

Algebras Defined by Monic Gröbner Bases over Rings

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Abstract

Let R be a commutative ring and $R\langle X \rangle = R\langle X_1, \dots, X_n \rangle$ the free algebra of n generators over R . It is shown that if \mathcal{G} is a monic Gröbner basis for the ideal $I = \langle \mathcal{G} \rangle$ in $R\langle X \rangle$, then, as that done over a field in the literature, many global structure properties of the R -algebra $A = R\langle X \rangle / \langle \mathcal{G} \rangle$ may be determined via a constructive PBW theory (PBW basis plus different types of PBW isomorphism) over R .

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1 Introduction

In the structure theory and the representation theory of associative algebras over a ground field K , it is well known that numerous popularly studied algebras have defining relations which form a Gröbner basis \mathcal{G} in the sense of ([4], [13], [26]), and such algebras can be studied in a computational way via their Gröbner defining relations (e.g., [1], [8], [11], [13], [18], [19], [29], [30]); also we know that algebras defined by the relations of the same type as provided by \mathcal{G} over a commutative ring R are equally important, for instance, those R -algebras considered in [31], [3], [7], and [22]. In this paper we show that the principle and methods of using Gröbner bases in the structure theory of algebras over a field, which were developed in [23] and [19], may be generalized to study algebras defined by monic Gröbner bases over rings. More precisely, let $R\langle X \rangle = R\langle X_1, \dots, X_n \rangle$ be the free R -algebra of n

generators over a commutative ring R . In Section 2, with a little modification we briefly review from the literature ([4], [13], [26]) several well-known fundamental results on monic Gröbner bases in $R\langle X \rangle$, and in consideration of the fact that Gröbner bases over a field are algorithmically constructible (checkable or computable), we indicate how Gröbner bases over a field and monic Gröbner bases over a ring are related to each other (for more details on the results presented in this section, the reader is referred to the original version of this paper at <http://arxiv.org/abs/0906.4396>). In Section 3, after strengthening and generalizing ([13], Proposition 2.14) and ([18], CH.III, Theorem 1.5), we demonstrate, by presenting some examples, how PBW R -bases and monic Gröbner bases of certain type can determine each other. In the final Section 4, we show that the working strategy via different types of PBW isomorphism developed in [23] and [19] can be generalized to study quotient algebras of $R\langle X \rangle$, so that many global structural properties of R -algebras defined by monic Gröbner bases may be determined in terms of their \mathbb{N} -leading homogeneous algebras and \mathcal{B}_R -leading homogeneous algebras.

Unless otherwise stated, rings considered in this paper are associative rings with multiplicative identity 1, ideals are meant two-sided ideals, and modules are unitary left modules. For a subset U of a ring S , we write $\langle U \rangle$ for the ideal generated by U . Moreover, we use \mathbb{N} , respectively \mathbb{Z} , to denote the set of nonnegative integers, respectively the set of integers.

2 Gröbner Bases over K vs Monic Gröbner Bases over R

Let R be an arbitrary *commutative ring*, $R\langle X \rangle = R\langle X_1, \dots, X_n \rangle$ the free R -algebra of n generators, and \mathcal{B}_R the standard R -basis of $R\langle X \rangle$ consisting of monomials (words in alphabet $X = \{X_1, \dots, X_n\}$, including empty word which is identified with the multiplicative identity element 1 of $R\langle X \rangle$). Considering an R -algebra $A = R\langle X \rangle / \langle S \rangle$ with the set of defining relations S consisting of monic elements of the form $g_\sigma = W_\sigma - f_\sigma$, where $W_\sigma \in \mathcal{B}_R$, and with respect to a semigroup partial ordering \prec on \mathcal{B}_R , each f_σ is a linear combination of monomials $\prec W_\sigma$, then, it is well-known that Bergman's diamond lemma [4] tackles the resolvability of ambiguities (or overlaps) of pairs $(g_{\sigma_i}, g_{\sigma_j})$ with $g_{\sigma_i}, g_{\sigma_j} \in S$, and consequently answers when the set of *normal monomials* (mod S) forms an R -basis for the algebra A . It is equally well-known that if $R = K$ is a field, then, because of the feasibility of a division algorithm in $K\langle X \rangle = K\langle X_1, \dots, X_n \rangle$, the celebrated Buchberger's termination theorem and Buchberger Algorithm in the commutative Gröbner basis theory over K ([5], [6]) had been successfully generalized to develop an algorithmic noncommutative Gröbner basis theory for $K\langle X \rangle$ ([26], [13]), in which Bergman's diamond

lemma is equivalent to the noncommutative version of Buchberger's termination theorem. More precisely, let $K\langle X \rangle = K\langle X_1, \dots, X_n \rangle$ be the free associative K -algebra of n generators over a field K , and let \mathcal{B} be the standard K -basis of $K\langle X \rangle$ consisting of monomials (words in alphabet $X = \{X_1, \dots, X_n\}$). Given a monomial ordering \prec on \mathcal{B} (i.e. a well-ordering \prec on \mathcal{B} satisfying: $u \prec v$ implies $wus \prec wvs$, and $w = uv$ implies $u \preceq w$ and $v \preceq w$, for all $w, u, v, s \in \mathcal{B}$), and $f, g \in K\langle X \rangle - \{0\}$, if there are monomials $u, v \in \mathcal{B}$ such that

$$(1) \quad \mathbf{LM}(f)u = v\mathbf{LM}(g), \text{ and}$$

$$(2) \quad \mathbf{LM}(f) \not\prec v \text{ and } \mathbf{LM}(g) \not\prec u,$$

then the element

$$o(f, u; v, g) = \frac{1}{\mathbf{LC}(f)}(f \cdot u) - \frac{1}{\mathbf{LC}(g)}(v \cdot g)$$

is referred to as an overlap element of f and g where, with respect to \prec , $\mathbf{LM}(\)$ denotes the function taking the leading monomial on elements of $K\langle X \rangle$, and $\mathbf{LC}(\)$ denotes the function taking the leading coefficient on elements of $K\langle X \rangle$ respectively. For a subset $S \subset K\langle X \rangle$, write $\mathbf{LM}(S) = \{\mathbf{LM}(f) \mid f \in S\}$ for the set of leading monomials of S with respect to \prec . A subset $\mathcal{G} \subset K\langle X \rangle$ is said to be a *Gröbner basis* for the ideal $I = \langle \mathcal{G} \rangle$ generated by \mathcal{G} if $\langle \mathbf{LM}(I) \rangle = \langle \mathbf{LM}(\mathcal{G}) \rangle$, or equivalently, if $0 \neq f \in I$, then there is some $g \in \mathcal{G}$ such that $\mathbf{LM}(g) \mid \mathbf{LM}(f)$. With notations and terminology as above, the termination theorem then states that

- if \mathcal{G} is an LM-reduced subset of $K\langle X \rangle$ (i.e., $\mathbf{LM}(g_i) \not\prec \mathbf{LM}(g_j)$ for $g_i, g_j \in \mathcal{G}$ with $i \neq j$), then \mathcal{G} is a Gröbner basis for the ideal $I = \langle \mathcal{G} \rangle$ if and only if for each pair $g_i, g_j \in \mathcal{G}$, including $g_i = g_j$, every overlap element $o(g_i, u; v, g_j)$ of g_i and g_j has the property $\overline{o(g_i, u; v, g_j)}^{\mathcal{G}} = 0$, that is, by the division by \mathcal{G} , every $o(g_i, u; v, g_j)$ has the zero remainder;

and it follows that there is a noncommutative analogue of Buchberger Algorithm for constructing a (possibly infinite) Gröbner basis starting with a given finite subset in $K\langle X \rangle$.

Note that the algorithmic feasibility of the above criterion lies in the fact that

- (a) for each pair (g_i, g_j) there are only finitely many associated overlap elements, and
- (b) there is no trouble with taking the inverse of a nonzero coefficient when the division algorithm is performed, for, K is a field.

But if the field K is replaced by a commutative ring R (or even if R is an arithmetic ring as considered by [12]), and if $\mathcal{G} \subset R\langle X \rangle = R\langle X_1, \dots, X_n \rangle$ is taken such that $\mathbf{LC}(g)$ is not invertible for some g 's, then a pair (g_i, g_j) of elements in \mathcal{G} may have infinitely many overlap elements, thereby there seems no an “algorithmically realizable” termination theorem (as we mentioned above) for \mathcal{G} . Nevertheless, we observe, in the case where R is a subring of the field K with the same identity element, that the usual division by monic elements (i.e. elements with leading coefficient 1) can be implemented in $R\langle X \rangle$ exactly as in $K\langle X \rangle$, and this fact implies immediately that if a subset $\mathcal{G} \subset R\langle X \rangle$ consisting of monic elements forms a Gröbner basis for the ideal $\langle \mathcal{G} \rangle$ in $K\langle X \rangle$ with respect to some monomial ordering \prec on \mathcal{B} , then, with respect to the same type of monomial ordering \prec on the standard R -basis \mathcal{B}_R of $R\langle X \rangle$, \mathcal{G} is a Gröbner basis for the ideal $\langle \mathcal{G} \rangle$ in $R\langle X \rangle$. More generally, for our purpose of generalizing the principle and methods in using Gröbner bases over a field K [19] to using Gröbner bases over a ring in a computational way, we summarize below several fundamental results concerning monic Gröbner bases over a commutative ring, which are comprehensively stemming from [4], [26], and [13], and furthermore indicate how Gröbner bases over a field and monic Gröbner bases over a ring are related to each other

Let R be an arbitrary commutative ring, $R\langle X \rangle = R\langle X_1, \dots, X_n \rangle$ the free R -algebra of n generators, and \mathcal{B}_R the standard R -basis of $R\langle X \rangle$ consisting of monomials (words in alphabet $X = \{X_1, \dots, X_n\}$). Unless otherwise stated, monomials in \mathcal{B}_R are denoted by lower case letters u, v, w, s, t, \dots . First note that all monomial orderings used for free algebras over a field can be well defined on the standard R -basis \mathcal{B}_R of $R\langle X \rangle$. In particular, by an \mathbb{N} -graded monomial ordering on \mathcal{B}_R , denoted \prec_{gr} , we mean a monomial ordering on \mathcal{B}_R which is defined subject to a well-ordering \prec on \mathcal{B}_R , that is, for $u, v \in \mathcal{B}_R$, $u \prec_{gr} v$ if either $\deg u < \deg v$ or $\deg u = \deg v$ but $u \prec v$, where $\deg(\)$ denotes the degree function on elements of $R\langle X \rangle$ with respect to a fixed *weight* \mathbb{N} -gradation of $R\langle X \rangle$ (i.e. each X_i is assigned a positive degree n_i , $1 \leq i \leq n$). For instance, the usual \mathbb{N} -graded (reverse) lexicographic ordering is a popularly used \mathbb{N} -graded monomial ordering.

