A Gröbner-Sylvester Hybrid Method for Closed-Form Displacement Analysis of Mechanisms

A. K. Dhingra

A. N. Almadi[1]

D. Kohli

Department of Mechanical Engineering,
University of Wisconsin-Milwaukee,
Milwaukee, WI 53201

The displacement analysis problem for planar and spatial mechanisms can be written as a system of multivariate polynomial equations. Elimination theory based on resultants and polynomial continuation are some of the methods that have been used to solve this problem. This paper presents a new approach to displacement analysis using the reduced Gröbner basis form of a system of equations under degree lexicographic (dlex) term ordering of its monomials and Sylvester's Dialytic elimination method. Using the Gröbner-Sylvester hybrid approach, a finitely solvable system of equations F is transformed into its reduced Gröbner basis G using dlex term ordering. Next, using the entire or a subset of the set of generators in G, the Sylvester's matrix is assembled. The vanishing of the resultant, given as the determinant of Sylvester's matrix, yields the necessary condition for polynomials in G (as well as F) to have a common factor. The proposed approach appears to provide a systematic and rational procedure to the problem discussed by Roth, dealing with the generation of (additional) equations for constructing the Sylvester's matrix. Three examples illustrating the applicability of the proposed approach to displacement analysis of planar and spatial mechanisms are presented. The first and second examples address the forward displacement analysis of the general 6-6 Stewart mechanism and the 6-6 Stewart platform, whereas the third example deals with the determination of the I/O polynomial of an 8-link 1-DOF mechanism that does not contain any 4-link loop. [S1050-0472(00)01204-6]

1 Introduction

Kinematic motion analysis and design of mechanical systems lead naturally to system of nonlinear algebraic and/or transcendental equations. One of the most frequently occurring problems in kinematics is to find solutions to this system of equations. The solution approaches for such equations can be broadly divided into two classes: numerical (iterative) methods and closed-form (analytical) techniques. Numerical techniques rely heavily on numerical iteration while closed-form techniques are based on analytical expressions and often require massive algebraic manipulations. Using numerical methods, a kinematics problem is considered solved if a tight upper bound on the number of solutions can be established, and an efficient algorithm for computing all solutions can be implemented. The commonly used iterative methods are variants of either the Newton or conjugate gradient methods. These methods require an initial guess at the solution. If the initial guess is not close enough to a solution, the iterations may converge slowly, converge to an unacceptable solution or may diverge altogether. However, the Newton's method is a valuable tool and is used as the building block for numerical continuation methods.

Numerical continuation (homotopy) methods have been used in solving kinematic equations of motion for planar as well as spatial mechanisms. They are based on the concept that a system of polynomial equations can undergo small changes in the system parameters producing small changes in the solutions. If the system of equations to be solved can be cast in a polynomial form, numerical continuation methods are capable of finding all possible solutions and eliminate the need for a good initial estimate to the solution [1].

Analytical or closed-form solutions to kinematic equations can be obtained using elimination theories based on resultants [2] or Gröbner bases [3]. In general, there are two types of eliminants. The first is concerned with the elimination of one variable in two polynomials. The commonly used resultant matrices for such problems are those of Sylvester and Bézout. These resultants are particularly effective in eliminations involving non-homogeneous polynomials. For a system of n polynomials in two or more variables, the resultant is defined as the greatest common divisor of a set of determinants (resultants). Hence, through successive or repeated application of resultants it may be ''possible'' to reduce a multivariate system of polynomials to one or more univariate polynomials.

For resultants of the second type, $n-1$ variables are eliminated simultaneously from a system of n polynomials. The eliminant is a matrix whose columns are indexed by the monomials in the polynomial ring and whose rows are indexed by monomial products of the n polynomials. The resultant is a polynomial in the remaining variable. For a system of three or more equations, no general conditions exist which express the resultant as a determinant, except for special class of systems of equations. However, it may be possible to express the resultant as a quotient of one determinant divided by another. The divisor is the extraneous factor. Since it is difficult to identify whether or not extraneous factors exist, it is not possible to insure that a resultant is devoid of extraneous solutions. For certain equation structures, it may be possible to derive the resultant using Sylvester's dialytic elimination method. In other cases, extraneous factors can be identified and eliminated as demonstrated by Macaulay [4] for homogeneous systems. However, problems arising in synthesis and analysis of mechanisms often result in a system of non-homogeneous

[1]Presently at Space Research Institute, KACST, Riyadh 11442, Saudi Arabia.

Contributed by the Mechanisms Committee for publication in the JOURNAL OF MECHANICAL DESIGN. Manuscript received Aug. 1999. Associate Technical Editor: C. W. Wampler II.

polynomials. Hence, Macaulay's approach is not suitable for such problems because the system of equations has extraneous solutions including solutions at infinity, and thus the eliminant matrix is degenerate.

Elimination theories for solving multivariate polynomial equations such as Gröbner bases are related in their basic form to Gaussian elimination's for linear equations and Euclid's algorithm for solving nonlinear univariate polynomials [5]. Just as Gaussian elimination triangulates a system of linear equations, elimination based on Gröbner basis under lexicographic term ordering of variables, i.e., $x_n < \ldots < x_1$ produces a triangular set of nonlinear polynomials. Generally, the resulting Gröbner basis can be partitioned into sets of nonlinear polynomials where the last set involves only one polynomial in x_n with minimal degree, the set before the last involves only the last two variables, x_{n-1} and x_n, etc. The complete solution set for the original multivariate system of equations can then be obtained by first solving the last univariate polynomial in x_n, next substitute each solution of x_n in the set of polynomials immediately preceding it, which are now univariate polynomials in the variable x_{n-1}, and so on, [5]. However, for kinematics problems of ''reasonable'' complexity, the Gröbner basis method under lex term ordering has been quite inefficient because of exploding intermediate results and excessive computation times [6]. For a comprehensive review of the state-of-art in solving polynomial systems arising in kinematics, see Raghavan and Roth [7].

