KYBERNETIKA — VOLUME 47 (2011), NUMBER 5, PAGES 790–811

CONSTRUCTION OF A CONTROLLER WITH A GENERALIZED LINEAR IMMERSION

Javier Diaz-Vargas, Dennis Tuyub-Puc* and Celia Villanueva-Novelo

Gröbner bases for modules are used to calculate a generalized linear immersion for a plant whose solutions to its regulation equations are polynomials or pseudo-polynomials. After calculating the generalized linear immersion, we build the controller which gives the robust regulation.

Keywords: output regulation, robust, nonlinear, generalized linear immersion, Gröbner basis, module, Noetherian, monomial ordering, normal form, syzygy

Classification: 93C10, 93D15, 13E05, 13P10

1. INTRODUCTION

The problem of output robust regulation by error feedback has already been solved by establishing a theorem with the necessary and sufficient conditions; it includes two main points [13]:

1. The existence of a solution to the so-called regulation equations, which provides the regulation law for all parameters in some neighborhood, and;
2. The existence of an immersion of the exosystem to a dynamic system whose output is the law of regulation, with favorable properties of detectability and stability.

However, the theorem and its proof, speaks only about the existence of a solution without describing how to build it. Thus, the need to understand methods that help in the construction arises. Our purpose here is to give a systematic method for constructing the generalized linear immersion, which, once obtained, allows to build the plant controller, which gives the robust regulation.

The first results on this subject were found in the 70's when Francis and Whonam [5] developed and presented to the scientific community the solution to the problem in the linear case, and established the separation principle that is used to construct the controller once that we have the linear immersion.

Isidori and Byrnes [8] generalized these results, in 1990, to non-linear systems. This is when they define the concept of immersion of a dynamic system into another,

*Supported by the Mexican Council for Science and Technology (CONACyT).

which is one of the key points in solving these problems and the matter of this article. Later Villanueva et al [13] extended the ideas of Isodori [8] and defined the concept of generalized linear immersion. Moreover, they showed how to construct a controller using vector spaces.

The next step is, then, to devise a construction for more general situations than those presented in [8]. In [4], we have an advance as the authors give the conditions under which the problem can become a problem of stabilizing an augmented system; however, none of them describes how to find the solution. This paper presents an algorithm to construct a generalized linear immersion considering an algebraic structure which generalizes the notion of vector space; this structure is called a *module*. Modules are a generalization of vector spaces because the scalars are elements of a ring instead of a field (which is a special case of ring). In our case, we use elements of the ring of polynomials in several variables as scalars. For the construction of the controller, the elements of the modules and exosignals are considered. An interesting type of this combination are pseudo-polynomials (polynomials with trigonometric coefficients) that allow to model signals used in practice.

To construct the generalized linear immersion, we take successive Lie derivatives of the solution of the regulation equations with respect to the exosystem. We show that this process ends (finiteness condition). Once it is known that this process ends, we then need to know when it ends. To do this, we use Gröbner bases and Buchberger's criterion, which will allow to decide when this happens (solving the so called membership problem). Then, we express the last derivative obtained as a linear combination with polynomial coefficients of the previous derivatives. Here, we need the notion of syzygies. All these concepts and algorithms are computational tools of commutative algebra which, fortunately, have been implemented in computer programs, such as *Singular*©, the program that we use and is freely available and easy to use ([6]).

To this end, Section 2 provides the relevant theory of regulation of the output. In the next section, we give the algebraic theory necessary for our goals; finally, in Section 4, the two theories are integrated in the construction of the generalized linear immersion and its corresponding controller.

2. OUTPUT REGULATION

2.1. Linear systems

To solve the problem of output regulation via error feedback requires a construction based on the *principle of separation.* This principle can be summarized as follows: if we want to stabilize a (stabilizable and detectable) linear system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}$$

via output feedback, it is possible to proceed as follows:

1. We need to construct a law $u = Kx$ of stabilization by feedback (where K is such that $A + BK$ has all eigenvalues with negative real part).

2. To construct an asymptotic observer

$$\dot{\xi} = (A - GC)\,\xi + Gy + Bu$$

(where G is such that $A - GC$ has all eigenvalues with negative real part).

3. "Consider ξ as if it were x" in the law of stabilization found in 1 and in the observer of 2, that is, set $u = K\xi$ and define the dynamic output feedback controller as follows:

$$\begin{aligned}\dot{\xi} &= (A - GC + BK)\,\xi + Gy\\ u &= K\xi.\end{aligned}$$

We follow a strategy similar to the previous procedure in the design of a law of control for the problem of output regulation via error feedback. For more on linear systems, see [9].

2.2. Nonlinear systems

Throughout this paper we use the standard notation of a differential geometric approach to nonlinear systems, see e. g. [7].

Consider a nonlinear plant described by

$$\begin{aligned}\dot{x} &= f(x, w, u, \mu)\\ e &= h(x, w, \mu).\end{aligned} \tag{1}$$

The first equation of (1) describes the dynamics of a *plant*, whose *state* x is defined in a neighborhood U of the origin in $\mathbb{R}^n$, with *control input* $u \in \mathbb{R}^m$ and subject to a set of *exogenous* input variables $w \in \mathbb{R}^r$, which includes *disturbances* (to be rejected) and/or *references* (to be tracked), and $\mu \in \mathbb{R}^p$ is a vector of unknown parameters. The second equation defines an *error* variable $e \in \mathbb{R}^m$, which is expressed as a function of the state x, the exogenous input w, and the vector of unknown parameters μ.

It is assumed that the family of the exogenous inputs $w(\cdot)$, that affect the plant, and for which the asymptotic decay of the error is to be achieved, is the family of all functions of time which are solution of a (possibly nonlinear) homogeneous differential equation

$$\dot{w} = s(w) \tag{2}$$

with initial condition $w(0)$ ranging on some neighborhood W of the origin of $\mathbb{R}^r$. This system, seen as a mathematical model of a *"generator"* of all possible exogenous input functions, is called *exosystem*. Throughout this paper, (2) is assumed to be neutrally stable, that is, its linearization has all eigenvalues with real part equal zero, which is a standard assumption for exogenous systems. For a detailed definition of neutral stability, see [8] and [2].

Briefly, the robust output regulation problem is aimed at designing a control law so that for all sufficiently small $w \in \mathbb{R}^r$ and all sufficiently small $\mu \in \mathbb{R}^p$, the solution of the closed-loop is bounded and the tracking error approaches 0 asymptotically.

2.3. Robust regulation of the output via error feedback and measurement of the states of the exosystem

The problem of robust regulation of the output via error feedback and measurement of the states of the exosystem, can be precisely formulated as follows: Given a nonlinear system of the form (1) and a neutrally stable exosystem as in (2), find, if possible, an integer ν, a neighborhood $\Xi \subset \mathbb{R}^\nu$, and two mappings

$$\begin{aligned} \theta(\xi, w), \theta : \Xi \times \mathbb{R}^r \to \mathbb{R}^m \\ \eta(\xi, e, w), \eta : \Xi \times \mathbb{R}^m \times \mathbb{R}^r \to \mathbb{R}^\nu \end{aligned}$$

satisfying the two following conditions of stability and regulation.