Let \prec be a monomial ordering on \mathcal{B}_R . We say that a subset $G \subset R\langle X \rangle$ is *monic* if the leading coefficient $\mathbf{LC}(g) = 1$ for all $g \in G$. For $u, v, w, s \in \mathcal{B}_R$, if $u = wvs$ then we say that v *divides* u , denoted $v|u$. The division of monomials naturally extends to a division algorithm by a monic subset G in $R\langle X \rangle$, and this leads to the following definition.

Definition 2.1 *Let \prec be a fixed monomial ordering on \mathcal{B}_R , and I an ideal of $R\langle X \rangle$. A monic Gröbner basis of I is a subset $\mathcal{G} \subset I$ satisfying:*

- (1) \mathcal{G} is monic; and
- (2) $f \in I$ and $f \neq 0$ implies $\mathbf{LM}(g)|\mathbf{LM}(f)$ for some $g \in \mathcal{G}$.

By the division algorithm, it is clear that a monic Gröbner basis of I is first of all a generating set of the ideal I , i.e., $I = \langle \mathcal{G} \rangle$, and moreover, a monic Gröbner basis of I can be characterized as follows.

Proposition 2.2 *Let \prec be a fixed monomial ordering on \mathcal{B}_R , and I an ideal of $R\langle X \rangle$. For a monic subset $\mathcal{G} \subset I$, the following statements are equivalent:*

- (i) \mathcal{G} is a monic Gröbner basis of I ;
- (ii) Each nonzero $f \in I$ has a Gröbner representation:

$$f = \sum_{i,j} \lambda_{ij} u_{ij} g_j v_{ij}, \text{ where } \lambda_{ij} \in R, u_{ij}, v_{ij} \in \mathcal{B}_R, g_j \in \mathcal{G}, \\ \text{satisfying } \mathbf{LM}(u_{ij} g_j v_{ij}) \preceq \mathbf{LM}(f) \text{ whenever } \lambda_{ij} \neq 0,$$

or equivalently, $\overline{f}^{\mathcal{G}} = 0$, where $\overline{f}^{\mathcal{G}}$ denotes the remainder of f on division by \mathcal{G} ;

- (iii) $\langle \mathbf{LM}(\mathcal{G}) \rangle = \langle \mathbf{LM}(I) \rangle$.

□

Let \prec be a monomial ordering on the standard R -basis \mathcal{B}_R of $R\langle X \rangle$, and let G be a monic subset of $R\langle X \rangle$. We call an element $f \in R\langle X \rangle$ a *normal element (mod G)* if $f = \sum_j \mu_j v_j$ with $\mu_j \in R$, $v_j \in \mathcal{B}_R$, and f has the property that $\mathbf{LM}(g) \not\prec v_j$ for every $g \in G$ and every $\mu_j \neq 0$. The set of normal monomials in $\mathcal{B}_R \pmod{G}$ is denoted by $N(G)$, i.e.,

$$N(G) = \{u \in \mathcal{B}_R \mid \mathbf{LM}(g) \not\prec u, g \in G\}.$$

Thus, an element $f \in R\langle X \rangle$ is normal (mod G) if and only if $f \in \sum_{u \in N(G)} Ru$.

Proposition 2.3 *Let \mathcal{G} be a monic Gröbner basis of the ideal $I = \langle \mathcal{G} \rangle$ in $R\langle X \rangle$ with respect to some monomial ordering \prec on \mathcal{B}_R . Then each nonzero $f \in R\langle X \rangle$ has a finite presentation*

$$f = \sum_{i,j} \lambda_{ij} s_{ij} g_i w_{ij} + r_f, \quad \lambda_{ij} \in R, s_{ij}, w_{ij} \in \mathcal{B}_R, g_i \in \mathcal{G},$$

where $\mathbf{LM}(s_{ij} g_i w_{ij}) \preceq \mathbf{LM}(f)$ whenever $\lambda_{ij} \neq 0$, and either $r_f = 0$ or r_f is a unique normal element (mod \mathcal{G}). Hence, $f \in I$ if and only if $r_f = 0$, solving the “membership problem” for I .

□

The foregoing results enable us to obtain further characterization of a monic Gröbner basis \mathcal{G} , which, in turn, gives rise to the fundamental decomposition theorem of the R -module $R\langle X \rangle$ by the ideal $I = \langle \mathcal{G} \rangle$, and thereby yields a free R -basis for the R -algebra $R\langle X \rangle/I$.

Theorem 2.4 Let $I = \langle \mathcal{G} \rangle$ be an ideal of $R\langle X \rangle$ generated by a monic subset \mathcal{G} . With notation as above, the following statements are equivalent.

- (i) \mathcal{G} is a monic Gröbner basis of I .
- (ii) The R -module $R\langle X \rangle$ has the decomposition

$$R\langle X \rangle = I \oplus \sum_{u \in N(\mathcal{G})} Ru = \langle \mathbf{LM}(I) \rangle \oplus \sum_{u \in N(\mathcal{G})} Ru.$$

- (iii) The canonical image $\overline{N(\mathcal{G})}$ of $N(\mathcal{G})$ in $R\langle X \rangle / \langle \mathbf{LM}(I) \rangle$ and $R\langle X \rangle / I$ forms a free R -basis for $R\langle X \rangle / \langle \mathbf{LM}(I) \rangle$ and $R\langle X \rangle / I$ respectively.

□

Before stating the termination theorem, which is a version of Bergman's diamond lemma but is modified in the algorithmic Gröbner basis language in the sense of ([26], [13]) for verifying an LM-reduced monic Gröbner basis in $R\langle X \rangle$ (see the definition below), we need a little more preparation.

Given a monomial ordering \prec on \mathcal{B}_R , we say that a subset $G \subset R\langle X \rangle$ is *LM-reduced* if

$$\mathbf{LM}(g_i) \not\prec \mathbf{LM}(g_j) \text{ for all } g_i, g_j \in G \text{ with } g_i \neq g_j.$$

If a subset $G \subset R\langle X \rangle$ is both LM-reduced and monic, then we call G an *LM-reduced monic subset*. Thus we have the notion of an *LM-reduced monic Gröbner basis*.

Let I be an ideal of $R\langle X \rangle$. If \mathcal{G} is a monic Gröbner basis of I and $g_1, g_2 \in \mathcal{G}$ such that $g_1 \neq g_2$ but $\mathbf{LM}(g_1) \mid \mathbf{LM}(g_2)$, then clearly g_2 can be removed from \mathcal{G} and the remained subset $\mathcal{G} - \{g_2\}$ is again a monic Gröbner basis for I . Hence, in order to have a better criterion for monic Gröbner basis we need only to consider the subset which is both LM-reduced and monic.

Let \prec be a monomial ordering on \mathcal{B}_R . For two *monic* elements $f, g \in R\langle X \rangle - \{0\}$, including $f = g$, if there are monomials $u, v \in \mathcal{B}_R$ such that

- (1) $\mathbf{LM}(f)u = v\mathbf{LM}(g)$, and
- (2) $\mathbf{LM}(f) \not\prec v$ and $\mathbf{LM}(g) \not\prec u$,

then the element

$$o(f, u; v, g) = f \cdot u - v \cdot g$$

is called an *overlap element* of f and g .

From the definition it is clear that there are only finitely many overlap elements for each pair (f, g) of monic elements in $R\langle X \rangle$. So, for a finite subset of monic elements $\mathcal{G} \subset R\langle X \rangle$, actually as in the classical case ([26], [13]), the termination theorem below enables us to check, by using the division algorithm, whether \mathcal{G} is a Gröbner basis of I or not.

Theorem 2.5 (*Termination theorem*) *Let \prec be a fixed monomial ordering on \mathcal{B}_R . If \mathcal{G} is an LM-reduced monic subset of $R\langle X \rangle$, then \mathcal{G} is an LM-reduced monic Gröbner basis for the ideal $I = \langle \mathcal{G} \rangle$ if and only if for each pair $g_i, g_j \in \mathcal{G}$, including $g_i = g_j$, every overlap element $o(g_i, u; v, g_j)$ of g_i, g_j has the property $\overline{o(g_i, u; v, g_j)}^{\mathcal{G}} = 0$, that is, by division by \mathcal{G} , every $o(g_i, u; v, g_j)$ is reduced to zero.*

□

Remark (i) Obviously, if $\mathcal{G} \subset R\langle X \rangle$ is an LM-reduced subset with the property that each $g \in \mathcal{G}$ has the leading coefficient $\mathbf{LC}(g)$ which is invertible in R , then Theorem 2.5 is also valid for \mathcal{G} .

(ii) It is obvious as well that Theorem 2.5 does not necessarily induce an analogue of Buchberger Algorithm as in the classical case.

(iii) It is not difficult to see that all results we presented so far are valid for getting monic Gröbner bases in a commutative polynomial ring $R[x_1, \dots, x_n]$ over an arbitrary commutative ring R where overlap elements are replaced by S -polynomials.

By virtue of Theorem 2.5, the following two propositions are obtained.