To overcome the difficulties of existing approaches, this paper presents a new method for solving algebraic system of equations which utilizes the reduced Gröbner basis form of the system of equations under total degree term ordering of its monomials and Sylvester's Dialytic elimination method. Using the proposed hybrid approach, the system of equations *F* is first transformed into its reduced Gröbner basis representation *G*. Next, using the entire or a subset of the set of generators in *G*, the Sylvester's matrix is assembled. The vanishing of the Sylvester's determinant yields the necessary condition for the polynomials in *G* as well as *F* to have a common factor.

The proposed method provides a systematic approach for constructing Sylvester's matrix. Normally, the construction of Sylvester's matrix requires an exhaustive search involving multiplication of the polynomials in *F* by appropriate monomials (power products) until a new system of equations *H* is generated such that the number of polynomials in *H* is equal to the number of monomials in *H*. The Sylvester's matrix can then be constructed directly using *H*. For a general system of equations, no systematic procedure exists for constructing additional equations and setting up the Sylvester's matrix [6]. Further, even if a resultant matrix can be obtained, it is not possible to conclude whether or not extraneous factors are present in the resultant. In an attempt to outline a general procedure for constructing a resultant matrix for a system of *n* equations in *n* unknowns, Macaulay [4] presented an approach applicable strictly to homogeneous system of equations. In addition, the proposed approach may lead to the introduction of extraneous factors as alluded to by Macaulay. The Gröbner-Sylvester hybrid approach may prove to be an important step towards a general approach for constructing a resultant matrix devoid of extraneous factors. In this method, the step for generating additional equations is based on a division approach similar to the *S*-polynomial and normal form reductions of Gröbner bases method rather than the common approach based on polynomial-monomial multiplications. Through three numerical examples, it is shown that the proposed hybrid approach can be used to perform closed-form displacement analysis of spatial and planar mechanisms devoid of any extraneous roots.

2 Gröbner Bases: Basic Concepts

If *F* is a system of polynomial equations, the Gröbner bases method reduces the problem of solving *F* to manipulating the monomials in *F*. This transforms *F* into its Gröbner basis representation *G* that generates the same ideal as *F* but is easier to solve. In brief, the Gröbner basis for ideal *I* generated by $F \in k[x_1, \ldots, x_n]$ over field *k*, is a set of generators $G \in k[x_1, \ldots, x_n]$ that generate the same ideal *I*, but are simpler in form and easier to solve. The Buchberger's algorithm is a simple and effective method for computing Gröbner bases. Since Gröbner bases computations involve manipulating the monomials in *F*, the notion of term ordering, denoted $<_T$, on the set of monomials in *F* must first be introduced.

A polynomial equation $f_i \in F$ can be viewed as a finite sum of nonzero terms with distinct monomials with scalar coefficients. Hence, it is necessary to fix a term ordering $<_T$ on the set of monomials in each $f_i \in F$. Two term orderings which play an important role in solving polynomial systems include the lexicographic (lex) order and degree lexicographic (dlex) order.

Lexicographic Ordering (lex): Let the term ordering on the variables x,y,w be fixed as $w <_T y <_T x$. This yields $1 <_{lex} w \cdots <_{lex} y^2 w <_{lex} y^3 <_{lex} x <_{lex} xw \cdots <_{lex} x^3$. In *lex* order, a variable dominates any monomial involving only smaller variables, regardless of its total degree.

Degree Lexicographic Ordering (dlex): Let the term ordering on the variables x,y,w be fixed as $w <_T y <_T x$. Using dlex order, the terms are first ordered by total degree and within a given degree lexicographically. This yields $1 <_{dlex} w <_{dlex} y <_{dlex} x <_{dlex} w^2 <_{dlex} yw \cdots <_{dlex} x^2 y <_{dlex} x^3$.

Since a polynomial is uniquely expressible as a finite sum of nonzero terms involving distinct monomials, a term ordering permits a comparison between every pair of monomials in the polynomial to establish their relative positions. Thus every polynomial $f \in F$ must be arranged such that its monomials are ordered as a descending sequence under the term ordering $<_T$.

2.1 Computation of Gröbner Basis. The computation of Gröbner basis is based on the following properties of polynomial ideals:

Property 1:

If $F \in k[x_1, \ldots, x_n]$, then for any $f \in F$, constant *c* and $u \in k[x_1, \ldots, x_n]$, $f, cuf \in I = Ideal(F)$.

Property 2:

If $f_i, f_j \in F$ and *r* is the division remainder of f_i, f_j, then $r \in I = Ideal(F)$.

Based on properties 1 and 2, Buchberger [3] introduced the notion of a *S*-polynomial for eliminating leading monomials, and the concept of normal form as a reduction algorithm for also eliminating monomials. The Buchberger's algorithm consists of two steps: the computation of *S*-polynomials and their normal form reduction.

The *S*-polynomial, designed to produce cancellation of leading terms, is defined as follows: let $F \in k[x_1, \ldots, x_n]$ a system of polynomial equations and $I = Ideal(F)$. If $f_i, f_j \in F$, then their *S*-polynomial, denoted $Spoly(f_i, f_j)$, is given as

$$h = Spoly(f_i, f_j) = -Spoly(f_j, f_i) = u_i f_i - c_j u_j f_j \tag{1}$$

where $c_j = Lc(f_i)/Lc(f_j)$, $u_\alpha = l_{ij}/m_\alpha = Lcm(Lm\{f_i, f_j\})/Lm(f_\alpha)$. Here $Lc(\cdot)$, $Lcm(\cdot)$ and $Lm(\cdot)$ denote leading coefficient, least common multiple and leading monomial, respectively.