(*Stability*) The equilibrium $(x, \xi) = (0, 0)$ of

$$\begin{aligned} \dot{x} &= f(x, 0, \theta(\xi, w), \mu) \\ \dot{\xi} &= \eta(\xi, h(x, 0), w) \end{aligned}$$

is asymptotically stable in the first approximation,

(*Regulation*) There exists a neighborhood $V \subset U \times \Xi \times W$ of $(0, 0, 0)$ such that, for each initial condition $(x(0), \xi(0), w(0)) \in V$, the solution $(x(t), \xi(t), w(t))$ of

$$\begin{aligned} \dot{x} &= f(x, w, \theta(\xi, w), \mu) \\ \dot{\xi} &= \eta(\xi, h(x, w), w) \\ \dot{w} &= s(w) \end{aligned}$$

has the property

$$\lim_{t \to \infty} h(x(t), w(t), \mu) = 0.$$

Now, we introduce the concept of *immersion* of a system into another system, as in [7].

Let us consider a pair of smooth autonomous systems with outputs

$$\dot{x} = f(x), \quad y = h(x)$$

and

$$\dot{\widetilde{x}} = \widetilde{f}(\widetilde{x}), \quad y = \widetilde{h}(\widetilde{x})$$

defined on two different state spaces, X and $\widetilde{X}$, but having the same output space $Y = \mathbb{R}^m$. Assume, as usual, $f(0) = 0, h(0) = 0$, and $\widetilde{f}(0) = 0, \widetilde{h}(0) = 0$ and let the two systems in question be denoted, for convenience, by $\{X, f, h\}$ and $\{\widetilde{X}, \widetilde{f}, \widetilde{h}\}$, respectively.

Definition 2.1. The system $\{X, f, h\}$ is said to be immersed into the system $\{\widetilde{X}, \widetilde{f}, \widetilde{h}\}$ if there exists a C^k mapping $\tau : X \to \widetilde{X}$, with $k \geq 1$, satisfying $\tau(0) = 0$ and

$$h(x) \neq h(z) \Rightarrow \widetilde{h}(\tau(x)) \neq \widetilde{h}(\tau(z)),$$

for all $x, z \in X$, and it is such that

$$\frac{\partial \tau}{\partial x} f(x) = \widetilde{f}(\tau(x))$$
$$h(x) = \widetilde{h}(\tau(x))$$

for all $x \in X$.

We may notice that the two conditions indicated in this definition express that any output response generated by $\{X, f, h\}$ is also an output response of $\{\widetilde{X}, \widetilde{f}, \widetilde{h}\}$.

In particular, consider a real valued function h and a vector field f, both defined on a subset U of $\mathbb{R}^n$. The *Lie derivative of* h *along* f is a new function denoted $L_f h$ and defined as

$$L_f h = dh(x) \cdot f(x) = \sum_{i=1}^{n} \frac{\partial h}{\partial x_i} f_i(x)$$

at each x of U. In the case that $h = (h_1, h_2, \dots, h_m)^T$ is a vector valued function, we put

$$L_f h := (L_f h_1, L_f h_2, \dots, L_f h_m)^T.$$

If h is differentiated k times along f, the notation $L_f^k h$ is used

$$L_f^k h := \frac{\partial (L_f^{k-1} h)}{\partial x} f(x), k > 1,$$

with $L_f^0 h = h(x)$.

We extend the exosystem in order to include the unknown parameters μ. The purpose is to obtain a control law to solve the problem of output regulation for each set of values of these parameters. The "augmented" exosystem is

$$\begin{bmatrix} \dot{w} \\ \dot{\mu} \end{bmatrix} = \begin{bmatrix} s(w) \\ 0 \end{bmatrix}$$

with an output $c^a(w, \mu)$ (described later) that leads to a smooth immersion of the form:

$$\begin{bmatrix} \dot{w} \\ \dot{\xi} \end{bmatrix} = \begin{bmatrix} s(w) \\ \Phi(w)\xi \end{bmatrix}$$

which is called *Generalized Linear Immersion.*

Remark 2.2. It may be seen that if we take a neutrally stable linear exosystem $s(w) = Sw$, then the eigenvalues of S have a zero real part.

For the generalized immersion of a neutrally stable linear exosystem, it is necessary to find an integer q such that if we take $\xi_1 = c^a(w, \mu)$ we obtain:

$$\begin{aligned}
&\dot{\xi}_1 = \xi_2 = L_s c^a(w, \mu) \\
&\dot{\xi}_2 = \xi_3 = L_s^2 c^a(w, \mu) \\
&\vdots \\
&\dot{\xi}_q = L_s^q c^a(w, \mu) = a_0(w) c^a(w, \mu) + a_1(w) L_s c^a(w, \mu) + \cdots + a_{q-1}(w) L_s^{q-1} c^a(w, \mu) \\
&u = \begin{pmatrix} 1, & 0, & \dots, & 0 \end{pmatrix} \xi, \text{where } \xi = (\xi_1, \dots, \xi_q)^T.
\end{aligned}$$

Or, in matrix form, we have that $\dot{\xi} = \Phi(w)\xi$, where

$$\Phi(w) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_0(w) & a_1(w) & a_2(w) & \cdots & a_{q-1}(w) \end{pmatrix}.$$

The last condition requires the output u of the Generalized Linear System, to be the same as in the "augmented" exosystem.

Such immersion allows the removal of the vector of parameters and, thus, it can be shown that it solves the problem of robust regulation of the output via error feedback and measurement of the states of the exosystem, namely, using a dynamic feedback. This is shown in the following theorem.

Theorem 2.3. The problem of robust regulation of the output via error feedback and measurement of the states of the exosystem can be solved if and only if there are mappings $x = \pi^a(w, \mu)$ and $u = c^a(w, \mu)$, with $\pi^a(0, 0) = 0$ and $c^a(0, 0) = 0$, both defined in a neighborhood $W^0 \times \mathcal{P} \subset W \times \mathbb{R}^p$ of the origin, satisfying the conditions

$$\frac{\partial \pi^a}{\partial w} s(w) = f(\pi^a(w), w, c^a(w), \mu) \tag{3}$$

$$0 = h(\pi^a(w, \mu), w, \mu) \tag{4}$$

for all $(w, \mu) \in W^0 \times \mathcal{P}$, and such that the autonomous system with output $\{W^0 \times \mathcal{P}, s^a, c^a\}$ is immersed into a system of the form:

$$\begin{bmatrix} \dot{w} \\ \dot{\xi} \end{bmatrix} = \begin{bmatrix} Sw \\ \Phi(w)\xi \end{bmatrix}$$

defined in a neighborhood Ξ of the origin in $\mathbb{R}^\nu$, such that the pair (here denoted: $\Phi(0) = \Phi$.)

$$\begin{bmatrix} A & 0 \\ NC & \Phi \end{bmatrix}, \quad \begin{bmatrix} B \\ 0 \end{bmatrix}$$

can be stabilized for some choice of N, and the pair

$$\begin{bmatrix} C & 0 \end{bmatrix}, \quad \begin{bmatrix} A & B\begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \\ 0 & \Phi \end{bmatrix}$$

is detectable.

The proof of this theorem can be found in [13]. See also [3].

Now, to find $\Phi(w)$, we need to work with modules instead of vector spaces because the coefficients $a_i(w)$ are polynomials in w and they form a ring, not a field. We also need to determine if the process of taking successive derivatives stops (finiteness condition), when it stops (membership problem) and how to find the coefficients $a_i(w)$ (syzygies). We will discuss all these algebraic problems in the next section.