Proposition 2.6 *Let $K\langle X \rangle = K\langle X_1, \dots, X_n \rangle$ be the free algebra of n generators over a field K , and let $R\langle X \rangle = R\langle X_1, \dots, X_n \rangle$ be the free algebra of n generators over an arbitrary commutative ring R . With notation as before, fixing the same monomial ordering \prec on both $K\langle X \rangle$ and $R\langle X \rangle$, the following statements hold.*

(i) *If a monic subset $\mathcal{G} \subset K\langle X \rangle$ is a Gröbner basis for the ideal $\langle \mathcal{G} \rangle$ in $K\langle X \rangle$, then, taking a counterpart of \mathcal{G} in $R\langle X \rangle$ (if it exists), again denoted by \mathcal{G} , \mathcal{G} is a monic Gröbner basis for the ideal $\langle \mathcal{G} \rangle$ in $R\langle X \rangle$.*

(ii) *If a monic subset $\mathcal{G} \subset R\langle X \rangle$ is a Gröbner basis for the ideal $\langle \mathcal{G} \rangle$ in $R\langle X \rangle$, then, taking a counterpart of \mathcal{G} in $K\langle X \rangle$ (if it exists), again denoted by \mathcal{G} , \mathcal{G} is a Gröbner basis for the ideal $\langle \mathcal{G} \rangle$ in $K\langle X \rangle$.*

□

Proposition 2.7 *Let R be a commutative ring and R' a subring of R with the same identity element 1. Considering the free R -algebra $R\langle X \rangle = R\langle X_1, \dots, X_n \rangle$ and the free R' -algebra $R'\langle X \rangle = R'\langle X_1, \dots, X_n \rangle$, the following two statements are equivalent for a subset $\mathcal{G} \subset R'\langle X \rangle$:*

(i) *\mathcal{G} is an LM-reduced monic Gröbner basis for the ideal $I = \langle \mathcal{G} \rangle$ in $R'\langle X \rangle$ with respect to some monomial ordering \prec on the standard R' -basis $\mathcal{B}_{R'}$ of $R'\langle X \rangle$;*

(ii) *\mathcal{G} is an LM-reduced monic Gröbner basis for the ideal $J = \langle \mathcal{G} \rangle$ in $R\langle X \rangle$ with respect to the monomial ordering \prec on the standard R -basis \mathcal{B}_R of $R\langle X \rangle$, where \prec is the same monomial ordering used in (i).*

□

Let K be a field. From the literature we know that numerous well-known K -algebras, such as Weyl algebras over K , enveloping algebras of K -Lie algebras, exterior K -algebras, Clifford K -algebras, down-up K -algebras, quantum binomial K -algebras, most popularly studied quantum groups over K , etc., all have defining relations that form an LM-reduced monic Gröbner basis in free K -algebras (cf. [18], [16], [10]). Hence, by Proposition 2.6, if the field K is replaced by a commutative ring R , then all of these R -algebras (if they exist) have defining relations that form an LM-reduced monic Gröbner basis in a free R -algebra. We end this section by giving another example illustrating Theorem 2.5 and Proposition 2.6 (more examples are given in the next section in connection with PBW R -bases).

Example 1. Let R be a commutative ring. Consider in $R\langle X \rangle = R\langle X_1, \dots, X_n \rangle$ the subset $\mathcal{G} = \Omega \cup \mathcal{R}$ consisting of

$$\begin{aligned} \Omega &\subseteq \{g_i = X_i^p \mid 1 \leq i \leq n\} \text{ with } p \geq 2 \text{ a fixed integer,} \\ \mathcal{R} &= \{g_{ji} = X_j X_i - \lambda_{ji} X_i X_j \mid 1 \leq i < j \leq n\} \text{ with } \lambda_{ji} \in R, \\ &\text{that is, } \lambda_{ji} \text{ may be zero.} \end{aligned}$$

In the case that $R = K$ is a field, it was verified in ([20], Example 4) that, under the \mathbb{N} -graded lexicographic ordering \prec_{gr} such that $X_1 \prec_{gr} X_2 \prec_{gr} \dots \prec_{gr} X_n$, \mathcal{G} forms an LM-reduced monic Gröbner basis for the ideal $I = \langle \mathcal{G} \rangle$ in $K\langle X \rangle$. Hence, by Proposition 2.6, \mathcal{G} is an LM-reduced monic Gröbner basis for the ideal $I = \langle \mathcal{G} \rangle$ in $R\langle X \rangle$. Furthermore, the division by $\mathbf{LM}(\mathcal{G})$ yields

$$N(\mathcal{G}) = \left\{ X_1^{\alpha_1} X_2^{\alpha_2} \cdots X_n^{\alpha_n} \mid \alpha_i \in \mathbb{N} \text{ and } 0 \leq \alpha_s \leq p - 1 \text{ if } X_s^p \in \Omega \right\}.$$

It follows from Theorem 2.4 that both the algebras $R\langle X \rangle/I$ and $R\langle X \rangle/\langle \mathbf{LM}(I) \rangle$ have the free R -basis

$$\overline{N(\mathcal{G})} = \left\{ \overline{X_1^{\alpha_1} X_2^{\alpha_2} \cdots X_n^{\alpha_n}} \mid \alpha_i \in \mathbb{N} \text{ and } 0 \leq \alpha_s \leq p - 1 \text{ if } X_s^p \in \Omega \right\},$$

where each $\overline{X_i}$ is the canonical image of X_i in $R\langle X \rangle/I$ and $R\langle X \rangle/\langle \mathbf{LM}(I) \rangle$ respectively.

Let us point out here that this example covers two families of special R -algebras, that is, in the case where $\Omega = \emptyset$, the R -algebra $R\langle X \rangle/I$ is similar to the coordinate ring of a quantum affine n -space over a field (such a quantum coordinate ring over a field is defined with all the $\lambda_{ji} \neq 0$); and in the case where $\Omega = \{g_i = X_i^2 \mid 1 \leq i \leq n\}$, the algebra $R\langle X \rangle/I$ is similar to the quantum grassmannian (or quantum exterior) algebra over a field in the sense of [24] (such a quantum grassmannian algebra over a field is defined with all the $\lambda_{ji} \neq 0$).

3 PBW R -bases vs Specific Monic Gröbner Bases

Let R be a commutative ring and $A = R[a_1, \dots, a_n]$ a finitely generated R -algebra with generators a_1, \dots, a_n . If the set $B = \{a_1^{\alpha_1} a_2^{\alpha_2} \cdots a_n^{\alpha_n} \mid \alpha_j \in \mathbb{N}\}$ forms a free R -basis of A , that is, A is, as an R -module, free with the basis B , then, in honor of the classical PBW (Poincaré-Birkhoff-Witt) theorem for enveloping algebras of Lie algebras over a ground field K , the set B is usually referred to as a *PBW R -basis* of A . Presenting A as a quotient algebra of the free R -algebra $R\langle X \rangle = R\langle X_1, \dots, X_n \rangle$, i.e., $A = R\langle X \rangle / I$ with I an ideal of $R\langle X \rangle$, the aim of this section is to show, under a mild condition, that A has a PBW R -basis is equivalent to that I has a specific monic Gröbner basis. This result enables us to obtain PBW R -bases by means of monic Gröbner bases on one hand; and on the other hand, since it is well known that in practice there are different ways to find a PBW basis of a given algebra provided it exists (e.g., [28], [31], [27], [3]), this result also enables us to obtain monic Gröbner bases via already known PBW R -bases.

Throughout this section, we let $R\langle X \rangle = R\langle X_1, \dots, X_n \rangle$ be the free algebra of n generators over a commutative ring R , and \mathcal{B}_R the standard R -basis of $R\langle X \rangle$. All notations and notions concerning monic Gröbner bases in $R\langle X \rangle$ are maintained as before.

Let I be an ideal of $R\langle X \rangle$ such that the R -algebra $A = R\langle X \rangle / I$ has the PBW R -basis $B = \{\bar{X}_1^{\alpha_1} \bar{X}_2^{\alpha_2} \cdots \bar{X}_n^{\alpha_n} \mid \alpha_i \in \mathbb{N}\}$, where each \bar{X}_i is the canonical image of X_i in A . Then I contains necessarily a subset G consisting of $\frac{n(n-1)}{2}$ elements of the form:

$$g_{ji} = X_j X_i - \sum_{\alpha} \lambda_{\alpha} w_{\alpha}, \text{ where } 1 \leq i < j \leq n, \lambda_{\alpha} \in R, w_{\alpha} = X_1^{\alpha_1} X_2^{\alpha_2} \cdots X_n^{\alpha_n}.$$

In light of Theorem 1.4 and the observation made above, below we give the main result of this section which, indeed, strengthens and generalizes ([13], Proposition 2.14) and ([18], CH.III, Theorem 1.5).

Theorem 3.1 *Let I be an ideal of $R\langle X \rangle$, $A = R\langle X \rangle / I$. Suppose that I contains a monic subset of $\frac{n(n-1)}{2}$ elements $G = \{g_{ji} \mid 1 \leq i < j \leq n\}$ such that, with respect to some monomial ordering \prec on the standard R -basis \mathcal{B}_R of $R\langle X \rangle$, $\mathbf{LM}(g_{ji}) = X_j X_i$ for $1 \leq i < j \leq n$. The following two statements are equivalent.*

- (i) *The R -algebra A has the PBW R -basis $B = \{\bar{X}_1^{\alpha_1} \bar{X}_2^{\alpha_2} \cdots \bar{X}_n^{\alpha_n} \mid \alpha_j \in \mathbb{N}\}$ where each \bar{X}_i is the canonical image of X_i in A .*
- (ii) *Any monic subset \mathcal{G} of I containing G is a monic Gröbner basis for I with respect to \prec .*

Proof (i) \Rightarrow (ii) Let \mathcal{G} be a monic subset of I containing G , and let

$$N(\mathcal{G}) = \{u \in \mathcal{B}_R \mid \mathbf{LM}(g) \not\prec u, g \in \mathcal{G}\}$$

be the set of normal monomials in $\mathcal{B}_R \pmod{\mathcal{G}}$. If $f \in I$ and $f \neq 0$, then, after implementing the division of f by \mathcal{G} (with respect to the given monomial ordering \prec) we have

$$\begin{aligned} f &= \sum_{i,j} \lambda_{ij} u_{ij} g_i v_{ij} + r_f, \text{ where } \lambda_{ij} \in R, u_{ij}, v_{ij} \in \mathcal{B}_R, g_i \in \mathcal{G}, \\ &\text{satisfying } \mathbf{LM}(w_{ij} g_i v_{ij}) \preceq \mathbf{LM}(f) \text{ whenever } \lambda_{ij} \neq 0, \\ &\text{and } r_f = \sum_p \lambda_p w_p \text{ with } \lambda_p \in R \text{ and } w_p \in N(\mathcal{G}). \end{aligned}$$

Note that $g_{ji} \in G \subseteq \mathcal{G}$ and $\mathbf{LM}(g_{ji}) = X_j X_i$ by the assumption. It follows that $N(\mathcal{G}) \subseteq \{X_1^{\alpha_1} X_2^{\alpha_2} \cdots X_n^{\alpha_n} \mid \alpha_j \in \mathbb{N}\}$. Thus, since B is a free R -basis of A , $r_f = \sum_p \lambda_p w_p = f - \sum_{i,j} \lambda_{ij} u_{ij} g_i v_{ij} \in I$ implies $\lambda_p = 0$ for all p . Consequently $r_f = 0$. This shows that every nonzero element of I has a Gröbner representation by the elements of \mathcal{G} . Hence \mathcal{G} is a monic Gröbner basis for I by Proposition 1.2. (ii) \Rightarrow (i) By (ii), the subset G itself is a monic Gröbner basis of I with respect to \prec . Let $N(G)$ be the set of normal monomials in $\mathcal{B}_R \pmod{G}$. Noticing that $\mathbf{LM}(g_{ji}) = X_j X_i$ for every $g_{ji} \in G$, it follows that $N(G) = \{X_1^{\alpha_1} X_2^{\alpha_2} \cdots X_n^{\alpha_n} \mid \alpha_j \in \mathbb{N}\}$, and thereby the algebra A has the desired PBW R -basis B by Theorem 1.4. \square

We illustrate Theorem 3.1 by several examples. The first four examples given below serve to obtain monic Gröbner bases by means of already known PBW R -bases which are obtained in the literature without using the theory of Gröbner basis.