The definition of $Spoly(f_i, f_j)$ in Eq. (1) makes use of both properties 1 and 2. Since $c_j \in k$, $u_i, u_j \in k[x_1, \ldots, x_n]$ and $f_i, f_j \in I$ $u_i f_i, c_j u_j f_j \in I$ and therefore the division remainder $h \in I$. Note that $u_i f_i$ and $c_j u_j f_j$ have the same leading term $Lc(f_i) l_{ij} \in k[x_1, \ldots, x_n]$ which is eliminated using $Spoly(f_i, f_j)$. Hence, $Spoly(f_i, f_j)$ may be viewed as one step in a generalized division and *h* is the division remainder.

The normal form reduction algorithm is designed to eliminate monomials such that no leading monomial of one polynomial divides the monomials of another. For instance, let $f, g \in k[x_1, \ldots, x_n]$ and $Lm(f) \leqslant Lm(g)$, then *g* is said to be reducible to *h* if and only if a monomial $m_i \in \langle m(g) \rangle$ is divisible by $Lm(f)$. Here $\langle m(g) \rangle$ denotes the monomial ideal of *g*. Hence,

$$h = g - c_i u_i f \tag{2}$$

where c_i=coefficient of m_i in $g/Lc(f)$, $u_i = m_i/Lm(f)$, and the term involving the monomial m_i in g has been deleted, i.e., $\langle m(h)\rangle = \langle m(g) - m_i\rangle$. If there exists an appropriate $c_i \in k$ and $u_i \in k[x_1, \ldots, x_n]$ such that

$$g - c_1 u_1 f \rightarrow \cdots h' - c_i u_i f \cdots \rightarrow h$$

and no further reductions of *h* are possible w.r.t. *f*, then *h* is said to be the normal form of *g* modulo *f*. Thus no monomial in *h* is divisible by $Lm(f)$. In general, if there exists an appropriate $c_i \in k$, $u_i \in k[x_1, \ldots, x_n]$ and $f_j \in F$ such that

$$g - c_1 u_1 f_\alpha \rightarrow \cdots h' - c_i u_i f_j \cdots \rightarrow h$$

and no further reductions of *h* are possible w.r.t. $f_j \in F$, then *h* is said to be the normal form of *g* modulo *F*, denoted $h = NForm(g,F)$, and no monomial in *h* is divisible by $\langle Lm(f_j) | f_j \in F\rangle$, the leading monomial ideal of *F*. With the *S*-polynomial computation and the normal form algorithm properly defined, the algorithmic criterion for Gröbner bases is formulated next.

Let $F = \{f_1, \ldots, f_s\} \in k[x_1, \ldots, x_n]$ be a system of polynomial equations, then *F* is a Gröbner basis if and only if for $1 \leqslant i < j \leqslant s$, $NForm(Spoly(f_i, f_j), F) = 0$. Since *F* is a finite set of polynomials, one has to consider finitely many pairs $f_i, f_j \in F$ and compute the $Spoly(f_i, f_j)$ polynomial to see whether $NForm(Spoly(f_i, f_j), F) = 0$. This is done as follows. Let $F = \{f_1, \ldots, f_m\} \in k[x_1, \ldots, x_n]$ and for $1 \leqslant i < j \leqslant m$, compute $h = NForm(Spoly(f_i, f_j), F)$. If $h \neq 0$, then $h \in Ideal(F)$ and can be adjoined to the list of polynomials in the generating set *F* without changing the ideal generated by *F*. Hence,

$$m = m + 1$$

$$F = F \cup \{h\}$$

Repeat this step until for all $1 \leqslant i < j \leqslant m$, $NForm(Spoly(f_i, f_j), F) = 0$. Upon termination of this algorithm yields the Gröbner basis.

A system of polynomial equations may have many Gröbner basis representations w.r.t. a fixed term order $<_T$. For instance, the number and form of generators in *G* is sensitive to the order in which $f_i, f_j \in F$ pairs are selected for the computation of each $Spoly(f_i, f_j)$. Further, using the *S*-polynomial and the normal form algorithm outlined previously, the resulting Gröbner basis *G* for a system of polynomial equations *F* may contain more generators than necessary. Therefore, some generators in *G* may be eliminated without changing the ideal generated by *F*. However, for any system of polynomial equations *F*, there exists a unique Gröbner basis, called *reduced Gröbner basis*, defined as: *G* is a reduced Gröbner basis if and only if for each $g_i \in G$, $Lc(g_i) = 1$ and no monomial of g_i lies in $\langle Lm(G - \{g_i\})\rangle$, i.e., no monomial of g_i is divisible by the leading monomial of any $g_j \in \{G - \{g_i\}\}$.

3 Gröbner-Sylvester Hybrid Method

If $F \in k[x_1, \ldots, x_n]$ is a finitely solvable system of equations, the corresponding reduced Gröbner basis *G* under lex term ordering of the monomials with $x_n <_T \cdots <_T x_1$ contains a univariate polynomial $g_p \in k[x_n]$ with minimal degree [3]. However, the computation of the reduced Gröbner basis under lex term ordering is very sensitive to permutation of the variables whereas the reduced Gröbner basis computation using dlex ordering is more stable. Further, the computation of the reduced Gröbner basis under dlex ordering has the advantage of being more efficient w.r.t. computation times and memory requirements. In fact, for three example problems considered herein, a univariate polynomial was obtained using the Gröbner-Sylvester hybrid approach where other methods such as the Gröbner bases under lex term ordering have failed.