3. COMPUTATIONAL COMMUTATIVE ALGEBRA

We introduce now the basic definitions and algorithms that we need for our work. This section is based on the books [1] and [6]. Throughout this paper, the word *ring* means commutative ring with identity element.

Definition 3.1. Let R be a ring. Let M be a set with two operations $+ : M \times M \to M$ (*addition*) and $\cdot : R \times M \to M$ (*scalar multiplication*). We say that M is an *R-module* if for $a, b \in R$ and $x, y \in M$ the following properties are satisfied:

- $(M, +)$ is an abelian group.
- $a \cdot (x + y) = a \cdot x + a \cdot y$.
- $(a + b) \cdot x = a \cdot x + b \cdot x$
- $(ab) \cdot x = a \cdot (b \cdot x)$
- $1 \cdot x = x$ where 1 is the identity of the ring.

From now on, ax is written instead of $a \cdot x$.

Definition 3.2. Let R be a ring and M a module on R. Let M_0 be a subset of M. It is said that M_0 is a *submodule* of M if it is an abelian subgroup and it is true that for each $r \in R$ and every $m \in M_0$, $rm \in M_0$; that is, it is a subgroup that is closed under the multiplication by elements of R.

Definition 3.3. If X is a subset of a module M over a ring R, then the intersection of all the submodules of M containing X is called a *submodule generated* by X.

The submodule generated by X is also a submodule of M.

Definition 3.4. If X is finite and X generates a module M, we say that M is *finitely generated*. If x_1, ..., x_n generate M, we write $M = \langle x_1, \ldots, x_n \rangle$.

Just as for the subspace generated by a finite set of vectors, if $M = \langle x_1, \ldots, x_n \rangle$, then $M = \{\sum_{i=1}^{n} r_i x_i : r_i \in R\}$ is the set of linear combinations of the generators.

Definition 3.5. If $\Lambda = \{M_i | i \in I\}$ is a family of submodules of M, then the submodule generated by $X = \bigcup_{i \in I} M_i$ is called the *sum* of the modules M_i. If the set of indexes is a finite number n, the sum is denoted $M_1 + M_2 + \cdots + M_n$. Then, if $m \in M_1 + M_2 + \cdots + M_n$, we can write $m = \sum_{i=1}^{n} m_i$, where $m_i \in M_i$ for all $i = 1, 2, \ldots, n$.

In particular, if $M = \langle x_1, \ldots, x_n \rangle$, then

$$M = M_1 + M_2 + \cdots + M_n,$$

where $M_i = \langle x_i \rangle$ is the *cyclic* submodule generated by the element x_i.

If M, N are R-modules, their *direct sum* $M \oplus N$ is the set of all pairs (x, y) with $x \in M, y \in N$. This is an R-module if we define addition and scalar multiplication in the obvious way:

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$$
$$a(x, y) = (ax, ay).$$

More generally, if $(M_i)_{i \in I}$ is any family of R-modules, we can define their *direct sum* $\oplus_{i \in I} M_i$; its elements are families $(x_i)_{i \in I}$ such that $x \in M_i$ for each $i \in I$ and almost all x_i are 0.

Let M, N be R-modules. A mapping $f : M \to N$ is an *R-module homomorphism* (or is *A-linear*) if

$$f(x + y) = f(x) + f(y)$$
$$f(ax) = af(x)$$

for all $a \in R$ and for all $x, y \in M$. When R is a field, an R-module homomorphism is a linear transformation of vector spaces. A bijective homomorphism of modules is called an *isomorphism.* M is called *isomorphic* to N, denoted $M \cong N$, if there exists an isomorphism $M \to N$.

Unlike the vector spaces, not all modules have a basis. Those who do have it, get a special name, they are called free modules. A *free* R-module is one which is isomorphic to an R-module of the form $\oplus_{i \in I} M_i$ where each $M_i \cong R$ (as an R-module). A finitely generated free R-module is therefore isomorphic to $R \oplus \cdots \oplus R$ (n summands), which is denoted by R^n.

In this article, we only use the ring $R = K[x_1, \ldots, x_n]$ of polynomials in several variables, and free modules over it.

3.1. Finiteness condition

In this subsection we establish the Hilbert's basis theorem for rings and modules which will allow us to show that the process of taking successive Lie derivatives must stop.

Definition 3.6. A set Σ is called *partially ordered* by a relation $\leq$ if given $x, y, z \in \Sigma$ it is true that:

1. $x \leq x$,
2. if $x \leq y$ and $y \leq x$, then $x = y$ and
3. if $x \leq y$ and $y \leq z$, then $x \leq z$, called reflexive, antisymmetric and transitive properties, respectively.

Theorem 3.7. Given a set Σ partially ordered, the following conditions are equivalent:

1. Each increasing sequence $x_1 \leq x_2 \leq \ldots$ in Σ is stationary, that is, there is a number n such that $x_n = x_{n+j}$, $j \in \mathbb{N}$.

2. Each non empty subset of Σ has a maximal element, that is, there is an element a in the subset such that for any other element s in it, you have $s \leq a$.

The proof of this theorem is very easy, see [1, p. 74].

Remark 3.8. If Σ is the set of submodules of a module M, ordered by the relation $\subseteq$, then the subparagraph 1) is called "*ascending chain condition*" and the part 2) is called "*the maximal condition.*"

Definition 3.9. A module M satisfying any of these conditions is called *Noetherian* (in honor of Emmy Noether).

Proposition 3.10. *M is a Noetherian R-module if and only if any submodule of M is finitely generated.*

For a proof of this proposition see [1, p. 75].

A ring R is itself an R-module. A submodule of R, considered as an R-module, is what is called an *ideal.*

Definition 3.11. A ring R is Noetherian if R satisfies the ascending chain condition for ideals, or if every ideal is finitely generated.

The Noetherian rings are among the most important classes of rings.

Let $R[x_1, x_2, \ldots, x_n]$ be the polynomial ring in $x_1, x_2, \ldots, x_n$ with coefficients in R. The Hilbert's basis theorem for rings is as follows (see [1, p. 81]):

Theorem 3.12. If R is a Noetherian ring, then so is $R[x_1, x_2, \ldots, x_n]$.

If $R = \mathbb{R}$, the field of real numbers or $R = \mathbb{C}$, the field of complex numbers, we will have polynomials with coefficients in a field; in particular, a field is a Noetherian ring, since it does not have proper ideals. The only ideals of a field are $\langle 0 \rangle$ and $R = \langle 1 \rangle$, which are finitely generated (by one element). Then $R[x_1, x_2, \ldots, x_n]$ will be a Noetherian ring when $R = \mathbb{R}$ or $\mathbb{C}$, due to Hilbert's basis theorem. In general, we have proved the following very well-known result.

Corollary 3.13. The ring $K[x_1, x_2, \ldots, x_n]$ of polynomials in n variables with coefficients in K, K a field, is a Noetherian ring.

Regarding modules, we have the corresponding theorem (see [1, p. 76]).

Theorem 3.14. *Hilbert's basis theorem for modules:* If R is a Noetherian ring, then each finitely generated R-module M is Noetherian.