Example 1. (This is a special case of Example 3 given later.) Let $\mathfrak{g} = R[V]$ be the R -Lie algebra defined by the free R -module $V = \bigoplus_{i=1}^n R x_i$ and the bracket product $[x_j, x_i] = \sum_{\ell=1}^n \lambda_{ji}^\ell x_\ell$, $1 \leq i < j \leq n$, $\lambda_{ji}^\ell \in R$. By the classical PBW theorem, the universal enveloping algebra $U(\mathfrak{g})$ of \mathfrak{g} has the PBW R -basis $B = \{x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} \mid \alpha_j \in \mathbb{N}\}$. If, with respect to the natural \mathbb{N} -gradation of $R\langle X \rangle = R\langle X_1, \dots, X_n \rangle$, we use an \mathbb{N} -graded monomial ordering \prec_{gr} on the standard R -basis \mathcal{B}_R of $R\langle X \rangle$ such that $X_1 \prec_{gr} X_2 \prec_{gr} \cdots \prec_{gr} X_n$ (i.e., $\deg X_i = 1$, $1 \leq i \leq n$), then the set of defining relations

$$\mathcal{G} = \left\{ g_{ji} = X_j X_i - X_i X_j - \sum_{\ell=1}^n \lambda_{ji}^\ell X_\ell \mid 1 \leq i < j \leq n \right\}$$

of $U(\mathfrak{g})$ satisfies $\mathbf{LM}(g_{ji}) = X_j X_i$ for $1 \leq i < j \leq n$. Hence, by Theorem 3.1, \mathcal{G} is a monic Gröbner basis for the ideal $I = \langle \mathcal{G} \rangle$ in $R\langle X \rangle$.

Example 2. Let $U_q^+(A_N)$ be the $(+)$ -part of the Drinfeld-Jimbo quantum group of type A_N over a commutative ring R , where $q \in R$ is invertible and

$q^8 \neq 1$. This example shows that the defining relations (Jimbo relations) of $U_q^+(A_N)$ over R form a monic Gröbner basis in a free R -algebra. By Proposition 2.6, we reach this property over a field K .

In [28] and [31], without using the Gröbner basis technique, it was proved that over a field K the algebra $U_q^+(A_N)$ has a PBW K -basis. Now, by using Theorem 3.1 we will easily see that the Jimbo relations form a Gröbner basis for the defining ideal of $U_q^+(A_N)$.

Recall that the Jimbo relations (as described in [31]) are given by

$$\begin{aligned} x_{mn}x_{ij} - q^{-2}x_{ij}x_{mn}, & \quad ((i, j), (m, n)) \in C_1 \cup C_3, \\ x_{mn}x_{ij} - x_{ij}x_{mn}, & \quad ((i, j), (m, n)) \in C_2 \cup C_6, \\ x_{mn}x_{ij} - x_{ij}x_{mn} + (q^2 - q^{-2})x_{in}x_{mj}, & \quad ((i, j), (m, n)) \in C_4, \\ x_{mn}x_{ij} - q^2x_{ij}x_{mn} + qx_{in}, & \quad ((i, j), (m, n)) \in C_1 \cup C_3, \end{aligned}$$

where with $\Lambda_N = \{(i, j) \in \mathbb{N} \times \mathbb{N} \mid 1 \leq i < j \leq N + 1\}$,

$$\begin{aligned} C_1 &= \{((i, j), (m, n)) \mid i = m < j < n\}, & C_2 &= \{((i, j), (m, n)) \mid i < m < n < j\}, \\ C_3 &= \{((i, j), (m, n)) \mid i < m < j = n\}, & C_4 &= \{((i, j), (m, n)) \mid i < m < j < n\}, \\ C_5 &= \{((i, j), (m, n)) \mid i < j = m < n\}, & C_6 &= \{((i, j), (m, n)) \mid i < j < m < n\}. \end{aligned}$$

By [31], for $q^8 \neq 1$, $U_q^+(A_N)$ has the PBW basis consisting of elements

$$x_{i_1j_1}x_{i_2j_2} \cdots x_{i_kj_k} \text{ with } (i_1, j_1) \leq (i_2, j_2) \leq \cdots \leq (i_k, j_k), \quad k \geq 0,$$

where $(i_\ell, j_\ell) \in \Lambda_N$ and $<$ is the lexicographic ordering on Λ_N . If we use the \mathbb{N} -graded monomial ordering \prec_{gr} (on the standard K -basis \mathcal{B} of the corresponding free algebra) subject to

$$x_{ij} \prec_{gr} x_{mn} \iff (i, j) < (m, n),$$

then it is clear that for each pair $((i, j), (m, n)) \in C_i$ with $(i, j) < (m, n)$, the leading monomial of the corresponding relation is of the form $x_{mn}x_{ij}$ as required by Theorem 3.1.

Example 3. With $R\langle X \rangle = R\langle X_1, \dots, X_n \rangle$, where R is an arbitrary commutative ring, recall from [3] that a q -algebra $A = R\langle X \rangle / \langle \mathcal{G} \rangle$ over R is defined by the set \mathcal{G} of quadric relations

$$\begin{aligned} g_{ji} &= X_jX_i - q_{ji}X_iX_j - \{X_j, X_i\}, \quad 1 \leq i < j \leq n, \text{ where } q_{ji} \in R - \{0\}, \\ &\text{and } \{X_j, X_i\} = \sum \alpha_{ji}^{k\ell} X_kX_\ell + \sum \alpha_h X_h + c_{ji}, \quad \alpha_{ji}^{k\ell}, \alpha_h, c_{ji} \in R, \\ &\text{satisfying if } \alpha_{ji}^{k\ell} \neq 0, \text{ then } i < k \leq \ell < j, \text{ and } k - i = j - \ell. \end{aligned}$$

Define two R -submodules of the free R -module $R\langle X \rangle$:

$$\begin{aligned} \mathcal{E}_1 &= R\text{-Span} \left\{ g_{ji} \mid 1 \leq i < j \leq n \right\}, \\ \mathcal{E}_2 &= R\text{-Span} \left\{ X_i g_{ji}, g_{ji} X_i, X_j g_{ji}, g_{ji} X_j \mid 1 \leq i < j \leq n \right\}. \end{aligned}$$

If, for $1 \leq i < j < k \leq n$, every Jacobi sum

$$\begin{aligned}
 J(X_k, X_j, X_i) = & \{X_k, X_j\}X_i - \lambda_{ki}\lambda_{ji}X_i\{X_k, X_j\} - \\
 & -\lambda_{ji}\{X_k, X_i\}X_j + \lambda_{kj}X_j\{X_k, X_i\} + \\
 & +\lambda_{kj}\lambda_{ki}\{X_j, X_i\}X_k - X_k\{X_j, X_i\}
 \end{aligned}$$

is contained in $\mathcal{E}_1 + \mathcal{E}_2$, then A is called a q -enveloping algebra. Clearly, enveloping algebras of R -Lie algebras are special q -enveloping algebras with $q = 1$. In [3], a q -PBW theorem for q -enveloping algebras over a commutative ring was obtained along the line similar to the classical argument on enveloping algebras of Lie algebras as given in [14], that is, if A is a q -enveloping R -algebra then A has the PBW R -basis $B = \{\overline{X}_1^{\alpha_1} \overline{X}_2^{\alpha_2} \cdots \overline{X}_n^{\alpha_n} \mid \alpha_j \in \mathbb{N}\}$.

Now, if we use the \mathbb{N} -graded monomial ordering $X_1 \prec_{gr} X_2 \prec_{gr} \cdots \prec_{gr} X_n$ on \mathcal{B}_R with respect to the natural \mathbb{N} -gradation of $R\langle X \rangle$ (i.e., $\deg X_i = 1, 1 \leq i \leq n$), then \mathcal{G} satisfies $\mathbf{LM}(g_{ji}) = X_j X_i$ for all $1 \leq i < j \leq n$. Hence, by Theorem 3.1, the set \mathcal{G} of the defining relations of a q -enveloping R -algebra is a monic Gröbner basis for the ideal $I = \langle \mathcal{G} \rangle$ in $R\langle X \rangle$. In particular, all quantum algebras over $R = \mathbb{C}[[h]]$ which are q -enveloping algebras appeared in [3] are defined by monic Gröbner bases.

Remark It is necessary to point out that if $R = K$ is a field, then the fact that the set of defining relations \mathcal{G} of a q -enveloping K -algebra A forms a Gröbner basis of the ideal $I = \langle \mathcal{G} \rangle$ was proved in ([18], CH.III) directly by using the termination theorem through the division algorithm. Here our last example provides the general result for all q -enveloping algebras over an arbitrary commutative ring.

Example 4. This example generalizes the previous three examples but uses an ad hoc monomial ordering. As an application we show that, over a commutative ring R , the PBW generators of the quantum algebra $U_q^+(A_N)$ derived in [27] provides another set of Gröbner defining relations for $U_q^+(A_N)$.