The Gröbner-Sylvester hybrid approach *for solving polynomial systems arising in kinematics problems* is outlined as follows. Let *F* be a finitely solvable system of equations given as $F = \{f_1, \ldots, f_m\} \in k[x_1, \ldots, x_n]$ where *k* is an arbitrary field. Using the Buchberger's algorithm and dlex term ordering with $x_n <_T \cdots <_T x_1$, the reduced Gröbner basis is given as $G = \{g_1, \ldots, g_p\} \in k[x_1, \ldots, x_n]$. With *G* known, it may be possible to the construct Sylvester's matrix using *G* or a subset of *G*.

If the entire set of polynomials in *G* is used to setup the Sylvester's matrix, then there may exist a variable x_i such that each $g_i \in G$ can be expressed as

$$g_i = \sum_j a_{ij} m_j, \quad a_{ij} \in k[x_i],$$
$$m_j \in k[x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n] \tag{3}$$

and the number of monomials m_j in *G* (including 1) may equal *p* (the number of polynomials in *G*). For such a case, *G* can be viewed as a system of linear polynomials in the unknown monomials $m_1 = 1, m_2, \ldots, m_p$. In matrix form, this linear system is given as

$$\begin{bmatrix} g_1 \\ \vdots \\ \\ g_p \end{bmatrix} = \begin{bmatrix} p \times p \\ a_{ij} \in k[x_i] \end{bmatrix} \begin{bmatrix} m_p \\ \vdots \\ m_2 \\ m_1 = 1 \end{bmatrix} \tag{4}$$

Theorem: Let *F* be a finitely solvable system of equations and $G = \{g_1, \ldots, g_p\}$ is the corresponding reduced Gröbner basis under *dlex* term order. Then the vanishing of the determinant of the coefficient matrix given by Eq. (4) gives necessary condition for polynomials in *F* and *G* to have common solutions.

Proof: Since $G = \{g_1, \ldots, g_p\} = 0$, Eq. (4) reduces to $SX = 0$ where $S \in k[x_i]$ is the $p \times p$ Sylvester's (coefficient) matrix and $X \in k[x_1, \ldots, x_{i-1}, x_{i+1}, \ldots x_n]$ is the $p \times 1$ column matrix of the unknown monomials. For this homogeneous system of equations to admit a non-trivial solution, *S* must be singular, i.e., $R = |S| = 0$.[1] Thus $R = 0$ yields the necessary condition for the polynomials in *G* to have common solutions. Since $ideal(G) = ideal(F)$, the vanishing of the resultant *R* yields the necessary condition for the polynomials in *F* to have common solutions.

It may be noted that the sufficiency of the above condition is not guaranteed because the possibility of extraneous solutions at infinity cannot be ruled out from *G*. However, for polynomial systems arising in our work on kinematics of mechanisms, it was seen that the proposed approach, in every instance, yields an I/O polynomial of correct degree devoid of any extraneous roots.

With x_i known from *R*, the other variables are solved as follows. Since Sylvester's matrix is singular, $rank(S) \leqslant p - 1$.[2] If rank $(S) = p - 1$, by deleting the last row and column of Sylvester's matrix *S* and the last row of the monomials column matrix *X*, Eq. (4) yields

$$\begin{bmatrix} (p-1) \times (p-1) \\ a_{ij} \in k[x_i] \end{bmatrix} \begin{bmatrix} m_p \\ \vdots \\ m_3 \\ m_2 \end{bmatrix} = \begin{bmatrix} -a_{1,p} \\ \vdots \\ -a_{p-2,p} \\ -a_{p-1,p} \end{bmatrix} \tag{5}$$

or simply $S'X' = -S_p$ where $S_p = [a_{1,p}, \ldots, a_{p-2,p}, a_{p-1,p}]^T$ is the last column of Sylvester's matrix *S* with the last row deleted. Since $S', S_p \in k[x_i]$, and $X' \in k[x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n]$, Eq. (5) can be explicitly solved for $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots x_n$ in terms of x_i.

[1]The determinant is called the resultant *R* of the system of equations in *G*.

[2]If $rank(S) = r < p - 1$, a system of nonlinear equations needs to be solved to determine m_i, $i = 2, \ldots, p$. By elementary row operations, matrix *S* can be reduced to an upper triangular form where all matrix entries in rows $r+1$ through *p* are identically zero. The first monomial to be solved will correspond to the *r*th row and the remaining monomials can be solved using back-substitution.

On the other hand, if a subset of G is used to setup the Sylvester's matrix, then there may exist q polynomials $g_i \in G$ defined in terms of $\{x_\alpha, \ldots, x_\beta\} \subset \{x_1, \ldots, x_n\}$. If each $h_i = g_i \in k[x_\alpha, \ldots, x_\beta]$ is expressed as

$$h_i = \sum_j a'_{ij} m'_j, \quad a'_{ij} \in k[x_i],$$

$$m'_j \in k[x_\alpha, \ldots, x_{i-1}, x_{i+1}, \ldots, x_\beta] \tag{6}$$

and the number of monomials m'_j in $H = \{h_1, \ldots, h_q\}$ including 1 is equal to q (the number of polynomials in H), then H can be viewed as a system of linear polynomials in the unknown monomials $m'_1 = 1, m'_2, \ldots, m'_q$. In matrix form, this linear system is given as

$$\begin{bmatrix} h_1 \\ \vdots \\ \\ h_q \end{bmatrix} = \begin{bmatrix} q \times q \\ a'_{ij} \in k[x_i] \end{bmatrix} \begin{bmatrix} m'_q \\ \vdots \\ m'_2 \\ m'_1 = 1 \end{bmatrix} \tag{7}$$

