_2x_4} - \frac{x_2}{x_4x_5} - \frac{x_4}{x_2x_5}\Big),\\
r_6 &= \frac{1}{2x_6} + \frac{1}{16}\Big(\frac{x_6}{x_1x_5} - \frac{x_1}{x_5x_6} - \frac{x_5}{x_1x_6}\Big) + \frac{1}{16}\Big(\frac{x_6}{x_3x_4} - \frac{x_3}{x_4x_6} - \frac{x_4}{x_3x_6}\Big).
\end{aligned}$$

A G_2-invariant Riemannian metric on the full flag manifold G_2/T is Einstein if and only if there is a positive constant k such that

$$r_1 = k, \quad r_2 = k, \quad r_3 = k, \quad r_4 = k, \quad r_5 = k, \quad r_6 = k, \tag{3.5}$$

where r_i $(i = 1,\ldots,6)$ are given in Proposition 3.4.

4. Proof of Theorem A

Note that the action of the Weyl group of $\mathfrak{g}_2$ on its root system (cf. Figure 2) induces an action on the components of the G_2-invariant metric (3.3). In particular, if $(x_1, x_2, x_3, x_4, x_5, x_6) = (a_1, a_2, a_3, a_4, a_5, x_6)$ is a solution for the system of equations (3.5), then

$$(x_1, x_2, x_3, x_4, x_5, x_6) = (a_5, a_2, a_4, a_3, a_1, a_6)$$

is also a solution of the system (3.5). In fact, if w is a reflection about $2\alpha_1 + 3\alpha_2$ in the root diagram of $\mathfrak{g}_2$, then $w(\alpha_1) = \alpha_1 + 3\alpha_2$, $w(\alpha_1 + \alpha_2) = \alpha_1 + 2\alpha_2$. This induces an interchange of x_1 with x_5 and x_3 with x_4, and keeps x_2, x_6 fixed. Similarly we see that

$$\begin{aligned}
(x_1, x_2, x_3, x_4, x_5, x_6) &= (a_6, a_3, a_4, a_2, a_1, a_5),\\
(x_1, x_2, x_3, x_4, x_5, x_6) &= (a_1, a_3, a_2, a_4, a_6, a_5),\\
(x_1, x_2, x_3, x_4, x_5, x_6) &= (a_5, a_4, a_2, a_3, a_6, a_1),\\
(x_1, x_2, x_3, x_4, x_5, x_6) &= (a_6, a_4, a_3, a_2, a_5, a_1)
\end{aligned}$$

are also solutions of system (3.5). These metrics are all isometric to each other.

The above analysis using the Weyl group suggests splitting the study of solutions for the system (3.5) into two cases: Case A: $(x_1 - x_5)(x_1 - x_6)(x_5 - x_6) = 0$ and Case B: $(x_1 - x_5)(x_1 - x_6)(x_5 - x_6) \neq 0$.

Note that the system of equations (3.5) is equivalent to the equations

$$r_1 - r_2 = 0, \quad r_2 - r_3 = 0, \quad r_3 - r_4 = 0, \quad r_4 - r_5 = 0, \quad r_5 - r_6 = 0. \tag{4.1}$$

Moreover, we normalize our equations by setting $x_1 = 1$. Then the system of equations (4.1) is equivalent to the equations