With $R\langle X \rangle = R\langle X_1, \dots, X_n \rangle$, consider the R -algebra $A = R\langle X \rangle / \langle \mathcal{G} \rangle$ defined by the subset \mathcal{G} consisting of $\frac{n(n-1)}{2}$ elements

$$\begin{aligned}
 g_{ji} = & X_j X_i - q_{ji} X_i X_j - \sum_{\alpha} \lambda_{\alpha} X_{i_1}^{\alpha_1} X_{i_2}^{\alpha_2} \cdots X_{i_s}^{\alpha_s} + \lambda_{ji}, \quad 1 \leq i < j \leq n, \\
 & \text{where } q_{ji}, \lambda_{\alpha}, \lambda_{ji} \in R, \alpha_k \in \mathbb{N}, i < i_1 \leq i_2 \leq \cdots \leq i_s < j.
 \end{aligned}$$

It is well-known that numerous iterated skew polynomial algebras over R are defined subject to such relations, and consequently they have the PBW R -basis $B = \{\overline{X}_1^{\alpha_1} \overline{X}_2^{\alpha_2} \cdots \overline{X}_n^{\alpha_n} \mid \alpha_j \in \mathbb{N}\}$. Under the assumption that A has the PBW R -basis as described we aim to show that \mathcal{G} is a monic Gröbner basis of $\langle \mathcal{G} \rangle$. In view of Theorem 3.1, it is sufficient to introduce a monomial ordering on \mathcal{B}_R so that $\mathbf{LM}(g_{ji}) = X_j X_i$ for all $1 \leq i < j \leq n$. To this end, let $R[t] = R[t_1, \dots, t_n]$ be the commutative polynomial R -algebra of n variables.

Consider the canonical algebra epimorphism $\pi: R\langle X \rangle \rightarrow R[t]$ with $\pi(X_i) = t_i$. If we fix the lexicographic ordering $X_1 <_{lex} X_2 <_{lex} \cdots <_{lex} X_n$ on \mathcal{B}_R of $R\langle X \rangle$ (note that $<_{lex}$ is not a monomial ordering on \mathcal{B}_R) and fix an arbitrarily chosen monomial ordering \prec on the standard R -basis $\mathbb{B}_R = \{t_1^{\alpha_1} t_2^{\alpha_2} \cdots t_n^{\alpha_n} \mid \alpha_j \in \mathbb{N}\}$ of $R[t]$, respectively, then, as in [9], a monomial ordering \prec_{et} on \mathcal{B}_R , which is called the *lexicographic extension* of the given monomial ordering \prec on \mathbb{B}_R , may be obtained as follows: for $u, v \in \mathcal{B}_R$,

$$u \prec_{et} v \text{ if } \begin{cases} \pi(u) \prec \pi(v), \\ \text{or} \\ \pi(u) = \pi(v) \text{ and } u <_{lex} v \text{ in } \mathcal{B}_R. \end{cases}$$

In particular, with respect to the monomial ordering \prec_{et} obtained by using the lexicographic ordering $t_n <_{lex} t_{n-1} <_{lex} \cdots <_{lex} t_1$ on \mathbb{B}_R , we see that $\mathbf{LM}(g_{ji}) = X_j X_i$ for all $1 \leq i < j \leq n$, as required by Theorem 3.1.

In [27] it was proved that $U_q^+(A_N)$ has $m = \frac{N(N+1)}{2}$ generators x_1, \dots, x_m satisfying the relations:

$$x_j x_i = q^{v_{ji}} x_i x_j - r_{ji}, \quad 1 \leq i < j \leq m,$$

where $v_{ji} = (wt(x_i), wt(x_j))$, and r_{ji} is a linear combination of monomials of the form $x_{i+1}^{\alpha_{i+1}} x_{i+2}^{\alpha_{i+2}} \cdots x_{j-1}^{\alpha_{j-1}}$, and that $U_q^+(A_N)$ is an iterated skew polynomial algebra generated by x_1, \dots, x_m subject to the above relations. Thus $U_q^+(A_N)$ has the PBW basis $\{x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_m^{\alpha_m} \mid \alpha_j \in \mathbb{N}\}$, and consequently $\mathcal{G} = \{g_{ji} = x_j x_i - q^{v_{ij}} x_i x_j + r_{ji} \mid 1 \leq i < j \leq m\}$ forms a monic Gröbner defining set of $U_q^+(A_N)$ with respect to the monomial ordering \prec_{et} as described before.

Remark If, in the defining relations given in the last example, the condition $i < i_1 \leq i_2 \leq \cdots \leq i_s < j$ is replaced by $1 \leq i_1 \leq i_2 \leq \cdots \leq i_s \leq i - 1$, then a similar result holds.

The next three examples provide monic Gröbner bases which are not necessarily the type as described in previous Examples 3 – 4, but they all give rise to PBW R -bases.

Example 5. Let R be a commutative ring, and let I be the ideal of the free R -algebra $R\langle X \rangle = R\langle X_1, X_2 \rangle$ generated by the single element

$$g_{21} = X_2 X_1 - q X_1 X_2 - \alpha X_2 - f(X_1),$$

where $q, \alpha \in R$, and $f(X_1)$ is a polynomial in the variable X_1 . Assigning to X_1 the degree 1, then in either of the following two cases:

- (a) $\deg f(X_1) \leq 2$, and X_2 is assigned the degree 1;
- (b) $\deg f(X_1) = n \geq 3$, and X_2 is assigned the degree n ,

$\mathcal{G} = \{g_{21}\}$ forms an LM-reduced monic Gröbner basis for I . For, in both cases we may use the \mathbb{N} -graded lexicographic ordering $X_1 \prec_{gr} X_2$ with respect to the natural \mathbb{N} -gradation of $K\langle X \rangle$, respectively the weight \mathbb{N} -gradation of $R\langle X \rangle$ with weight $\{1, n\}$, such that $\mathbf{LM}(g_{21}) = X_2X_1$, and then we see that the only overlap element of \mathcal{G} is $o(g_{21}, 1; 1, g_{21}) = 0$. Thus, by Theorem 3.1 in both cases the algebra $A = R\langle X \rangle/I$ has the PBW R -basis $B = \{\overline{X_1}^\alpha \overline{X_2}^\beta \mid \alpha, \beta \in \mathbb{N}\}$.

Example 6. Let R be a commutative ring, and let $R\langle X \rangle = R\langle X_1, X_2, X_3 \rangle$ be the free R -algebra generated by $X = \{X_1, X_2, X_3\}$. This example provides a family of algebras similar to the enveloping algebra $U(\mathfrak{sl}(2, R))$ of the R -Lie algebra $\mathfrak{sl}(2, R)$, that is, we consider the algebra $A = R\langle X \rangle/\langle \mathcal{G} \rangle$ with \mathcal{G} consisting of

$$\begin{aligned} g_{31} &= X_3X_1 - \lambda X_1X_3 + \gamma X_3, \\ g_{12} &= X_1X_2 - \lambda X_2X_1 + \gamma X_2, \\ g_{32} &= X_3X_2 - \omega X_2X_3 + f(X_1), \end{aligned}$$

where $\lambda, \gamma, \omega \in R$, and $f(X_1)$ is a polynomial in the variable X_1 . It is clear that $A = U(\mathfrak{sl}(2, R))$ in case $\lambda = \omega = 1$, $\gamma = 2$ and $f(X_1) = -X_1$.

Suppose $f(X_1)$ has degree $n \geq 1$. Then we can always equip $R\langle X \rangle$ with a weight \mathbb{N} -gradation by assigning to X_1, X_2 and X_3 the positive degree n_1, n_2, n_3 respectively (for instance, $(1, 1, 1)$ if $\deg f(X_1) = n \leq 2$; $(1, n, n)$ if $\deg f(X_1) = n > 2$), such that $\mathbf{LM}(\mathcal{G}) = \{X_3X_1, X_1X_2, X_3X_2\}$ with respect to the \mathbb{N} -graded monomial ordering $X_2 \prec_{gr} X_1 \prec_{gr} X_3$ on \mathcal{B}_R . In the case that $R = K$ is a field, it was verified in ([20], Example 7) that \mathcal{G} is a Gröbner basis for the ideal $\langle \mathcal{G} \rangle$ in $K\langle X \rangle$ with respect to the same \prec_{gr} . Hence, by Proposition 2.6, \mathcal{G} is a Gröbner basis for the ideal $\langle \mathcal{G} \rangle$ in $R\langle X \rangle$. It follows from Theorem 3.1 that the algebra $A = R\langle X \rangle/\langle \mathcal{G} \rangle$ has the PBW R -basis $B = \{\overline{X_2}^{\alpha_2} \overline{X_1}^{\alpha_1} \overline{X_3}^{\alpha_3} \mid \alpha_j \in \mathbb{N}\}$.

Let us point out that in the case that $f(X_1)$ has degree ≤ 2 , i.e., $f(X_1)$ is of the form

$$f(X_1) = aX_1^2 + bX_1 + c \text{ with } a, b, c \in R,$$

if $\deg X_1 = \deg X_2 = \deg X_3 = 1$ is used, the algebra A provides R -versions of some popularly studied algebras over a field K , for instance,

- (a) let $\zeta \in R$ be invertible, and put $\lambda = \zeta^4$, $\omega = \zeta^2$, $\gamma = -(1 + \zeta^2)$, $a = 0 = c$, and $b = -\zeta$, then A is just the R -version of the Woronowicz's deformation of $U(\mathfrak{sl}(2, K))$ introduced in the noncommutative differential calculus;
- (b) if $\lambda\gamma\omega b \neq 0$ and $c = 0$, then A is just the R -version of Le Bruyn's conformal $\mathfrak{sl}(2, K)$ enveloping algebra [15] which provides a special family of Witten's deformation of $U(\mathfrak{sl}(2, K))$ in quantum group theory.

Example 7. Let \mathcal{G} be the subset of the free R -algebra $R\langle X \rangle = K\langle X_1, X_2, X_3 \rangle$

consisting of

$$\begin{aligned} g_{21} &= X_2X_1 - X_1X_2, \\ g_{31} &= X_3X_1 - \lambda X_1X_3 - \mu X_2X_3 - \gamma X_2, \quad \lambda, \mu, \gamma \in R, \\ g_{32} &= X_3X_2 - X_2X_3. \end{aligned}$$

Then, under the \mathbb{N} -graded lexicographic ordering $X_1 \prec_{gr} X_2 \prec_{gr} X_3$ with respect to the natural \mathbb{N} -gradation of $R\langle X \rangle$, $\mathbf{LM}(g_{ji}) = X_jX_i$, $1 \leq i < j \leq 3$, and the only nontrivial overlap element of \mathcal{G} is $S_{321} = o(g_{32}, X_1; X_3, g_{21}) = -X_2X_3X_1 + X_3X_1X_2$. One checks easily that $\overline{S_{321}}^{\mathcal{G}} = 0$. By Theorem 3.1, \mathcal{G} is an LM-reduced monic Gröbner basis for the ideal $\langle \mathcal{G} \rangle$. Hence, by Theorem 3.1 the algebra $A = R\langle X \rangle / \langle \mathcal{G} \rangle$ has the PBW R -basis $B = \{X_1^{\alpha_1} X_2^{\alpha_2} X_3^{\alpha_3} \mid \alpha_j \in \mathbb{N}\}$.