Here too, $H = \{h_1, \ldots, h_q\} = 0$ and Eq. (7) reduces to $SX = 0$ where $S \in k[x_i]$ is the $q \times q$ Sylvester's matrix and $X \in k[x_\alpha, \ldots, x_{i-1}, x_{i+1}, \ldots x_\beta]$ is the $q \times 1$ column matrix of the unknown monomials. From Eq. (7), the resultant (R) is given as the determinant of the Sylvester's matrix S. The vanishing of this resultant yields the necessary condition for the polynomials in G as well as F to have common solutions since G generates the same ideal as F. With x_i known from R, the solution for the other variables $x_\alpha, \ldots, x_{i-1}, x_{i+1}, \ldots x_\beta$ is obtained as follows. If rank $(S) = q-1$, upon deletion of the last row and last column of S and the last row of column X, Eq. (7) leads to

$$\begin{bmatrix} (q-1) \times (q-1) \\ a'_{ij} \in k[x_i] \end{bmatrix} \begin{bmatrix} m'_q \\ \vdots \\ m'_3 \\ m'_2 \end{bmatrix} = \begin{bmatrix} -a'_{1,q} \\ \vdots \\ -a'_{q-2,q} \\ -a'_{q-1,q} \end{bmatrix} \tag{8}$$

or simply $S'X' = -S_q$ where $S_q = [a'_{1,q}, \ldots, a'_{q-2,q}, a'_{q-1,q}]^T$ is the qth column of S with the qth row deleted. Since $S', S_q \in k[x_i]$ and $X' \in k[x_\alpha, \ldots, x_{i-1}, x_{i+1}, \ldots, x_\beta]$, Eq. (8) yields expressions for $x_\alpha, \ldots, x_{i-1}, x_{i+1}, \ldots x_\beta$ explicitly in terms of x_i. With $x_\alpha, \ldots, x_\beta$ known, the remaining polynomials in G can be used to derive explicit expressions for the remaining unknown variables $\{x_1, \ldots, x_n\} - \{x_\alpha, \ldots, x_\beta\}$ in terms of $x_\alpha, \ldots, x_\beta$. The case when $rank(S) < q-1$ is handled in a manner discussed earlier.

It should be noted that the form and number of generators in the reduced Gröbner basis is quite sensitive to the term ordering $<_T$ on the set of monomials. Even though the reduced Gröbner basis G for a system of equations F is always unique w.r.t. a single term ordering $<_T$, different term orderings can yield completely different G's. Therefore, even though the system of equations processed by the hybrid approach is not unique (because it depends on the term order $<_T$), the resultant expressed in x_i is always unique and is independent of the term order. Further, it is worth noting that even though the choice of hidden variable (x_i) is arbitrary, there may exist term orderings for which an eliminant cannot be derived from G regardless of the choice of x_i.

The application of the Gröbner-Sylvester hybrid method is demonstrated next through forward kinematic analysis of the general 6-6 Stewart mechanism and platform. The forward kinematic analysis of these parallel manipulators is quite complex. Nevertheless, using the proposed approach, the univariate I/O polynomial can be derived quite easily. In addition, the displacement analysis of a general 8-link 1-DOF planar mechanism is also performed using this approach leading to the closed-form I/O polynomial for the mechanism. For all three examples presented below, the *grobner* package available in Maple V (Release 3) was used for Gröbner basis calculations. All computations were performed either symbolically or using rational arithmetic. Therefore, no numerical error is incurred in calculating the reduced Gröbner basis G. Once G is obtained, due to computational limitations, sometimes it was not possible to proceed further using symbolic manipulations and/or rational number arithmetic. Consequently, at this stage, floating-point calculations with anywhere from 30-200 digits were used to set up and expand the Sylvester's matrix to determine the univariate polynomial.

4 Forward Displacement Analysis of General Stewart Mechanism

Consider the 6-DOF general Stewart mechanism shown in Fig. 1. The six inputs are provided at the prismatic joints in each leg, which in turn controls the location and orientation of the upper platform. For both moving and fixed platforms, the spheric joints P_i and X_i, $i = 1, \ldots, 6$, are not restricted to lie in a plane. The notation and the loop-closure equations used herein are adopted from Dhingra et al. [8] and are given as follows (see Fig. 2).

Let $\underline{x}_j$ denote the vector from the origin of the global system to the grounded spheric pair X_j, $\underline{p}_j$ denote the vector (expressed in moving frame) from the origin of the coordinate frame attached to the platform at P_1 to the spheric pair P_j, $\underline{l}_j$ denote the vector from the ground spheric pair X_j to moving spheric pair P_j expressed in the base coordinate frame with l_{xj}, l_{yj} and the l_{zj} being the x-, y- and z-components of vector $\underline{l}_j$, $[R]$ is the 3×3 rotational matrix (of direction cosines) denoting the orientation of the mov-