3.2. Membership problem

First, we define an order over the monomials in $K[x_1, \ldots, x_n]$. Note that the map $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mapsto \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ establishes a bijective correspondence between the monomials in $K[x_1, \ldots, x_n]$ and $\mathbb{N}^n$. Therefore, any order $>$ that we define on the space $\mathbb{N}^n$ will lead to an order on the monomial in $K[x_1, \ldots, x_n]$: if $\alpha > \beta$, according to this order, we will say that $x^\alpha > x^\beta$. There are several orders on $\mathbb{N}^n$, but for our purposes, several of them are not useful because the orders need to be compatible with the algebraic structure of the polynomial ring $K[x_1, \ldots, x_n]$. With this consideration, we have the following definition.

Definition 3.15. A *global monomial ordering* on $K[x_1, \ldots, x_n]$ is any relation $>$ on $\mathbb{N}^n$, or equivalently, on the set of monomials x^α, $\alpha \in \mathbb{N}^n$, satisfying:

1. $>$ is a total (or linear) ordering on $\mathbb{N}^n$.
2. If $\alpha > \beta$ and $\gamma \in \mathbb{N}^n$, then $\alpha + \gamma > \beta + \gamma$.
3. $>$ is a well-ordering on $\mathbb{N}^n$.

The condition (3), about well-ordering of the relation, means that every non empty subset of $\mathbb{N}^n$ has a smallest element with respect to $>$.

The usual order on $\mathbb{N}$, $\cdots > m + 1 > m > \cdots > 1 > 0$, satisfies the three conditions. Therefore, the order given with respect to degree for monomials in $K[x]$ is a global monomial ordering. An ordering on $\mathbb{N}^n$ (with $n > 1$) is given below.

Definition 3.16. *Lexicographic ordering lp.*
Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}^n$. We will say that $\alpha >_{\text{lp}} \beta$ if, in the difference vector $\alpha - \beta \in \mathbb{Z}^n$, the left-most nonzero entry is positive. We will write $x^\alpha >_{\text{lp}} x^\beta$ if $\alpha >_{\text{lp}} \beta$.

For example, in $\mathbb{N}^3$, $\alpha = (1, 2, 0) >_{\text{lp}} \beta = (0, 3, 4)$ since $\alpha - \beta = (1, -1, -4)$.

Proposition 3.17. The lexicographic ordering $>_{\text{lp}}$ is a global monomial ordering on $\mathbb{N}^n$.

There are other global monomial orderings over $\mathbb{N}^n$ such as the *graded lexicographic order* and *graded reverse lexicographic order* (for details the reader may consult eg [6]).

For the intended applications, we have to extend the notion of global monomial orderings to the free module $K[x_1, \ldots, x_n]^r$. We call

$$x^\alpha e_i = (0, \ldots, x^\alpha, \ldots, 0) \in K[x_1, \ldots, x_n]^r$$

a *monomial (involving component i)*, where

$$e_i = (0, \ldots, 1, \ldots, 0) \in K[x_1, \ldots, x_n]^r$$

denotes the ith canonical basis vector of $K[x_1, \ldots, x_n]^r$ with 1 at the ith place.

Definition 3.18. Let $>$ be a global monomial ordering on $K[x_1, \ldots, x_n]$. A *(module) global monomial ordering* or a *global module ordering* on $K[x_1, \ldots, x_n]^r$ is a total ordering $>_m$ on the set of monomials $\{x^\alpha e_i \mid \alpha \in \mathbb{N}^n, i = 1, \ldots, r\}$, which is compatible with the $K[x_1, \ldots, x_n]$-module structure including the ordering $>$, that is, satisfying:

1. $x^\alpha e_i >_m x^\beta e_j \Rightarrow x^{\alpha+\gamma} e_i >_m x^{\beta+\gamma} e_j$,
2. $x^\alpha > x^\beta \Rightarrow x^\alpha e_i >_m x^\beta e_i$,

for all $\alpha, \beta, \gamma \in \mathbb{N}^n, i, j = 1, \ldots, r$.

A global module ordering of particular practical interest is:

$$x^\alpha e_i > x^\beta e_j \text{ if and only if } i < j \text{ or } (i = j \text{ and } x^\alpha >_{\text{lp}} x^\beta),$$

giving priority to the components, denoted by $(c, >_{\text{lp}})$.

Now we fix a global module ordering $>$. Since any vector $f \in K[x_1, \ldots, x_n]^r \setminus \{0\}$ can be written uniquely as

$$f = cx^\alpha e_i + f^*$$

with $c \in K \setminus \{0\}$ and $x^\alpha e_i > x^{\alpha^*} e_j$ for any nonzero term $c^* x^{\alpha^*} e_j$ of f^*, we can define

$$\begin{aligned} \text{LM}(f) :=& x^\alpha e_i, \\ \text{LC}(f) :=& c \end{aligned}$$

and call it the *leading monomial* and *leading coefficient*, respectively, of f. We say that $x^\beta e_j$ is *divisible* by $x^\alpha e_i$ if $i = j$ and $x^\alpha | x^\beta$. For $G \subset K[x_1, \ldots, x_n]^r$ we call

$$L(G) := \langle \text{LM}(g) \mid g \in G \setminus \{0\} \rangle$$

the *leading submodule* of $\langle G \rangle$. In particular, if $I \subset K[x_1, \ldots, x_n]^r$ is a submodule, then $L(I)$ is called the *leading module* of I. A finite set $G \subset I$ is called a *Gröbner basis* of I if and only if $L(G) = L(I)$.

The following two definitions are important for our treatment of *Gröbner basis.*

Definition 3.19. Let $\mathcal{G}$ denote the set of all finite ordered subsets $G \subset K[x_1, \ldots, x_n]^r$. A map

$$\text{NF} : K[x_1, \ldots, x_n]^r \times \mathcal{G} \to K[x_1, \ldots, x_n]^r, (f, G) \mapsto \text{NF}(f \mid G),$$

is called a *weak normal form* on $K[x_1, \ldots, x_n]^r$ if for all $f \in K[x_1, \ldots, x_n]^r$ and $G \in \mathcal{G}$, $\text{NF}(0 \mid G) = 0$, and

a) $\text{NF}(f \mid G) \neq 0$, then $\text{LM}(\text{NF}(\text{f} \mid \text{G})) \notin L(G)$,

b) For each $f \in K[x_1, \dots, x_n]^r$ and each $G \in \mathcal{G}$ there exists a nonzero $u \in K$ such that $r = uf - \mathrm{NF}(f \mid G)$ has a *standard representation* with respect to G, that is, either $r = 0$, or

$$r = \sum_{i=i}^{s} a_i g_i, a_i \in K[x_1, \dots, x_n],$$

satisfying $\mathrm{LM}(f) \geq \mathrm{LM}(a_i g_i)$, for all i with $a_i g_i \neq 0$.