4 PBW Isomorphisms and Applications

In this section we show that the working principle via different types of PBW isomorphism developed in [23] and [19] over a field K can be generalized to study algebras defined by monic Gröbner bases over a commutative ring R . All notions and notations used in previous sections are maintained.

Let R be an arbitrary commutative ring, $R\langle X \rangle = R\langle X_1, \dots, X_n \rangle$ the free R -algebra of n generators, and \mathcal{B}_R the standard free R -basis of $R\langle X \rangle$. Consider a weight \mathbb{N} -gradation of $R\langle X \rangle$ subject to $\deg(X_i) = n_i > 0$, $1 \leq i \leq n$, that is, $R\langle X \rangle = \bigoplus_{p \in \mathbb{N}} R\langle X \rangle_p$ with $R\langle X \rangle_p = R\text{-span}\{w \in \mathcal{B} \mid \deg(w) = p\}$. For an element $f \in R\langle X \rangle$, say $f = F_0 + F_1 + \dots + F_p$ with $F_i \in R\langle X \rangle_i$ and $F_p \neq 0$, let $\mathbf{LH}_{\mathbb{N}}(f)$ denote the \mathbb{N} -leading homogeneous element of f , i.e., $\mathbf{LH}_{\mathbb{N}}(f) = F_p$. Then every ideal I of $R\langle X \rangle$ is associated to an \mathbb{N} -graded ideal $\langle \mathbf{LH}_{\mathbb{N}}(I) \rangle$ generated by the set of \mathbb{N} -leading homogeneous elements $\mathbf{LH}_{\mathbb{N}}(I) = \{\mathbf{LH}_{\mathbb{N}}(f) \mid f \in I\}$. Adopting the notion and notation as in [19], we call the \mathbb{N} -graded algebra $A_{\mathbf{LH}}^{\mathbb{N}} = R\langle X \rangle / \langle \mathbf{LH}_{\mathbb{N}}(I) \rangle$ the \mathbb{N} -leading homogeneous algebra of the algebra $A = R\langle X \rangle / I$. On the other hand, noticing that $R\langle X \rangle$ is also a B_R -graded algebra by the multiplicative monoid B_R , i.e., $R\langle X \rangle = \bigoplus_{w \in B_R} R\langle X \rangle_w$ with $R\langle X \rangle_w = Rw$, if \prec is a monomial ordering on \mathcal{B}_R and if $f = \sum_{i=1}^n \lambda_i w_i \in R\langle X \rangle$ with $w_1 \prec w_2 \prec \dots \prec w_n$, then the term $\lambda_n w_n$ is called the B_R -leading homogeneous element of f and is denoted by $\mathbf{LH}_{B_R}(f)$. Thus each ideal I of $R\langle X \rangle$ is associated to a B_R -graded ideal $\langle \mathbf{LH}_{B_R}(I) \rangle$ generated by the set of B_R -leading homogeneous elements $\mathbf{LH}_{B_R}(I) = \{\mathbf{LH}_{B_R}(f) \mid f \in I\}$, and similarly, the B_R -graded algebra $A_{\mathbf{LH}}^{B_R} = R\langle X \rangle / \langle \mathbf{LH}_{B_R}(I) \rangle$ is referred to as the B_R -leading homogeneous algebra of the algebra $A = R\langle X \rangle / I$. Furthermore, consider the \mathbb{N} -grading filtration $F^{\mathbb{N}}R\langle X \rangle$ of $R\langle X \rangle$ defined by

$$F_p^{\mathbb{N}}R\langle X \rangle = \bigoplus_{i \leq p} R\langle X \rangle_i, \quad p, i \in \mathbb{N},$$

and the \mathcal{B}_R -grading filtration $F^{\mathcal{B}_R}R\langle X \rangle$ of $R\langle X \rangle$ defined by

$$F_w^{\mathcal{B}_R}R\langle X \rangle = \bigoplus_{u \preceq w} R\langle X \rangle_u, \quad w, u \in \mathcal{B}_R.$$

If I is an ideal of $R\langle X \rangle$, then the algebra $A = R\langle X \rangle/I$ has the \mathbb{N} -filtration $F^{\mathbb{N}}A$ induced by $F^{\mathbb{N}}R\langle X \rangle$, i.e.,

$$F_p^{\mathbb{N}}A = (F_p^{\mathbb{N}}R\langle X \rangle + I)/I, \quad p \in \mathbb{N},$$

respectively the \mathcal{B}_R -filtration $F^{\mathcal{B}_R}A$ induced by $F^{\mathcal{B}_R}R\langle X \rangle$, i.e.,

$$F_w^{\mathcal{B}_R}A = (F_w^{\mathcal{B}_R}R\langle X \rangle + I)/I, \quad w \in \mathcal{B}_R.$$

Note that if each X_i has degree 1, $1 \leq i \leq n$, then the filtration $F^{\mathbb{N}}A$ is just the commonly used *natural* \mathbb{N} -filtration. Let $G^{\mathbb{N}}(A) = \bigoplus_{p \in \mathbb{N}} G^{\mathbb{N}}(A)_p$ with $G^{\mathbb{N}}(A)_p = F_p^{\mathbb{N}}A/F_{p-1}^{\mathbb{N}}A$ be the associated \mathbb{N} -graded algebra of A determined by $F^{\mathbb{N}}A$, respectively $G^{\mathcal{B}_R}(A) = \bigoplus_{w \in \mathcal{B}_R} G^{\mathcal{B}_R}(A)_w$ with $G^{\mathcal{B}_R}(A)_w = F_w^{\mathcal{B}_R}A/F_{\prec w}^{\mathcal{B}_R}A$ the associated \mathcal{B}_R -graded algebra of A determined by $F^{\mathcal{B}_R}A$, where $F_{\prec w}^{\mathcal{B}_R}A = \bigcup_{u \prec w} F_u^{\mathcal{B}_R}A$. We have the following analogue of ([19], Theorem 1.1). Since the proof of this result is similar to that given in loc. cit., we omit it here.

Theorem 4.1 *With notation as above, there are graded R -algebra isomorphisms:*

$$A_{\mathbf{LH}}^{\mathbb{N}} = R\langle X \rangle / \langle \mathbf{LH}_{\mathbb{N}}(I) \rangle \cong G^{\mathbb{N}}(A), \quad A_{\mathbf{LH}}^{\mathcal{B}_R} = R\langle X \rangle / \langle \mathbf{LH}_{\mathcal{B}_R}(I) \rangle \cong G^{\mathcal{B}_R}(A).$$

□

Since we are using an arbitrary commutative ring R (instead of a field) as the coefficient ring, the next lemma provides the *key bridge* that enables us to generalize the working principle of [23] and [19] to study quotient algebras of $R\langle X \rangle$ defined by monic Gröbner bases.

Lemma 4.2 *Let $R\langle X \rangle$ be equipped with the fixed weight \mathbb{N} -gradation as before, and I an ideal of $R\langle X \rangle$. Put $J = \langle \mathbf{LH}_{\mathbb{N}}(I) \rangle$. The following two statements hold.*

- (i) *If h is a nonzero homogeneous element of $R\langle X \rangle$, then $h \in J$ if and only if $h \in \mathbf{LH}_{\mathbb{N}}(I)$. Hence $\mathbf{LH}_{\mathbb{N}}(J) = \mathbf{LH}_{\mathbb{N}}(I)$.*
- (ii) *Let \prec_{gr} be an \mathbb{N} -graded monomial ordering on \mathcal{B}_R with respect to the fixed weight \mathbb{N} -gradation of $R\langle X \rangle$. Then $\mathbf{LH}_{\mathcal{B}_R}(J) = \mathbf{LH}_{\mathcal{B}_R}(I)$ and $\mathbf{LM}(J) = \mathbf{LM}(I)$.*
- (iii) *Let \prec_{gr} be an \mathbb{N} -graded monomial ordering on \mathcal{B}_R with respect to the fixed weight \mathbb{N} -gradation of $R\langle X \rangle$. If \mathcal{G} is a monic Gröbner basis of I , then*

$$\langle \mathbf{LH}_{\mathcal{B}_R}(J) \rangle = \langle \mathbf{LH}_{\mathcal{B}_R}(I) \rangle = \langle \mathbf{LM}(\mathcal{G}) \rangle = \langle \mathbf{LM}(I) \rangle = \langle \mathbf{LM}(J) \rangle.$$

Proof (i) Let h be a nonzero homogeneous element in $R\langle X \rangle$. If $h \in J$, then

$$h = \sum_{i,j} H_{ij} \mathbf{LH}_{\mathbb{N}}(f_i) T_{ij}, \text{ where } H_{ij}, T_{ij} \text{ are homogeneous elements and } f_i \in I.$$

If we write $f_i = \mathbf{LH}_{\mathbb{N}}(f_i) + f'_i$, where $\deg(f'_i) < \deg(f_i)$, then $f = \sum_{i,j} H_{ij} f_i T_{ij} \in I$ and

$$f = \sum_{i,j} H_{ij} \mathbf{LH}_{\mathbb{N}}(f_i) T_{ij} + \sum_{i,j} H_{ij} f'_i T_{ij} = h + \sum_{i,j} H_{ij} f'_i T_{ij}.$$

It follows immediately that $h = \mathbf{LH}_{\mathbb{N}}(f) \in \mathbf{LH}_{\mathbb{N}}(I)$. This shows that $\mathbf{LH}_{\mathbb{N}}(J) \subseteq \mathbf{LH}_{\mathbb{N}}(I)$ and hence the equality holds.