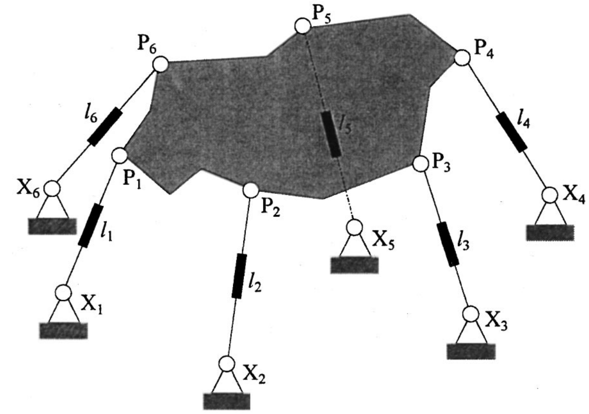

Fig. 1 A 6-DOF general Stewart mechanism

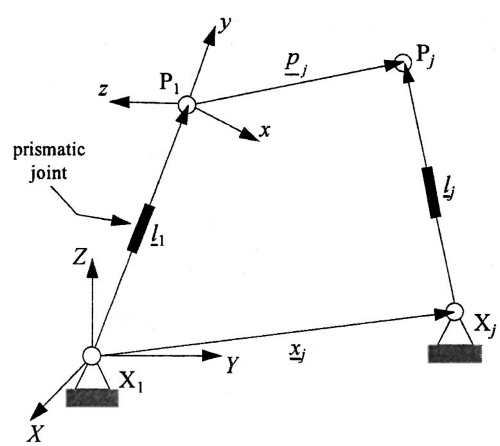

Fig. 2 A loop of general Stewart mechanism

ing frame relative to the base frame, and x_j, l_1, l_j and p_j denote the magnitudes of the vectors $\underline{x}_j$, $\underline{l}_1$, $\underline{l}_j$ and $\underline{p}_j$, respectively.

Then, the loop-closure equations for the mechanism are:

$$\underline{l}_j=\underline{l}_1+[R]\underline{p}_j-\underline{x}_j,\quad j=2,\ldots,6 \tag{9}$$

Equating the magnitudes of the vectors on the left- and right-hand-side of Eq. (9) as

$$\underline{l}_j\cdot\underline{l}_j=(\underline{l}_1+[R]\underline{p}_j-\underline{x}_j)\cdot(\underline{l}_1+[R]\underline{p}_j-\underline{x}_j) \tag{10}$$

leads to

$$l_j^2=l_1^2+x_j^2+p_j^2+2\underline{l}_1\cdot[R]\underline{p}_j-2\underline{x}_j\cdot[R]\underline{p}_j-2\underline{l}_1\cdot\underline{x}_j \tag{11}$$

Expanding Eq. (11) yields

$$\begin{aligned}&2h_1(p_{xj})+2h_2(p_{yj})+2h_3(p_{zj})+p_j^2+l_1^2+x_j^2-l_j^2-2[(p_{xj})r_1\\&+(p_{yj})r_2+(p_{zj})r_3]x_{xj}-2[(p_{xj})r_4+(p_{yj})r_5+(p_{zj})r_6]x_{yj}\\&-2[(p_{xj})r_7+(p_{yj})r_8+(p_{zj})r_9]x_{zj}-2l_{x1}(x_{xj})-2l_{y1}(x_{yj})\\&-2l_{z1}(x_{zj})=0,\quad j=2,\ldots,6\end{aligned} \tag{12}$$

where

$$\begin{aligned}h_1&=l_{x1}r_1+l_{y1}r_4+l_{z1}r_7\\h_2&=l_{x1}r_2+l_{y1}r_5+l_{z1}r_8\\h_3&=l_{x1}r_3+l_{y1}r_6+l_{z1}r_9\end{aligned} \tag{13}$$

and (r_1,r_2,r_3), (r_4,r_5,r_6), (r_7,r_8,r_9) represent the three rows of the rotation matrix $[R]$. Since $[R]$ is orthogonal, the rows and columns of $[R]$ satisfy the following dot- and cross-product relations

$$\begin{aligned}r_1^2+r_4^2+r_7^2&=1\\r_2^2+r_5^2+r_8^2&=1\\r_3^2+r_6^2+r_9^2&=1\\r_7^2+r_8^2+r_9^2&=1\end{aligned} \tag{14}$$

$$\begin{aligned}r_1r_2+r_4r_5+r_7r_8&=0\\r_1r_3+r_4r_6+r_7r_9&=0\\r_2r_3+r_5r_6+r_8r_9&=0\end{aligned} \tag{15}$$

$$\begin{aligned}r_7&=r_2r_6-r_3r_5\\r_8&=r_3r_4-r_1r_6\\r_9&=r_1r_5-r_2r_4\end{aligned} \tag{16}$$

The constant length condition of vector $\underline{l}_1$ is expressed as

$$l_1^2=l_{x1}^2+l_{y1}^2+l_{z1}^2 \tag{17}$$

The direct kinematics problem can now be stated as follows: for specified leg lengths l_j, for $j=1,\ldots,6$, determine all possible assembly configurations (real and complex) of the mechanism in terms of the *x*-, *y*- and *z*-components of the reference leg l_{x1}, l_{y1}, l_{z1} and the nine elements $r_1,\ldots,r_9$ of the rotational matrix $[R]$. The system of Eqs. (12)–(17) represents a system of 19 equations in 15 unknown variables, namely $r_1,\ldots,r_9$, l_{x1}, l_{y1}, l_{z1}, h_1, h_2 and h_3. Although only 15 equations (Eqs. (12)–(13), Eq. (17), and any 6 equations from Eqs. (14)–(16)) are needed to derive the closed-form I/O polynomial for this mechanism, the fact that additional equations are used may actually improve the reduced Gröbner basis computational efficiency, provided that the given overconstrained system of equations is finitely solvable. In fact, 11 additional relations can be derived from the dot- and cross products of the orthogonal vectors of the rotational matrix $[R]$ that can be adjoined to Eqs. (12)–(17) to derive the same reduced Gröbner basis. It should be emphasized that the reduced Gröbner basis and the number of generators used to setup the Sylvester's matrix is independent of the initial number equations used to derive the reduced Gröbner basis provided that the initial system of equations is finitely solvable.