$$\left.\begin{aligned}
f_1 &= -3{x_2}^2x_3x_6 - 6{x_2}^2x_4x_5x_6 - 4{x_2}^2x_5x_6 - 3x_2x_3x_4{x_5}^2\\
&\quad +24x_2x_3x_4x_5x_6 - 3x_2x_3x_4{x_6}^2 + 3x_2x_3x_4 + 4{x_3}^2x_5x_6 + 3x_3{x_4}^2x_6\\
&\quad -24x_3x_4x_5x_6 + 3x_3{x_5}^2x_6 + 4{x_4}^2x_5x_6 + 6x_4x_5x_6 = 0,\\
f_2 &= 3{x_2}^2x_3x_6 + 6{x_2}^2x_4x_5x_6 + 8{x_2}^2x_5x_6 - 3x_2{x_3}^2x_5 + 3x_2{x_4}^2x_5\\
&\quad -24x_2x_4x_5x_6 + 3x_2x_5{x_6}^2 - 6{x_3}^2x_4x_5x_6 - 8{x_3}^2x_5x_6 - 3x_3{x_4}^2x_6\\
&\quad +24x_3x_4x_5x_6 - 3x_3{x_5}^2x_6 = 0,\\
f_3 &= 3{x_2}^2x_3x_6 - 3{x_2}^2x_4x_5x_6 + 6x_2{x_3}^2x_5 - 24x_2x_3x_5x_6 - 6x_2{x_4}^2x_5\\
&\quad +24x_2x_4x_5x_6 + 3{x_3}^2x_4x_5x_6 + 8{x_3}^2x_5x_6 - 3x_3{x_4}^2x_6 + 3x_3{x_5}^2x_6\\
&\quad -8{x_4}^2x_5x_6 - 3x_4x_5x_6 = 0,\\
f_4 &= -4{x_2}^2x_5x_6 - 3x_2{x_3}^2x_5 - 3x_2x_3x_4{x_5}^2 + 3x_2x_3x_4{x_6}^2\\
&\quad -24x_2x_3x_4x_6 + 3x_2x_3x_4 + 24x_2x_3x_5x_6 + 3x_2{x_4}^2x_5 - 3x_2x_5{x_6}^2\\
&\quad -4{x_3}^2x_5x_6 + 6x_3{x_4}^2x_6 - 6x_3{x_5}^2x_6 + 4{x_4}^2x_5x_6 = 0,\\
f_5 &= -{x_2}^2x_3x_6 + x_2{x_3}^2x_5 + 2x_2x_3x_4{x_5}^2 - 8x_2x_3x_4x_5 - 2x_2x_3x_4{x_6}^2\\
&\quad +8x_2x_3x_4x_6 + x_2{x_4}^2x_5 - x_2x_5{x_6}^2 - x_3{x_4}^2x_6 + x_3{x_5}^2x_6 = 0
\end{aligned}\right\} \tag{4.2}$$

for solutions with $x_2x_3x_4x_5x_6 \neq 0$.

To find nonzero solutions of equations (4.2), we compute a Gröbner basis (see for example [9]) by using algebraic manipulations in a computer system.

Case A. We may assume that $x_1 = x_5 = 1$. If $x_6 = 1$, we consider a polynomial ring $R_1 = \mathbb{Q}[y, x_2, x_3, x_4, x_5, x_6]$ and an ideal I_1 generated by

$$\{f_1,\, f_2,\, f_3,\, f_4,\, f_5,\, x_5 - 1,\, x_6 - 1,\, y\, x_2x_3x_4x_5x_6 - 1\}.$$

We take a lexicographic order $>$ with $y > x_2 > x_3 > x_4 > x_5 > x_6$ for a monomial ordering on R_1. Then a Gröbner basis for the ideal I_1 is given by

$$\{x_6 - 1, x_5 - 1, 15{x_4}^2 - 20x_4 + 9, x_3 - x_4, x_2 - x_3, -2600 + 3975x_4 + 729y\}.$$

Now the equation $15{x_4}^2 - 20x_4 + 9 = 0$ has no real solutions. Thus there are no Einstein metrics for this case.

If $x_6 \neq 1$, we consider an ideal I_2 generated by

$$\{f_1,\ f_2,\ f_3,\ f_4,\ f_5,\ x_5 - 1,\ (x_6 - 1)\, y\, x_2x_3x_4x_5x_6 - 1\}.$$

We take a lexicographic order $>$ with $y > x_6 > x_5 > x_2 > x_3 > x_4$ for a monomial ordering on R_1. Then we see that $x_3 - x_4$ is an element of a Gröbner basis for the ideal I_2. Thus we obtain the following expression for the Ricci components in this case:

$$\begin{aligned}
r_1 = r_5 &= \frac{1}{2} + \frac{1}{16}\left(\frac{1}{x_2x_3} - \frac{x_2}{x_3} - \frac{x_3}{x_2}\right) - \frac{x_6}{16},\\
r_2 &= \frac{1}{2x_2} + \frac{1}{12}\left(\frac{x_2}{{x_3}^2} - \frac{2}{x_2}\right) + \frac{1}{8}\left(\frac{x_2}{x_3} - \frac{x_3}{x_2} - \frac{1}{x_2x_3}\right),\\
r_3 = r_4 &= \frac{1}{2x_3} + \frac{1}{16}\left(\frac{x_3}{x_2} - \frac{x_2}{x_3} - \frac{1}{x_2x_3}\right) - \frac{x_2}{12{x_3}^2} - \frac{x_6}{16{x_3}^2},\\
r_6 &= \frac{1}{2x_6} + \frac{1}{16}\left(x_6 - \frac{2}{x_6}\right) + \frac{1}{16}\left(\frac{x_6}{{x_3}^2} - \frac{2}{x_6}\right).
\end{aligned}$$