Definition 3.20. Let $f, g \in K[x_1, \dots, x_n]^r \setminus \{0\}$ with $\mathrm{LM}(f) := x^\alpha e_i, \mathrm{LM}(g) := x^\beta e_j$. Let

$$\gamma := lcm(\alpha, \beta) := (\max(\alpha_1, \beta_1), \dots, \max(\alpha_n, \beta_n))$$

be the least common multiple of α and β and define the S-polynomial of f and g to be

$$\mathrm{spoly}(f, g) := \begin{cases} x^{\gamma-\alpha} f - \frac{\mathrm{LC}(f)}{\mathrm{LC}(g)} x^{\gamma-\beta} g, & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

For a monomial $x^\alpha e_i \in K[x_1, \dots, x_n]^r$ set

$$\deg x^\alpha e_i := \deg x^\alpha = \alpha_1 + \cdots + \alpha_n.$$

For $f \in K[x_1, \dots, x_n]^r \setminus \{0\}$, let $\deg f$ be the maximal degree of all monomials occurring in f. We define the *ecart* of f as

$$\mathrm{ecart}(f) := \deg f - \deg \mathrm{LM}(f).$$

We can always find a weak normal form NF, the basic idea is due to Mora [10]. Let $>$ be any global monomial ordering on $K[x_1, \dots, x_n]^r$.

Algorithm 3.21. NFMora$(f \mid G)$

`Input:` $f \in K[x_1, \dots, x_n]^r, G = \{g_1, \dots, g_s\} \subset K[x_1, \dots, x_n]^r$.

`Output:` $h \in K[x_1, \dots, x_n]^r$ a weak normal form of f with respect to G, such that there exists a standard representation $uf - h = \sum_{i=i}^{s} a_i g_i, a_i \in K[x_1, \dots, x_n], u \in K^*$.

- $h = f$;
- $T = G$;
- while ($h \neq 0$ and $T_h = \{g \in T \mid \mathrm{LM}(g) \text{ divides } \mathrm{LM}(h)\} \neq \emptyset$)

    choose $g \in T_h$ with ecart(g) minimal;

    if ($\mathrm{ecart}(g) > \mathrm{ecart}(h)$)

        $T = T \cup \{h\}$;

    $h = \mathrm{spoly}(h, g)$;

- return h.

Let $>$ be any global monomial ordering on $K[x_1, \ldots, x_n]^r$. To find a Gröbner basis, we can use the following algorithm:

Algorithm 3.22. GröbnerBasis(G)

`Input:` $G \in \mathcal{G}$
`Output:` $S \in \mathcal{G}$ such that S is a Gröbner basis of $I = \langle G \rangle$, the submodule in $K[x_1, \ldots, x_n]^r$, generated by the elements of G.

- $S = G$;
- $P = \{(f, g) \mid f, g \in S, f \neq g\}$;
- while $(P \neq \emptyset)$
  choose $(f, g) \in P$;
  $P = P \setminus \{(f, g)\}$;
  $h = \mathbf{NFMora}(\text{spoly}(f, g) \mid S)$,
  If $h \neq 0$
    $P = P \cup \{(h, f) \mid f \in S\}$;
    $S = S \cup \{h\}$;
- return S.

The correctness of the *GröbnerBasis* algorithm follows from applying Buchberger's fundamental basis criterion below (see [6, p. 122]).

Theorem 3.23. *Buchberger's criterion.* Let $I \subset K[x_1, \ldots, x_n]^r$ be a submodule and $G = \{g_1, \ldots, g_s\} \subset I$. Let $\text{NF}(\_ \mid G)$ be a weak normal form on $K[x_1, \ldots, x_n]^r$ with respect to G. Then G is a Gröbner basis of I if and only if $\text{NF}(f \mid G) = 0$ for all $f \in I$. This is equivalent to the following statement: G generates I and $\text{NF}(\text{spoly}(g_i, g_j) \mid G) = 0$ for $i, j = 1, \ldots, s$.

The module membership problem can be formulated as follows:

Problem 3.24. *Membership problem.* Given polynomial vectors $f, f_1, \ldots, f_k \in K[x_1, \ldots, x_n]^r$, decide whether $f \in I := \langle f_1, \ldots, f_k \rangle \subset K[x_1, \ldots, x_n]^r$ or not.

We can solve this problem using Gröbner bases.

Solution. Compute a Gröbner basis G for the submodule I with respect to any global monomial ordering. Then

$$\boxed{f \in I \text{ if and only if } \mathbf{NFMora}(f \mid G) = 0.}$$

This can be proved using Buchberger's criterion.

3.3. Syzygies

The last problem that we need to solve in order to find the *Generalized Linear Immersion* is:

Problem 3.25. If $f \in I = \langle f_1, \dots, f_r \rangle \subset K[x_1, \dots, x_n]^r$, then express f as a linear combination $uf = \sum_{i=1}^{k} g_i f_i$ with $g_i \in K[x_1, \dots, x_n]$, $u \in K^*$.

Before solving this problem, we need to define the notion of *syzygy*.

Definition 3.26. A *syzygy* or *relation* between k elements $f_1, \dots, f_k$ of an R-module M is a k-tuple $(g_1, \dots, g_k) \in K[x_1, \dots, x_n]^k$ satisfying:

$$\sum_{i=1}^{k} g_i f_i = 0.$$

The set of all syzygies between $f_1, \dots, f_k$, $\operatorname{syz}(f_1, \dots, f_k)$, is a submodule of $K[x_1, \dots, x_n]^k$.

Now, we solve the problem 2 above.

Solution. Compute a Gröbner basis G of $\operatorname{syz}(f_1, \dots, f_k, f) \subset K[x_1, \dots, x_n]^{k+1}$ with respect to the ordering (c,lp). Now choose any vector $h = (-g_1, \dots, -g_k, u) \in G$ whose last component u satisfies $u \in K^*$. Then $uf = \sum_{i=1}^{k} g_i f_i$.

To find the module of syzygies, we need to find a set of its generators. This can be achieved with the following algorithm.

Algorithm 3.27. $\operatorname{SYZ}(f_1, \dots, f_k)$

`Input:` $f_1, \dots, f_k \in K[x_1, \dots, x_n]^r$.
`Output:` $S = \{s_1, \dots, s_l\} \subset K[x_1, \dots, x_n]^k$ such that $\langle s_1, \dots, s_l \rangle = \operatorname{syz}(f_1, \dots, f_k) \subset K[x_1, \dots, x_n]^k$.

- $F := \{f_1 + e_{r+1}, \dots, f_k + e_{r+k}\}$, where $e_1, \dots, e_{r+k}$ denote the canonical generators of $K[x_1, \dots, x_n]^{r+k}$;
- Compute a Gröbner basis G of $\langle f_1 + e_{r+1}, \dots, f_k + e_{r+k} \rangle \subset K[x_1, \dots, x_n]^{r+k}$ with respect to (c,lp);
- $G_0 := G \cap \oplus_{i=r+1}^{r+k} Re_i = \{g_1, \dots, g_l\}$, with $g_i = \sum_{j=1}^{k} a_{ij} e_{r+j}, i = 1, \dots, l$;
- $s_i := (a_{i1}, \dots, a_{ik}), i = 1, \dots, l$;
- return $S = \{s_1, \dots, s_l\}$.

4. CONSTRUCTION OF A CONTROLLER WITH A GENERALIZED LINEAR IMMERSION

We will show now that it is possible to find a generalized linear immersion using the algebraic theory established above.