Using the Buchberger's algorithm [3] and the *dlex* term ordering with $r_5<_Tr_4<_Tr_2<_Tr_9<_Tr_8<_Tr_7<_Tr_6<_Tr_3<_Tr_1<_Tl_{z1}<_Tl_{y1}<_Tl_{x1}<_Th_3<_Th_2<_Th_1$, yields a reduced Gröbner basis G with 68 polynomials. For the numerical data given below, the 68 polynomials in G are not reported herein due to space limitations, but they can be found in Almadi [9].

$p_{x1}=0$	$p_{y1}=0$	$p_{z1}=0$	$x_{x1}=0$	$x_{y1}=0$	$x_{z1}=0$	$l_1=12$
$p_{x2}=2$	$p_{y2}=3$	$p_{z2}=3$	$x_{x2}=2$	$x_{y2}=3$	$x_{z2}=0$	$l_2=12$
$p_{x3}=3$	$p_{y3}=5$	$p_{z3}=0$	$x_{x3}=3$	$x_{y3}=4$	$x_{z3}=4$	$l_3=10$
$p_{x4}=1$	$p_{y4}=0$	$p_{z4}=4$	$x_{x4}=5$	$x_{y4}=1$	$x_{z4}=2$	$l_4=14$
$p_{x5}=4$	$p_{y5}=2$	$p_{z5}=1$	$x_{x5}=0$	$x_{y5}=2$	$x_{z5}=3$	$l_5=12$
$p_{x6}=2$	$p_{y6}=1$	$p_{z6}=3$	$x_{x6}=4$	$x_{y6}=0$	$x_{z6}=5$	$l_6=10$

Suppressing the unknown r_5, G can be viewed as a linear system of 68 equations in 68 unknown monomials r_2^3, $l_{z1}r_4^2$, $r_1r_4^2$, $r_4^2r_3$, $r_4^2r_6$, $r_7r_4^2$, $r_9r_4^2$, r_4^3, l_{Z1}^2, r_1l_{z1}, r_1^2, r_3l_{z1}, r_1r_3, r_3^2, $l_{z1}r_6$, r_1r_6, r_3r_6, r_6^2, $l_{z1}r_7$, r_1r_7, r_3r_7, r_7r_6, r_7^2, $l_{z1}r_8$, r_1r_8, r_3r_8, r_8r_6, r_7r_8, r_8^2, $l_{z1}r_9$, r_1r_9, r_3r_9, r_9r_6, r_7r_9, r_8r_9, r_9^2, r_2l_{z1}, r_1r_2, r_2r_3, r_2r_6, r_2r_7, r_2r_8, r_2r_9, r_2^2, $l_{z1}r_4$, r_1r_4, r_4r_3, r_4r_6, r_7r_4, r_8r_4, r_9r_4, r_2r_4, r_4^2, h_1, h_2, h_3, l_{x1}, l_{y1}, l_{z1}, r_1, r_3, r_6, r_7, r_8, r_9, r_2, r_4 and 1, with the polynomial coefficients expressed in terms of r_5.

$$\begin{bmatrix}g_1\\\vdots\\\\g_{68}\end{bmatrix}=\begin{bmatrix}68\times68\\a_{ij}\in k[r_5]\end{bmatrix}\begin{bmatrix}r_2^3\\l_{z1}r_4^2\\\vdots\\1\end{bmatrix} \tag{18}$$

Since $G=\{g_1,\ldots,g_{68}\}=0$, Eq. (18) reduces to $SX=0$ where S is the 68×68 Sylvester's (or coefficient) matrix and X is the 68×1 column matrix of the unknown variables (the 68 monomials). The vanishing of the Sylvester's matrix determinant, i.e., $|S|=0$, yields the following univariate resultant

$$\begin{aligned}F_{\mathrm{I/O}}=&r_5^{40}-40.34068436r_5^{39}+2613.674295r_5^{38}+48741.59612r_5^{37}-428723.4299r_5^{36}-4274132.197r_5^{35}-83540302.79r_5^{34}\\&-1100933043r_5^{33}+0.1759214513e11r_5^{32}+0.1965350860e12r_5^{31}+0.1048048526e13r_5^{30}-0.2345420202e14r_5^{29}\\&+0.1519869349e15r_5^{28}-0.2532142007e16r_5^{27}+0.5763041011e16r_5^{26}-0.1262508992e18r_5^{25}+0.1708093186e19r_5^{24}\\&-0.1280226550e20r_5^{23}+0.1277021571e21r_5^{22}-0.1438435067e21r_5^{21}+0.2959635899e22r_5^{20}-0.5268108151e23r_5^{19}\\&+0.7036209479e23r_5^{18}-0.4607232515e24r_5^{17}+0.9532463122e25r_5^{16}-0.1808866634e26r_5^{15}+0.2100680317e27r_5^{14}\\&-0.2058612826e28r_5^{13}+0.6893379360e28r_5^{12}-0.3848331827e29r_5^{11}+0.1749687727e30r_5^{10}-0.3684514761e30r_5^{9}\\&+0.1092764420e31r_5^{8}-0.2122101006e31r_5^{7}-0.1577335438e31r_5^{6}+0.5167452234e31r_5^{5}+0.4545748611e31r_5^{4}\\&-0.9804725966e31r_5^{3}+0.9522001983e31r_5^{2}-0.1610650914e32r_5+0.9994869958e31=0\end{aligned} \tag{19}$$

Here $F_{I/O}$ is the 40th degree I/O polynomial for the forward kinematics problem of the general Stewart mechanism and it is devoid of any extraneous factors. This result is in agreement with the 40th degree polynomial obtained by Husty [10] where Soma coordinates were used to represent rigid-body displacements. However, unlike Husty's formulation where intermediate polynomials of high degree (up to 320) are generated, the hybrid approach directly yields the 40th degree I/O polynomial (Eq. (19)). For some non-constructive proofs regarding I/O polynomial degree, see Mourrain [11], Lazard [12] and Wampler [13]. These proofs are non-constructive because the forward displacement analysis problem was not reduced to a single univariate polynomial; instead the polynomial degree was reported using algebraic techniques.