Now the system of equations (3.5) is equivalent to the equations

$$r_1 = r_2, \quad r_2 = r_3, \quad r_3 = r_6. \tag{4.3}$$

Moreover, we see that the system of equations (4.3) is equivalent to the equations

$$\left.\begin{aligned}
&h_1 = -9{x_2}^2x_3 - 4{x_2}^2 - 3x_2{x_3}^2x_6 + 24x_2{x_3}^2 + 3{x_3}^3 - 16{x_3}^2 + 9x_3 = 0\\
&h_2 = 9{x_2}^2x_3 + 8{x_2}^2 - 24x_2x_3 + 3x_2x_6 - 9{x_3}^3 + 16{x_3}^2 - 3x_3 = 0\\
&h_3 = -3{x_2}^2x_3x_6 - 4{x_2}^2x_6 - 3x_2{x_3}^2{x_6}^2 - 12x_2{x_3}^2 + 24x_2x_3x_6\\
&\qquad -6x_2{x_6}^2 + 3{x_3}^3x_6 - 3x_3x_6 = 0,
\end{aligned}\right\} \tag{4.4}$$

for solutions with $x_2x_3x_6 \neq 0$.

We consider a polynomial ring $R_2 = \mathbb{Q}[y, x_2, x_3, x_6]$ and an ideal I_3 generated by

$$\{h_1,\ h_2,\ h_3,\ y\, x_2x_3x_6 - 1\}.$$

We take a lexicographic order $>$ with $y > x_2 > x_3 > x_6$ for a monomial ordering on R_2. Then a Gröbner basis for the ideal I_3 contains the following polynomials p_1, p_2, p_3:

$$\begin{aligned}
p_1 = {}& 28431{x_6}^{14} - 589032{x_6}^{13} + 5435343{x_6}^{12} - 29379024{x_6}^{11} + 100757208{x_6}^{10}\\
&- 224163176{x_6}^{9} + 336260186{x_6}^{8} - 371473808{x_6}^{7} + 339968604{x_6}^{6} - 262478048{x_6}^{5}\\
&+ 152856152{x_6}^{4} - 69550016{x_6}^{3} + 35706576{x_6}^{2} - 17407872x_6 + 3888000,
\end{aligned}$$

$$
\begin{aligned}
p_2 = {} & 5819853108320284739829203580542725270399576306963 2x_2 \\
& - 36431187984975954069625075825512020735715490014597{x_6}^{13} \\
& + 72992357388477268215374104374790339627732724331072{x_6}^{12} \\
& - 64656772775820793500227598662803317923006565266306 1{x_6}^{11} \\
& + 33215185790428453715523238606476025841096505537289 20{x_6}^{10} \\
& - 10630524684514537641000725361809530238649740680444344{x_6}^{9} \\
& + 21417364804945911429515190574637753191025839827487192{x_6}^{8} \\
& - 28389061171757812126136768127456927964712883615920638{x_6}^{7} \\
& + 28311617865989383607989773945214867782385295574349024{x_6}^{6} \\
& - 24774704999202893012898243740523073131413082414850260{x_6}^{5} \\
& + 17526790961102909129622834293267525297910502941466624{x_6}^{4} \\
& - 8188114481577095576234998176450007614578500056871240{x_6}^{3} \\
& + 3562379534276698939524030165567374089875873542732800{x_6}^{2} \\
& - 2298954881044018869226019836424141856363362783139696x_6 \\
& + 738157956056149928743880926430168536213084185530880,
\end{aligned}
$$

$$
\begin{aligned}
p_3 = {} & 242493879513345197492883482522613552933315679456 8x_3 \\
& + 19029972626061774836007867169218828586354518623 1{x_6}^{13} \\
& - 37726721802091649084429970484292312301086880157 08{x_6}^{12} \\
& + 33007596001063757829305936652578219133471910058553{x_6}^{11} \\
& - 167088331330227007688571325972415637397648450592985{x_6}^{10} \\
& + 52450842367029388490748344153895307456807516761375 0{x_6}^{9} \\
& - 1028643118190496545823481969284436484392588928255299{x_6}^{8} \\
& + 1321914168075901690582280884750861726955041116133678{x_6}^{7} \\
& - 1286826151972665839433699700223920764972986949833794{x_6}^{6} \\
& + 1102747968247342493561980802133113539094868778040040{x_6}^{5} \\
& - 748737830066525920856184078153848962211101215021298{x_6}^{4} \\
& + 334500258786115622392457312297818354307784956975224{x_6}^{3} \\
& - 155759212247584755088196238509822799941866625955256{x_6}^{2} \\
& + 954075532838413595542047169961244884467927948471 68x_6 \\
& - 280834152747250865327250246248554269292077786168 00.
\end{aligned}
$$

