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ON THE CONSTRUCTION OF EXPLICIT SOLUTIONS TO THE MATRIX EQUATION $X^2AX = AXA$*

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Abstract. In a previous article by Aihua Li and Duane Randall, the existence of solutions to certain matrix equations is demonstrated via nonconstructive methods. A recurring example appears in that work, namely the matrix equation $AXA = X^2AX$, where A is a fixed, square matrix with real entries and X is an unknown square matrix. In this paper, the solution space is explicitly constructed for all 2×2 complex matrices using Gröbner basis techniques. When A is a 2×2 matrix, the equation $AXA = X^2AX$ is equivalent to a system of four polynomial equations. The solution space then is the variety defined by the polynomials involved. The ideal of the underlying polynomial ring generated by the defining polynomials plays an important role in solving the system. In our procedure for solving these equations, Gröbner bases are used to transform the polynomial system into a simpler one, which makes it possible to classify all the solutions. In addition to classifying all solutions for 2×2 matrices, certain explicit solutions are produced in arbitrary dimensions when A is nonsingular. In higher dimensions, Gröbner bases are extraordinarily computationally demanding, and so a different approach is taken. This technique can be applied to more general matrix equations, and the focus here is placed on solutions coming from a particular class of matrices.

Key words. Matrix equation, Ideal, Gröbner bases.

AMS subject classifications. 39B42, 15A24, 12Y05, 13B25, 13F20.

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