(ii) Note that \prec_{gr} is an \mathbb{N} -graded monomial ordering on \mathcal{B}_R , every element of \mathcal{B}_R is an \mathbb{N} -homogeneous element, and thus for $f \in R\langle X \rangle$ we have

$$(*) \quad \mathbf{LH}_{\mathcal{B}_R}(f) = \mathbf{LH}_{\mathcal{B}_R}(\mathbf{LH}_{\mathbb{N}}(f)) \text{ and } \mathbf{LM}(f) = \mathbf{LM}(\mathbf{LH}_{\mathbb{N}}(f))$$

It follows from (i) and the above formula (*) that

$$\begin{aligned} \mathbf{LH}_{\mathcal{B}_R}(J) &= \mathbf{LH}_{\mathcal{B}_R}(\mathbf{LH}_{\mathbb{N}}(J)) = \mathbf{LH}_{\mathcal{B}_R}(\mathbf{LH}_{\mathbb{N}}(I)) = \mathbf{LH}_{\mathcal{B}_R}(I), \\ \mathbf{LM}(J) &= \mathbf{LM}(\mathbf{LH}_{\mathcal{B}_R}(J)) = \mathbf{LM}(\mathbf{LH}_{\mathcal{B}_R}(I)) = \mathbf{LM}(I). \end{aligned}$$

(iii) Let $f \in R\langle X \rangle$ be a monic element with respect to the fixed monomial ordering \prec_{gr} , say $f = w + \sum \lambda_i w_i$ with $w, w_i \in \mathcal{B}_R$, $\lambda_i \in R$ and $\mathbf{LM}(f) = w$. Then it is clear that

$$(**) \quad \mathbf{LM}(f) = w = \mathbf{LH}_{\mathcal{B}_R}(f).$$

So, if \mathcal{G} is a monic Gröbner basis of I with respect to \prec_{gr} , then the above formula $(**)$ implies $\mathbf{LM}(\mathcal{G}) = \mathbf{LH}_{\mathcal{B}_R}(\mathcal{G}) \subset \mathbf{LH}_{\mathcal{B}_R}(I)$. Hence, by (ii) and Proposition 2.2 we obtain the desired equalities:

$$\langle \mathbf{LH}_{\mathcal{B}_R}(J) \rangle = \langle \mathbf{LH}_{\mathcal{B}_R}(I) \rangle = \langle \mathbf{LM}(\mathcal{G}) \rangle = \langle \mathbf{LM}(I) \rangle = \langle \mathbf{LM}(J) \rangle.$$

□

Next, we show that an analogue of ([23], Theorem 2.3.2 (i) \Leftrightarrow (iii)) holds true for monic Gröbner bases in $R\langle X \rangle$.

Theorem 4.3 *Let I be an ideal of $R\langle X \rangle$. With notation as above, if \prec_{gr} is an \mathbb{N} -graded monomial ordering on \mathcal{B}_R with respect to a fixed weight \mathbb{N} -gradation of $R\langle X \rangle$, the following two statements are equivalent for a subset $\mathcal{G} \subset I$:*

- (i) \mathcal{G} is a monic Gröbner basis of I ;
- (ii) $\mathbf{LH}_{\mathbb{N}}(\mathcal{G}) = \{\mathbf{LH}_{\mathbb{N}}(g) \mid g \in \mathcal{G}\}$ is a monic Gröbner basis for the \mathbb{N} -graded ideal $\langle \mathbf{LH}_{\mathbb{N}}(I) \rangle$.

Proof Since we are using the \mathbb{N} -graded monomial ordering \prec_{gr} on \mathcal{B}_R , by Lemma 4.2 or its proof, a subset \mathcal{G} of $R\langle X \rangle$ is monic if and only if $\mathbf{LH}_{\mathbb{N}}(\mathcal{G})$ is monic, and we have

$$\langle \mathbf{LM}(I) \rangle = \langle \mathbf{LM}(\mathcal{G}) \rangle \text{ if and only if } \langle \mathbf{LM}(\langle \mathbf{LH}_{\mathbb{N}}(I) \rangle) \rangle = \langle \mathbf{LM}(\mathbf{LH}_{\mathbb{N}}(\mathcal{G})) \rangle.$$

It follows from Proposition 2.2 that \mathcal{G} is a monic Gröbner basis for the ideal I if and only if $\mathbf{LH}_{\mathbb{N}}(\mathcal{G})$ is a monic Gröbner basis for the \mathbb{N} -graded ideal $\langle \mathbf{LH}_{\mathbb{N}}(I) \rangle$, proving the equivalence of (i) and (ii). □

Remark We also point out that an analogue of ([23], Theorem 2.3.2 (i) \Leftrightarrow (ii)) works well for monic Gröbner bases when the homogenization of I in $R\langle X \rangle[t]$ is considered.

Combining the previous 4.1 – 4.3, we get immediately the result presenting the associated \mathbb{N} -graded algebra, respectively the associated \mathcal{B}_R -graded algebra via a monic Gröbner basis.

Theorem 4.4 *Let $R\langle X \rangle$ be equipped with a fixed weight \mathbb{N} -gradation as before, and I an ideal of $R\langle X \rangle$. If \mathcal{G} is a monic Gröbner basis of I with respect to*

an \mathbb{N} -graded monomial ordering \prec_{gr} on \mathcal{B}_R , then we have the graded algebra isomorphisms

$$\begin{aligned} A_{\mathbf{LH}}^{\mathbb{N}} &= R\langle X \rangle / \langle \mathbf{LH}_{\mathbb{N}}(I) \rangle = R\langle X \rangle / \langle \mathbf{LH}_{\mathbb{N}}(\mathcal{G}) \rangle \cong G^{\mathbb{N}}(A), \\ A_{\mathbf{LH}}^{\mathcal{B}_R} &= R\langle X \rangle / \langle \mathbf{LH}_{\mathcal{B}_R}(I) \rangle = R\langle X \rangle / \langle \mathbf{LM}(\mathcal{G}) \rangle \cong G^{\mathcal{B}_R}(A), \\ (A_{\mathbf{LH}}^{\mathbb{N}})_{\mathbf{LH}}^{\mathcal{B}_R} &= R\langle X \rangle / \langle \mathbf{LH}_{\mathcal{B}_R}(\mathbf{LH}_{\mathbb{N}}(I)) \rangle = R\langle X \rangle / \langle \mathbf{LM}(\mathcal{G}) \rangle \cong G^{\mathcal{B}_R}(A_{\mathbf{LH}}^{\mathbb{N}}). \end{aligned}$$

□

As in [19], we call the graded algebra isomorphisms presented in the last theorem respectively the \mathbb{N} -PBW isomorphism (for the first one) and the \mathcal{B}_R -PBW isomorphism (for the last two), determined by the given monic Gröbner basis \mathcal{G} .

Focusing on the first isomorphism of Theorem 4.4, typical examples can be given by using the Gröbner defining relations of Weyl algebras and enveloping algebras of Lie algebras, or more generally, the Gröbner defining relations of q -enveloping algebras determined in Example 3 of the last section, over a commutative ring. We specify several other examples below. In all examples given below, R is an arbitrary commutative ring.

Example 1. Let $X = \{X_i\}_{i \in J}$ and $\mathbf{C} = R\langle X \rangle / \langle \mathcal{G} \rangle$ the Clifford algebra over R , where \mathcal{G} consists of

$$\begin{aligned} g_i &= X_i^2 - q_i, & i \in J, q_i \in R, \\ g_{k\ell} &= X_k X_\ell + X_\ell X_k - q_{k\ell}, & k, \ell \in J, k > \ell, q_{k\ell} \in R. \end{aligned}$$

Note that if all the $q_i = 0$, $q_{k,\ell} = 0$, we get the defining relations of an R -exterior algebra. It is well known that if $R = K$ is a field, then, under the \mathbb{N} -graded lexicographic ordering \prec_{gr} such that $\deg X_i = 1$, $i \in J$, and

$$X_\ell \prec_{gr} X_k, \quad \ell, k \in J, \ell < k,$$

\mathcal{G} forms a Gröbner basis for the ideal $\langle \mathcal{G} \rangle$ in $K\langle X \rangle$ (e.g., see CH.II of [18]). It follows from Proposition 2.6 that \mathcal{G} is a Gröbner basis for the ideal $\langle \mathcal{G} \rangle$ in $R\langle X \rangle$. By Theorem 4.4, with respect to the natural \mathbb{N} -filtration $F^{\mathbb{N}}\mathbf{C}$ of \mathbf{C} , the associated \mathbb{N} -graded algebra $G^{\mathbb{N}}(\mathbf{C}) \cong R\langle X \rangle / \langle \mathbf{LH}_{\mathbb{N}}(\mathcal{G}) \rangle$ of \mathbf{C} is nothing but an exterior algebra \mathbf{E} over R .

Example 2. Let $A = R\langle X_1, X_2 \rangle / \langle \mathcal{G} \rangle$ be a down-up R -algebra in the sense of [2], where \mathcal{G} consists of

$$\begin{aligned} g_1 &= X_1^2 X_2 - \alpha X_1 X_2 X_1 - \beta X_2 X_1^2 - \gamma X_1, \\ g_2 &= X_1 X_2^2 - \alpha X_2 X_1 X_2 - \beta X_2^2 X_1 - \gamma X_2, \end{aligned} \quad \alpha, \beta \in R.$$

It is well known that if $R = K$ is a field, then, under the \mathbb{N} -graded lexicographic ordering \prec_{gr} such that $\deg X_1 = \deg X_2 = 1$ and $X_2 \prec_{gr} X_1$, \mathcal{G} forms a Gröbner basis for the ideal $\langle \mathcal{G} \rangle$ in $K\langle X \rangle$ (e.g., see CH.II of [18]). It follows from Proposition 2.6 that \mathcal{G} is a Gröbner basis for the ideal $\langle \mathcal{G} \rangle$ in $R\langle X \rangle$. By Theorem 4.4, with respect to the natural \mathbb{N} -filtration $F^{\mathbb{N}}A$ of A , the associated \mathbb{N} -graded algebra $G^{\mathbb{N}}(A) \cong R\langle X_1, X_2 \rangle / \langle \mathbf{LH}_{\mathbb{N}}(\mathcal{G}) \rangle$ of A is a down-up algebra over R with the set of defining relations $\mathbf{LH}_{\mathbb{N}}(\mathcal{G}) = \{\mathbf{LH}_{\mathbb{N}}(g_1) = X_1^2 X_2 - \alpha X_1 X_2 X_1 - \beta X_2 X_1^2, \mathbf{LH}_{\mathbb{N}}(g_2) = X_1 X_2^2 - \alpha X_2 X_1 X_2 - \beta X_2^2 X_1\}$; in particular, one sees that if $\alpha = 2$ and $\beta = -1$, then $G^{\mathbb{N}}(A)$ is nothing but the universal enveloping algebra of the $(-)$ -part (or $(+)$ -part) of the Kac-Moody R -Lie algebra associated to the Cartan matrix $\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$.