By solving the equation $p_1 = 0$ numerically, we obtain exactly two real solutions which are approximately given by $x_6 \approx 0.74403477990$ and $x_6 \approx 1.789600622$. We substitute these values for x_6 into equations $p_2 = 0$ and $p_3 = 0$, and we obtain two positive solutions approximately given by $x_2 \approx 0.21737$, $x_3 \approx 1.02343$ and $x_2 \approx 0.27624$, $x_3 \approx 1.03473$. Moreover, we obtain the value for k from (3.5). Thus we have the following:

Theorem 4.1. *The full flag manifold G_2/T admits two non-Kähler G_2-invariant Einstein metrics. These metrics are given approximately as follows:*

$$x_1 = 1, \quad x_2 \approx 0.2762, \quad x_3 = x_4 \approx 1.0347, \quad x_5 = 1, \quad x_6 \approx 1.7896, \quad k \approx 0.3560,$$
$$x_1 = 1, \quad x_2 \approx 0.2173, \quad x_3 = x_4 \approx 1.0234, \quad x_5 = 1, \quad x_6 \approx 0.7440, \quad k \approx 0.4269.$$

Case B. We consider a polynomial ring $R_1 = \mathbb{Q}[y, x_2, x_3, x_4, x_5, x_6]$ and an ideal I_4 generated by

$$\{f_1,\, f_2,\, f_3,\, f_4,\, f_5,\, (x_5-1)(x_6-1)\, y\, x_2x_3x_4x_5x_6 - 1\}.$$

We take a lexicographic order $>$ with $y > x_2 > x_3 > x_4 > x_5 > x_6$ for a monomial ordering on R_1. Then a Gröbner basis for the ideal I_4 contains a polynomial of the form

$$(x_6-3)(x_6-2)(2x_6-3)(2x_6-1)(3x_6-2)(3x_6-1)q_1(x_6),$$

where $q_1(x_6)$ is an explicitly given polynomial of degree 84 with integer coefficients.

For the case when $(x_6-3)(x_6-2)(2x_6-3)(2x_6-1)(3x_6-2)(3x_6-1) = 0$, we first consider an ideal I_5 generated by

$$\{f_1,\, f_2,\, f_3,\, f_4,\, f_5, (x_6-3), y\, x_2x_3x_4x_5x_6 - 1\}.$$

We take a lexicographic order $>$ with $y > x_2 > x_3 > x_4 > x_5 > x_6$ for a monomial ordering on R_1. Then a Gröbner basis for the ideal I_5 is given by

$$\{x_6-3, x_5-2, 3x_4-5, 3x_3-4, 3x_2-1, 40y-9\}.$$

We also compute the Gröbner basis for other cases and we obtain the following sets of solutions for equations (4.2):

$$(x_6 = 3, x_5 = 2, x_4 = \tfrac{5}{3}, x_3 = \tfrac{4}{3}, x_2 = \tfrac{1}{3}),\ (x_6 = 2, x_5 = 3, x_4 = \tfrac{5}{3},\ x_3 = \tfrac{1}{3}, x_2 = \tfrac{4}{3}),$$
$$(x_6 = \tfrac{3}{2}, x_5 = \tfrac{1}{2},\ x_4 = \tfrac{2}{3}, x_3 = \tfrac{5}{6}, x_2 = \tfrac{1}{6}),\ (x_6 = \tfrac{1}{2}, x_5 = \tfrac{3}{2}, x_4 = \tfrac{2}{3}, x_3 = \tfrac{1}{6}, x_2 = \tfrac{5}{6}),$$
$$(x_6 = \tfrac{2}{3}, x_5 = \tfrac{1}{3}, x_4 = \tfrac{1}{9}, x_3 = \tfrac{5}{9}, x_2 = \tfrac{4}{9}),\ (x_6 = \tfrac{1}{3}, x_5 = \tfrac{2}{3}, x_4 = \tfrac{1}{9}, x_3 = \tfrac{4}{9}, x_2 = \tfrac{5}{9}).$$

Note that these are the six Kähler-Einstein metrics of Theorem 3.1.