Consider the set $M = \{f : \mathbb{R}^n \to \mathbb{R}\}$ of continuous scalar fields. This set with the usual sum for functions is an abelian group. Here, we consider this group as a module over the Noetherian ring $\mathbb{R}[w_1, w_2, \ldots, w_n]$, considering a polynomial as a function and taking scalar multiplication as the usual multiplication of functions. Given a function $f_1(w_1, w_2, \ldots, w_n)$, we generate a module M_1 over the ring $\mathbb{R}[w_1, w_2, \ldots, w_n]$; this module is finitely generated and, by Theorem 3.14, is also Noetherian. Similarly, we can generate a module M_k with a finite subset of functions $X = \{f_1, f_2, \ldots, f_k | f_i : \mathbb{R}^n \to \mathbb{R}$, for all $i = 1, \ldots, k\}$; this module is also Noetherian.

We can observe the following: taking ξ_1 in M_k, let

$$
\begin{aligned}
&N_1 = \langle \xi_1 \rangle \text{, where } \xi_1 = u(w) \text{ the control law.}\\
&N_2 =< \xi_2 > + N_1 =< \xi_1, \xi_2 > \text{, where } \xi_2 = L_s \xi_1 \in M_k.\\
&N_3 =< \xi_3 > + N_2 =< \xi_1, \xi_2, \xi_3 > \text{, where } \xi_3 = L_s \xi_2 \in M_k.\\
&\quad \vdots\\
&N_q =< \xi_q > + N_{q-1} =< \xi_1, \xi_2, \xi_3, \ldots, \xi_q > \text{, where } \xi_q = L_s \xi_{q-1} \in M_k.
\end{aligned}
$$

Note that

$$N_1 \subseteq N_2 \subseteq \cdots \subseteq N_q \subseteq \cdots.$$

By definition of Noetherian module and the ascending chain condition, we have that there exists $q \in \mathbb{N}$ such that:

$$N_1 \subseteq N_2 \subseteq \cdots \subseteq N_q = N_{q+j}, j \in \mathbb{N}.$$

This means that $\xi_{q+1} \in N_q =< \xi_1, \xi_2, \xi_3, \ldots, \xi_q >$ and so, there are polynomials $\alpha_1, \alpha_2, \ldots, \alpha_q \in \mathbb{R}[w_1, w_2, \ldots, w_n]$ such that

$$\xi_{q+1} = \alpha_1 \xi_1 + \alpha_2 \xi_2 + \cdots + \alpha_q \xi_q.$$

Remark 4.1. It is important to emphasize the fact that the functions g_i must be in the finitely generated module M_k; which means that it is important to make a proper choice of the elements f_i that generate this module.

4.1. Example

Consider the mathematical model known as the inverted pendulum,

$$
\begin{aligned}
\dot{x}_1 &= x_2\\
\dot{x}_2 &= g \sin(x_1) - cu.\\
e &= x_1 - w_1,
\end{aligned}
$$

where x_1 is the angle between the pendulum and the vertical, x_2 is the speed, g is the gravity, c is the parameter that contains the friction and u is the control law we are trying to design, and an exosystem given by a linear oscillator:

$$\dot{w}_1 = \alpha w_2$$
$$\dot{w}_2 = -\alpha w_1$$

where $\alpha \in \mathbb{R}$.

The variables w_1 and w_2 are the paths to follow or reject and are supposed to be given at the beginning of the control problem, and are called exogenous variables.

The objective of the regulator is to make zero the error given by $y = e = x_1 - w_1$. The parameters c and g are approximately known, so you want the solution to be robust with respect to them.

Solving the equations of regulation (3) and (4) gives:

$$\pi_1 = w_1$$
$$\pi_2 = \alpha w_2$$
$$u = \frac{\alpha^2 w_1 + g\sin(w_1)}{c} = a w_1 + b\,\sin(w_1)$$

where $a = \frac{\alpha^2}{c}$, $b = \frac{g}{c}$ is the reparametrization of the parameters made to facilitate the following calculations. The nominal values of these parameters are $\alpha = 1$, $c = 1$, $g = -9.8$

To find the generalized linear immersion, we take the module

$$M = \langle 1,\, \sin(w_1),\ \cos(w_1)\rangle$$

generated by 1, $\sin(w_1)$, $\cos(w_1)$ over the ring of polynomials $\mathbb{R}[w_1, w_2]$; the elements of the $\mathbb{R}[w_1, w_2]$-module M are of the form

$$p_1(w_1, w_2) + p_2(w_1, w_2)\sin(w_1) + p_3(w_1, w_2)\cos(w_1),$$

where $p_i(w_1, w_2), i = 1, 2, 3 \in \mathbb{R}[w_1, w_2]$. Since M is a finitely generated module over a Noetherian ring, by the Hilbert's basis theorem for modules, M is a Noetherian module. In fact, M is a free module, $M \cong \mathbb{R}[w_1, w_2]^3$, via the obvious isomorphism,

$$p_1(w_1, w_2) + p_2(w_1, w_2)\sin(w_1) + p_3(w_1, w_2)\cos(w_1) \mapsto (p_1(w_1, w_2), p_2(w_1, w_2), p_3(w_1, w_2)).$$

So, we identify the elements of M with the corresponding vector of polynomials in $\mathbb{R}[w_1, w_2]^3$. For example, $\sin(w_1)$ corresponds to the vector $(0, 1, 0)$, $\cos(w_1)$ to $(0, 0, 1)$, etc.

We have then

$\xi_1 = w_1 - 9.8\sin(w_1) = u(w_1, w_2) \in M$. The control law.

Under the identification, ξ_1 corresponds to the vector $(w_1, -9.8, 0)$.

$\xi_2 = \dot{\xi}_1 = w_2(1 - 9.8\cos w_1) \in M \mapsto (w_2, 0, -9.8w_2)$.

In this case, we have that $N_1 = \langle \xi_1 \rangle$. We want to know if $\xi_2 \in N_1$, a membership problem. To respond, now we use the algorithms **NFMora** and **GröbnerBasis**. Specifically, we compute $\textbf{NFMora}(\xi_2, \textbf{GröbnerBasis}(N_1)) = \xi_2$, using the free software for computational commutative algebra *Singular*© (see [6]). Since $\xi_2 \neq 0$, we calculate ξ_3.

$$\xi_3 = \dot{\xi_2} = -w_1 + 9.8w_2^2 \sin(w_1) + 9.8w_1 \cos(w_1) \mapsto (-w_1, 9.8w_2^2, 9.8w_1).$$

Now $N_2 = \langle \xi_1, \xi_2 \rangle$ and since $\textbf{NFMora}(\xi_3, \textbf{GröbnerBasis}(N_2)) \neq 0$, we calculate ξ_4.

$$\begin{aligned} \xi_4 = \dot{\xi_3} &= -w_2 - 29.4w_1w_2 \sin(w_1) + \left(9.8w_2^3 + 9.8w_2\right) \cos(w_1) \\ &\mapsto (-w_2, -29.4w_1w_2, 9.8w_2^3 + 9.8w_2). \end{aligned}$$

We define $N_3 = \langle \xi_1, \xi_2, \xi_3 \rangle$ and get that $\textbf{NFMora}(\xi_4, \textbf{GröbnerBasis}(N_3)) \neq 0$,

$$\xi_5 = \dot{\xi_4} \mapsto \left(\ w_1, \quad -9.8w_2^4 - 39.2w_2^2 + 29.4w_1^2, \quad -58.8w_1w_2^2 - 9.8w_1 \ \right).$$