Example 3. Let $A = R\langle X_1, X_2 \rangle / \langle g_{21} \rangle$ be the R -algebra as given in (Section 2, Example 5). Then by Theorem 4.4, with respect to both the natural \mathbb{N} -filtration and the weight \mathbb{N} -filtration induced by the weight \mathbb{N} -grading filtration of $R\langle X_1, X_2 \rangle$, $G^{\mathbb{N}}(A) \cong R\langle X_1, X_2 \rangle / \langle X_2 X_1 - q X_1 X_2 \rangle$ as \mathbb{N} -graded algebras, which, in the case that q is invertible, is the coordinate ring of the quantum plane over R .

Example 4. Let $A = R\langle X_1, X_2, X_3 \rangle / \langle \mathcal{G} \rangle$ be the R -algebra as given in (Section 2, Example 6). In the case that $f(X_1)$ has degree ≤ 2 , i.e., $f(X_1)$ is of the form

$$f(X_1) = aX_1^2 + bX_1 + c \text{ with } a, b, c \in R,$$

then by Theorem 4.4, with respect to the natural \mathbb{N} -filtration $F^{\mathbb{N}}A$, A has the associated \mathbb{N} -graded algebra $G^{\mathbb{N}}(A) \cong R\langle X_1, X_2, X_3 \rangle / \langle \mathbf{LH}_{\mathbb{N}}(\mathcal{G}) \rangle$ with

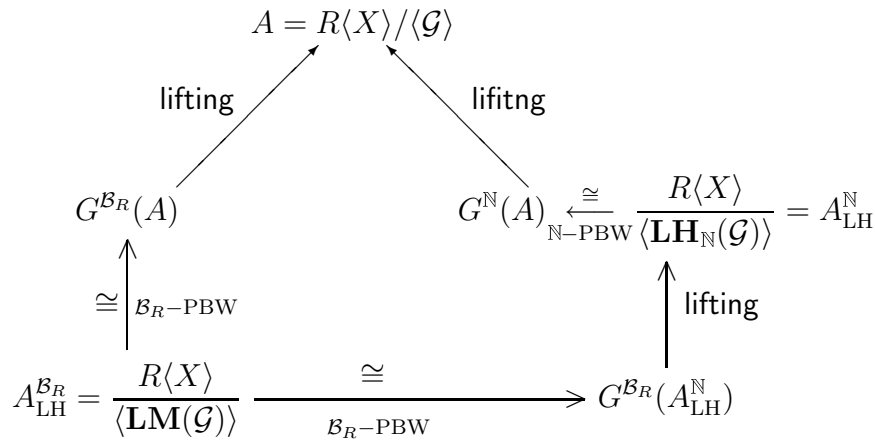
$$\mathbf{LH}_{\mathbb{N}}(\mathcal{G}) = \{X_3 X_1 - \lambda X_1 X_3, X_1 X_2 - \lambda X_2 X_1, X_3 X_2 - \omega X_2 X_3 + aX_1^2\};$$

while in the case that $f(X_1)$ has degree $n \geq 3$, if the weight $(1, n, n)$ is used, then by Theorem 4.4, with respect to the weight \mathbb{N} -filtration $F^{\mathbb{N}}A$ induced by the weight \mathbb{N} -grading filtration of $R\langle X_1, X_2 \rangle$, A has the associated \mathbb{N} -graded algebra $G^{\mathbb{N}}(A) \cong R\langle X_1, X_2, X_3 \rangle / \langle \mathbf{LH}_{\mathbb{N}}(\mathcal{G}) \rangle$ with

$$\mathbf{LH}_{\mathbb{N}}(\mathcal{G}) = \{X_3 X_1 - \lambda X_1 X_3, X_1 X_2 - \lambda X_2 X_1, X_3 X_2 - \omega X_2 X_3\}$$

By referring to the well-known filtered-graded comparison principle for algebras with an \mathbb{N} -filtration ([25], [17], [21], [19]), we now summarize, without proof, several applications of Theorem 4.3 and Theorem 4.4. Let R be an arbitrary commutative ring. For convenience, in what follows we let the free R -algebra $R\langle X \rangle = R\langle X_1, \dots, X_n \rangle$ be equipped with a fixed weight \mathbb{N} -gradation, $I = \langle \mathcal{G} \rangle$ an ideal of $R\langle X \rangle$ generated by a monic Gröbner basis \mathcal{G} with respect to an \mathbb{N} -graded monomial ordering \prec_{gr} on the standard R -basis \mathcal{B}_R of $R\langle X \rangle$,

and $A = R\langle X \rangle / I$. Then the following diagram may indicate how all results to be given will work:



Theorem 4.5 Under the respective canonical algebra epimorphism, the set $N(\mathcal{G})$ of normal monomials in $\mathcal{B}_R \pmod{\mathcal{G}}$, projects to a free R -basis for the algebras $A = R\langle X \rangle / I$, $A_{\text{LH}}^{\mathbb{N}} = R\langle X \rangle / \langle \mathbf{LH}_{\mathbb{N}}(I) \rangle$, and $A_{\text{LH}}^{\mathcal{B}_R} = R\langle X \rangle / \langle \mathbf{LM}(I) \rangle$ respectively, and thereby to a free R -basis for $G^{\mathbb{N}}(A)$, $G^{\mathcal{B}_R}(A)$, and $G^{\mathcal{B}_R}(A_{\text{LH}}^{\mathbb{N}})$, respectively.

□

Theorem 4.6 Bearing $A_{\text{LH}}^{\mathcal{B}_R} = R\langle X \rangle / \langle \mathbf{LM}(\mathcal{G}) \rangle$ in mind, the following statements hold.

- (i) If $A_{\text{LH}}^{\mathcal{B}_R}$ is a (semi-)prime ring, then $A_{\text{LH}}^{\mathbb{N}}$ is a (semi-)prime ring (hence $G^{\mathbb{N}}(A)$ is a (semi-)prime ring), and A is a (semi-)prime ring.
- (ii) If $A_{\text{LH}}^{\mathcal{B}_R}$ is \mathcal{B}_R -graded left Noetherian, that is, every \mathcal{B}_R -graded left ideal of $G(A)$ is finitely generated, then $A_{\text{LH}}^{\mathbb{N}}$ is left Noetherian (hence $G^{\mathbb{N}}(A)$ is left Noetherian), and A is left Noetherian.
- (iii) If $A_{\text{LH}}^{\mathcal{B}_R}$ is \mathcal{B}_R -graded left Artinian, that is, $A_{\text{LH}}^{\mathcal{B}_R}$ satisfies the descending chain condition for \mathcal{B}_R -graded left ideals, then $A_{\text{LH}}^{\mathbb{N}}$ is left Artinian (hence $G^{\mathbb{N}}(A)$ is left Artinian), and A is left Artinian.
- (iv) If $A_{\text{LH}}^{\mathcal{B}_R}$ is a \mathcal{B}_R -graded simple R -algebra, that is, $A_{\text{LH}}^{\mathcal{B}_R}$ does not have nontrivial \mathcal{B}_R -graded ideal, then $A_{\text{LH}}^{\mathbb{N}}$ is a simple R -algebra (hence $G^{\mathbb{N}}(A)$ is a simple R -algebra), and A is a simple R -algebra.
- (v) If the Krull dimension ($K.\dim$ in the sense of Gabriel and Rentschler, e.g. see [25] for the definition) of $A_{\text{LH}}^{\mathcal{B}_R}$ is well defined, then the Krull dimension of $A_{\text{LH}}^{\mathbb{N}}$ (hence of $G^{\mathbb{N}}(A)$) and A is well defined and $K.\dim A \leq K.\dim G^{\mathbb{N}}(A) = K.\dim A_{\text{LH}}^{\mathbb{N}} \leq K.\dim A_{\text{LH}}^{\mathcal{B}_R}$.
- (vi) If $A_{\text{LH}}^{\mathcal{B}_R}$ is semisimple (simple) Artinian, then $A_{\text{LH}}^{\mathbb{N}}$ is semisimple (simple) Artinian (hence $G^{\mathbb{N}}(A)$ is semisimple (simple) Artinian), and A is semisimple (simple) Artinian.

(vii) Let *gl.dim* abbreviate the phrase “global homological dimension”. We have $gl.dim A \leq gl.dim G^{\mathbb{N}}(A) = gl.dim A_{\text{LH}}^{\mathbb{N}} \leq gl.dim A_{\text{LH}}^{\mathbb{B}_R}$.

(viii) If $A_{\text{LH}}^{\mathbb{B}_R}$ is left hereditary, then $A_{\text{LH}}^{\mathbb{N}}$ is left hereditary (hence $G^{\mathbb{N}}(A)$ is left hereditary), and A is left hereditary.

(ix) Let *gl.wdim* abbreviate the phrase “global weak homological dimension”. We have $gl.wdim A \leq gl.wdim G^{\mathbb{N}}(A) = gl.wdim A_{\text{LH}}^{\mathbb{N}} \leq gl.wdim A_{\text{LH}}^{\mathbb{B}_R}$.

(x) If $A_{\text{LH}}^{\mathbb{B}_R}$ is a Von Neuman regular ring, then $A_{\text{LH}}^{\mathbb{N}}$ is Von Neuman regular ring (hence $G^{\mathbb{N}}(A)$ is a Von Neuman regular ring), and A is a Von Neuman regular ring.

□

Theorem 4.7 Bearing $A_{\text{LH}}^{\mathbb{N}} = R\langle X \rangle / \langle \mathbf{LH}_{\mathbb{N}}(\mathcal{G}) \rangle$ in mind, if the role of $A_{\text{LH}}^{\mathbb{B}_R}$ is replaced by $A_{\text{LH}}^{\mathbb{N}}$, then the analogues of Theorem 4.6 (i) – (x) hold true. Moreover, we have:

(i) If $A_{\text{LH}}^{\mathbb{N}}$ is a domain, then A is a domain.

(ii) If $A_{\text{LH}}^{\mathbb{N}}$ is a Noetherian domain and maximal order in its quotient ring (see e.g. [25] for the definition), then A is a Noetherian domain and maximal order in its quotient ring.

(iii) If $A_{\text{LH}}^{\mathbb{N}}$ is an Auslander regular ring (see e.g. [17], [21] for the definition), then A is an Auslander regular ring.

□

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