Now, by solving the equation $q_1(x_6) = 0$ numerically, we obtain 14 positive solutions, which are approximately given by

$$x_6 \approx 0.1101296649906623,\ x_6 \approx 0.1276467609933986,\ x_6 \approx 0.1654266507070432,$$
$$x_6 \approx 0.2010643285289733,\ x_6 \approx 0.3065328288396123,\ x_6 \approx 0.5181203151843693,$$
$$x_6 \approx 0.5477334830916693,\ x_6 \approx 1.825705440455314,\ x_6 \approx 1.930053639460474,$$
$$x_6 \approx 3.262293320377869,\ x_6 \approx 4.973532636625297,\ x_6 \approx 6.044975194298747,$$
$$x_6 \approx 7.834119661302769,\ x_6 \approx 9.080205592968872.$$

To get the solutions of equations (4.2) for the variables x_2, x_3, x_4, x_5 corresponding to the solution x_6, we compute a Gröbner basis under the condition $(x_6 - 3)(x_6-2)(2x_6-3)(2x_6-1)(3x_6-2)(3x_6-1) \neq 0$; that is, we consider an ideal I_6 generated by $\{f_1,\, f_2,\, f_3,\, f_4,\, f_5, (x_6-3)(x_6-2)(2x_6-3)(2x_6-1)(3x_6-2)(3x_6 - 1)\, y\, x_2x_3x_4x_5x_6 - 1\}$ and take a lexicographic order $>$ with $y > x_2 > x_3 > x_4 > x_5 > x_6$ for a monomial ordering on R_1. Then q_1 is contained in this Gröbner basis, and by examining the other elements of the obtained Gröbner basis, we see that the other variables x_2, x_3, x_4, x_5 can be expressed by polynomials of x_6 with degree 83. Let $x_2 = q_2(x_6)$, $x_3 = q_3(x_6)$, $x_4 = q_4(x_6)$, and $x_5 = q_5(x_6)$ be these polynomials. Now we substitute these 14 values for x_6 into the expressions of the polynomials q_2, q_3, q_4, q_5 of x_6 with degree 83. Then we get the following solutions which are approximately given by

$$
\begin{aligned}
&x_6 \approx 0.11013,\ x_5 \approx 0.547733,\ x_4 \approx 1.61358,\ x_3 \approx 0.399131,\ x_2 \approx -0.277481,\\
&x_6 \approx 0.127647,\ x_5 \approx -0.775539,\ x_4 \approx 0.202709,\ x_3 \approx 1.7601,\ x_2 \approx -0.203265,\\
&x_6 \approx 0.165427,\ x_5 \approx -0.021892,\ x_4 \approx 0.308989,\ x_3 \approx 0.00455279,\ x_2 \approx 0.5435,\\
&x_6 \approx 0.201064,\ x_5 \approx 1.82571,\ x_4 \approx 0.728695,\ x_3 \approx 2.94591,\ x_2 \approx -0.506599,\\
&x_6 \approx 0.306533,\ x_5 \approx -1.52438,\ x_4 \approx 0.207857,\ x_3 \approx 1.64949,\ x_2 \approx 5.33389,\\
&x_6 \approx 0.51812,\ x_5 \approx -0.100239,\ x_4 \approx -0.120371,\ x_3 \approx -2.58645,\ x_2 \approx -0.539579,\\
&x_6 \approx 0.547733,\ x_5 \approx 0.11013,\ x_4 \approx 1.61358,\ x_3 \approx -0.277481,\ x_2 \approx 0.399131,\\
&x_6 \approx 1.82571,\ x_5 \approx 0.201064,\ x_4 \approx 0.728695,\ x_3 \approx -0.506599,\ x_2 \approx 2.94591,\\
&x_6 \approx 1.93005,\ x_5 \approx -0.193467,\ x_4 \approx -1.04142,\ x_3 \approx -4.99198,\ x_2 \approx -0.232323,\\
&x_6 \approx 3.26229,\ x_5 \approx -4.97297,\ x_4 \approx 17.4007,\ x_3 \approx 5.38113,\ x_2 \approx 0.678092,\\
&x_6 \approx 4.97353,\ x_5 \approx 9.08021,\ x_4 \approx -2.51959,\ x_3 \approx 14.6516,\ x_2 \approx 3.62419,\\
&x_6 \approx 6.04498,\ x_5 \approx -0.132336,\ x_4 \approx 3.28544,\ x_3 \approx 0.0275215,\ x_2 \approx 1.86783,\\
&x_6 \approx 7.83412,\ x_5 \approx -6.07566,\ x_4 \approx -1.5924,\ x_3 \approx 13.7889,\ x_2 \approx 1.58805,\\
&x_6 \approx 9.08021,\ x_5 \approx 4.97353,\ x_4 \approx -2.51959,\ x_3 \approx 3.62419,\ x_2 \approx 14.6516.
\end{aligned}
$$

We see that at least one of the x_i's in these solutions is negative. Thus there are no invariant Einstein metrics for these cases, and this completes the proof of Theorem A.

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