Now $N_4 = \langle \xi_1, \xi_2, \xi_3, \xi_4 \rangle$ and we see that $\textbf{NFMora}(\xi_5, \textbf{GröbnerBasis}(N_4)) \neq 0$,

$$\xi_6 = \dot{\xi_5} \mapsto \left(\ w_2, \quad 98w_1w_2^3 + 147w_1w_2, \quad -9.8w_2^5 - 98w_2^3 + 147w_1^2w_2 - 9.8w_2 \ \right)$$

and define $N_5 = \langle \xi_1, \xi_2, \xi_3, \xi_4, \xi_5 \rangle$. This time we get

$$\textbf{NFMora}(\xi_6, \textbf{GröbnerBasis}(N_5)) = 0,$$

and this means we should stop and $\xi_6 \in N_5$. We need to express ξ_6 as a linear combination of $\xi_1, \ldots, \xi_5$. First we calculate the generators of the module of syzygies $S = \operatorname{syz}(\xi_1, \ldots, \xi_6)$, using the algorithm $\operatorname{SYZ}(\xi_1, \ldots, \xi_6)$. Then, we get a Gröbner basis of the module S. Again, we use the program *Singular*©. The generators of the module of syzygies that were obtained through this program are:

$$\begin{aligned} S_1 &= \left(w_2^3, -0.8w_1w_2^2 + 1.8w_1, 0.2w_2^3 + 1.8w_2, 0.2w_1, 0.2w_2\right) \\ S_2 &= \left(w_2^5, -3w_1^3 - w_1w_2^4, -3w_1^2w_2 + w_2^3, -w_1w_2^2\right) \\ S_3 &= \left(w_1w_2, -0.5333w_1^2 - 0.06667w_2^2 - 0.6, 0.4667w_1w_2, -0.06667w_2^2 - 0.6667, 0, \right. \\ &\qquad \left. - 0.06667\right). \end{aligned}$$

We do not need to compute the Gröbner basis of the module of syzygies S, the generator S_3 has a constant in its last component. Dividing all the components of the vector S_3 by this constant, we obtain that

$$\xi_6 = 15w_1w_2\xi_1 + (-8w_1^2 - w_2^2 - 9)\xi_2 + 7w_1w_2\xi_3 + (-w_2^2 - 10)\xi_4,$$

that is, the generalized linear immersion $\dot{\xi} = \Phi(w)\xi$ has been obtained, where $\xi = [\xi_1, \ldots, \xi_5]^T$ and

$$\Phi(w) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 15w_1w_2 & -8w_1^2 - w_2^2 - 9 & 7w_1w_2 & -w_2^2 - 10 & 0 \end{bmatrix}.$$

For another perspective on how to obtain the matrix $\Phi(w)$ and further details, see [12].

4.1.1. Construction of the controller

Once you find a linear or generalized linear immersion, the problem of regulating the output is now transformed into a stabilization problem for the system formed by the plant and the immersion that was found. The controller that solves the problem consists of the parallel connection of two subsystems [7]: The subcontroller

$$\begin{aligned}\dot{\xi} &= \varphi(\xi) + Ne \\ u_1 &= \gamma(\xi),\end{aligned}$$

which is our generalized linear immersion, and the subcontroller:

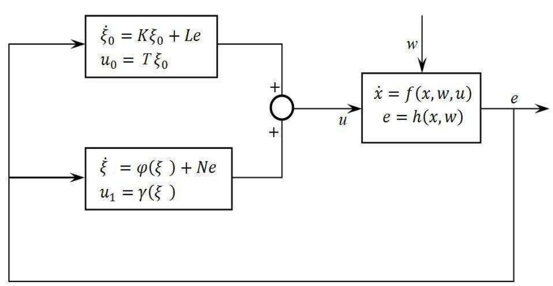
$$\begin{aligned}\dot{\xi}_0 &= K\xi_0 + Le \\ u_0 &= T\xi_0,\end{aligned}$$

which is a linear system and have the role of stabilizing in the first approximation the interconnection

$$\begin{aligned}\dot{x} &= f(x, w, \gamma(\xi) + u) \\ \dot{\xi} &= \gamma(\xi) + Nh(x, w) \\ e &= h(x, w).\end{aligned}$$

Here, $\varphi(\xi) = \Phi(w)\xi$, where $\Phi(w)$ is the matrix found in the previous section and $\xi = [\xi_1, \ldots, \xi_5]^T$ is the vector of Lie derivatives of the controller. Also, $\gamma = (1, 0, \ldots, 0)$ and $\gamma(\xi) = \gamma \cdot \xi = \xi_1$ is the controller. Finally, ξ_0 is a vector of auxiliary variables to implement the principle of separation.

This connection is illustrated in the following diagram:



One way to build the stabilization uses the principle of separation referred to in subsection 2.1: For the construction of a controller one works with the linearization of the system originally given in an equilibrium point of interest, taking into account that the behavior of a nonlinear system is diffeomorphic to the linear system behavior. Thus, since we work with the linear system we can use the separation principle.

So, the control of the linear system can be applied to the nonlinear system and we produce the desired results but only in a small neighborhood of the equilibrium point. How small is this neighborhood is still under study.

We need to find a matrix L such that the matrix

$$\begin{bmatrix} A & B\begin{bmatrix}1 & 0 & \cdots & 0\end{bmatrix} \\ 0 & \varphi(0) \end{bmatrix} - L\begin{bmatrix} C & 0 \end{bmatrix}$$

has all its eigenvalues with negative real part, and find a matrix T such that:

$$\begin{bmatrix} A & 0 \\ NC & \varphi(0) \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} T$$

also has all its eigenvalues with negative real part. For the selection of N, it is necessary to choose one that is distinct from zero and such that the pair

$$\begin{bmatrix} A & 0 \\ NC & \varphi(0) \end{bmatrix}, \begin{bmatrix} B \\ 0 \end{bmatrix}$$

is stabilizable. This is made many times choosing an N such that all entries are zeros except the last one is a 1. Then we define the matrix K as follows:

$$K = \begin{bmatrix} A & B\begin{bmatrix}1 & 0 & \cdots & 0\end{bmatrix} \\ NC & \varphi(0) \end{bmatrix} - L\begin{bmatrix} C & 0 \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} T.$$

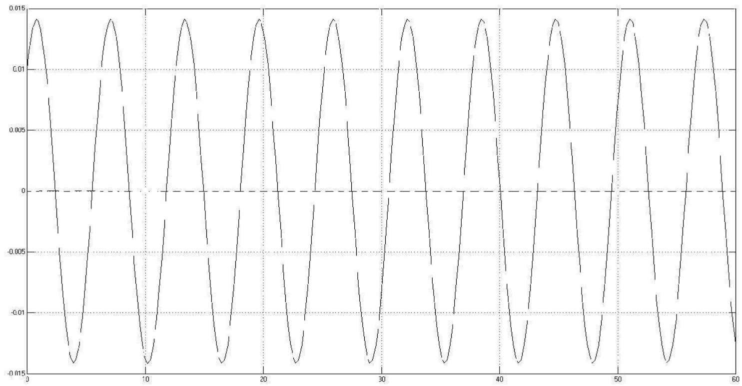
In our example, using the $Matlab^{©}$ command ACKER, command using Ackerman's formula, and the principle of duality (see [11], pp 821 and 748) we found that

$$L = \begin{bmatrix} 18 \\ 108 \\ -328 \\ 35 \\ 1912 \\ -372 \\ -16481 \end{bmatrix}, \; T = \begin{bmatrix} 108 & 17.5 & 315 & 22.5 & 1369.4 & -35 & 328.1 \end{bmatrix},$$

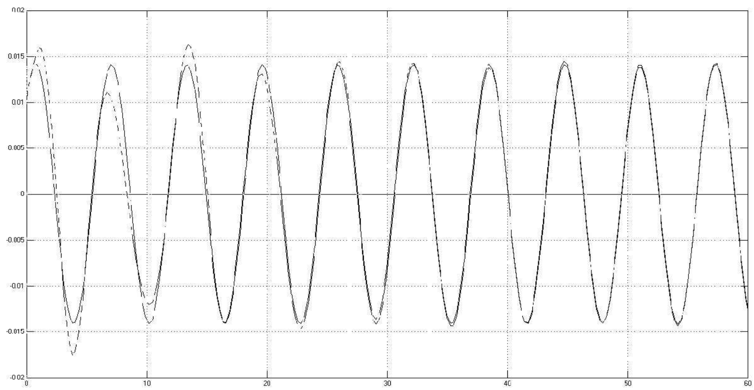
and hence

$$K = \begin{bmatrix} -18 & 1 & 0 & 0 & 0 & 0 & 0 \\ -226 & -18 & -316 & -22 & -1369 & 35 & -328 \\ 328 & 0 & 0 & 1 & 0 & 0 & 0 \\ -35 & 0 & 0 & 0 & 1 & 0 & 0 \\ -1912 & 0 & 0 & 0 & 0 & 1 & 0 \\ 372 & 0 & 0 & 0 & 0 & 0 & 1 \\ 16482 & 0 & 0 & -9 & 0 & -10 & 0 \end{bmatrix}.$$

Finally, we made a simulation of the system composed of the plant, the generalized linear immersion and stabilization. To do this we use the $Matlab^{©}$ Simulink program, first with nominal values, obtaining the figure below,



and then varying the parameters:



The parameters were varied by 10%, and the response of the system was adequate for tracking the trajectory. One of the unsolved problems is to find conditions and strategies for the variation of parameters to be greater.

In the graphs there should be three curves:

1. the curve whose behavior we want to imitate,
2. the controlled system behavior and
3. the error.

In the first graph, with nominal values of the inverted pendulum, the controller is so good that it follows the desired trajectory from initial values; so only one curve can be seen since the curve to follow and the controlled output are superimposed. The dotted straight line that is observed at the level of $y = 0$ is the error in monitoring. However, the error is so small that it looks like the line $y = 0$.

In the second graph, where we vary the parameters a and b of the inverted pendulum, the path to follow is continuous and the dotted curve is our controlled system. The error is not included in the graph, but we can see that from the time 30 seconds, the error begins to decrease until it becomes zero.

5. CONCLUSIONS

In this work, a systematic way to construct a generalized linear immersion for triangular systems with trigonometric functions is found. The reason to use triangular systems is that their structure facilitates the finding of solutions to the equations of regulation. However, the construction of the generalized immersion uses only as a hypothesis that the controller that solves the regulation equations is pseudopolynomial; therefore, the techniques described here can be applied to any controller that is pseudopolynomial, no matter whether it comes from a differential equations system that is triangular or not.

The generalized linear immersion is a system of differential equations that is added to the original system (plant); so, the question that arises is what the initial values for this new differential equations system are since the solution depends on these initial values.

The initial values used in the generalized linear immersion of the inverted pendulum were found by trial and error. Thus, to develop a technique to find such initial values remains for future work .

Once the initial values are known, the behavior of the plant, the exosystem and the generalized linear immersion may be simulated. This is where we found another problem for future work: To determine the best algorithm to simulate our augmented system of differential equations . In this work, it was observed that a small system of differential equations grows, considerably, when the generalized linear immersion and the stabilization are incorporated. Due to the size of the system, a slight error quickly causes destabilization of the system even when techniques for robustness have been used. Therefore, there is a need to determine an algorithm to solve differential equations that minimizes the approximation error at each iteration.

ACKNOWLEDGEMENT

We thank the referees for their comments and suggestions that helped greatly improve this article. Also, thanks to Luis Rodríguez for helping improve the English writing of this article.

(Received August 5, 2008)

REFERENCES

[1] M. F. Atiyah and I. G. MacDonald: Introduction to Commutative Algebra. Addison–Wesley Publishing Company, Inc., Reading, Massachusetts 1969.

[2] C. I. Byrnes and A. Isidori: Output regulation for nonlinear systems: an overview. Internat. J. Robust Nonlinear Control *10* (2000), 323–337.

[3] B. Castillo-Toledo, S. Čelikovský, and S. Di Gennaro: Generalized immersion and nonlinear robust output regulation problem. Kybernetika *40* (2004), 207–220.

[4] Z. Chen and J. Huang: A general formulation and sovability of the global output regulation problem. In: Proc. 42nd IEEE Conf. on Decision & Control, Mawui 2003, pp. 1071–1079.

[5] B. A. Francis and W. M. Wonham: The internal model principle of control theory. Automatica J. IFAC *12* (1976), 457–465.

[6] G.-M. Greuel and G. Pfister: A Singular Introduction to Commutative Algebra. Springer–Verlag, Berlin 2002.

[7] A. Isidori: Nonlinear Control Systems. Third edition. Springer–Verlag, Berlin 1995.

[8] A. Isidori and C. I. Byrnes: Output regulation of nonlinear systems. IEEE Trans. Automat. Control *35* (1990), 131–140.

[9] H.-W. Knobloch, A. Isidori, and D. Flockerzi: Topics in Control Theory. Birkhäuser Verlag, Basel 1993.

[10] F. Mora: An algorithm to compute the equations of tangents cones. In: Proc. EUROCAM 82, Lecture Notes In Comput. Sci. *144*, Springer, Berlin-New – York 1982, pp. 158–165.

[11] K. Ogata: Modern Control Engineering. Third edition. Prentice–Hall, Upper Saddle River, New Jersey 1997.

[12] D. R. Tuyub-Puc: Aplicación de las Bases de Gröbner a la construcción de un controlador con una Inmersión Lineal Generalizada. Master's thesis. Universidad Autónoma de Yucatán 2009.

[13] C. Villanueva-Novelo, S. Čelikovský, and B. Castillo-Toledo: Structurally stable design of output regulation for a class of nonlinear systems. Kybernetika *37* (2001), 547–564.

Javier Diaz-Vargas, Facultad de Matemáticas, Universidad Autónoma de Yucatán, Anillo Periférico Norte, Tablaje Catastral 13615, Mérida, Yucatán. México.
e-mail: jdvargas@uady.mx

Dennis Tuyub-Puc, Facultad de Matemáticas, Universidad Autónoma de Yucatán, Anillo Periférico Norte, Tablaje Catastral 13615, Mérida, Yucatán. México.
e-mail: dtuyub@uady.mx

Celia Villanueva-Novelo, Facultad de Matemáticas, Universidad Autónoma de Yucatán, Anillo Periférico Norte, Tablaje Catastral 13615, Mérida, Yucatán. México.
e-mail: vnovelo@uady.mx