Two items are worth mentioning at this point. First, it is not necessary that the smallest variable in the *dlex* term order should be hidden to set up the Sylvester's matrix even though it is the case with the example above where r_5 is suppressed. Second, it may be noted that the forty solutions could also have been obtained by solving the generalized eigenvalue problem (Eq. 18) instead of expanding the determinant of the Sylvester's matrix. The reason for expanding the determinant of S to derive the univariate I/O polynomial is that this polynomial contains a wealth of other information about Stewart's mechanism such as its singular and unstable configurations which can be determined by analyzing discriminants of Eq. (19).

The solution for the remaining unknowns proceeds as outlined earlier (see Eq. (5)). For instance, by using the first 67 equations in Eq. (18), it reduces to $S'X'=-S_P$ where S' is a 67×67 matrix formed by deleting the last row and last column of S; X' contains the first 67 elements of X; and S_P contains the first 67 elements of the last column of S. The unknown variables r_1, r_2, r_3, r_4, r_6, r_7, r_8, r_9, l_{x1}, l_{y1}, l_{z1}, h_1, h_2 and h_3 appear linearly in S_P. Since S_P is given in terms of r_5, by solving $S'X'=-S_P$, an explicit expression for each of the 14 unknown variables can be obtained in r_5. Hence, the Stewart mechanism has 40 possible solutions (real and complex) since the I/O polynomial in Eq. (19) is a 40th order polynomial and the remaining unknown variables are linear and expressed explicitly in r_5. It should be noted that since $[R]$ is a direction cosine matrix, the elements (r_i) of this matrix must satisfy the additional constraint $|r_i|\leqslant 1$. Otherwise, the solution is considered to be complex since $\cos^{-1}(r_i>1)$ or $\sin^{-1}(r_i>1)$ leads to complex values.

5 Direct Kinematic Analysis of General Stewart Platform

Consider the general Stewart platform shown in Fig. 3 where all grounded spheric joints are in the XY-plane of the base frame, i.e., $x_{zj}=0$ for $j=2,\ldots,6$. Likewise, all moving spheric joints are assumed to lie in the XY-plane of the moving frame, i.e. $p_{zj}=0$, for $j=2,\ldots,6$. Thus using $x_{zj}=p_{zj}=0$, Eq. (12) reduces to

$$2h_1(p_{xj})+2h_2(p_{yj})+p_j^2+l_1^2+x_j^2-l_j^2-2[(p_{xj})r_1+(p_{yj})r_2]x_{xj} -2[(p_{xj})r_4+(p_{yj})r_5]x_{yj}-2l_{x1}(x_{xj})-2l_{y1}(x_{yj})=0, \quad j=2,\ldots,6 \tag{20}$$

where

$$h_1=l_{x1}r_1+l_{y1}r_4+l_{z1}r_7$$
$$h_2=l_{x1}r_2+l_{y1}r_5+l_{z1}r_8 \tag{21}$$

The length of the vector $\underline{l}_1$ is given as (Eq. (17))

$$l_1^2=l_{x1}^2+l_{y1}^2+l_{z1}^2 \tag{22}$$

Since these equations (Eqs. (20)–(22)) are devoid of r_3, r_6 and r_9, the following equations

$$r_1^2+r_4^2+r_7^2=1$$
$$r_2^2+r_5^2+r_8^2=1 \tag{23}$$
$$r_1r_2+r_4r_5+r_7r_8=0$$

from the dot- and cross products of the orthogonal vectors of the rotational matrix $[R]$ are used. The forward kinematic analysis of the Stewart platform is identical to that of the Stewart mechanism. Thus given the leg lengths l_j, for $j=1,\ldots,6$, determine the position and orientation of the top platform, (Fig. 3). The system of equations, Eqs. (20)–(23) represents 11 equations in 11 unknowns r_1, r_2, r_4, r_5, r_7, r_8, l_{x1}, l_{y1}, l_{z1}, h_1 and h_2. Using the method of Gröbner basis and the dlex term ordering $r_5<_T r_4<_T r_2<_T r_8 <_T r_7<_T r_1<_T h_2<_T h_1<_T l_{z1}<_T l_{y1}<_T l_{x1}$, the reduced Gröbner basis G is derived. The CPU time needed to compute G was 41.29 sec on a P3-600 MHz PC with 64 Mb Ram. The reduced Gröbner basis for this mechanism contains 43 polynomials of which only 15 polynomials in three unknowns r_2, r_4 and r_5 can be used to set up the Sylvester's matrix. Due to space limitations, these polynomials are not included herein but for the numerical data given below, they can be found in Appendix G of Almadi [9].

$p_{x1}=0$	$p_{y1}=0$	$x_{x1}=0$	$x_{y1}=0$	$l_1=12$
$p_{x2}=2$	$p_{y2}=3$	$x_{x2}=2$	$x_{y2}=3$	$l_2=12$
$p_{x3}=3$	$p_{y3}=5$	$x_{x3}=3$	$x_{y3}=4$	$l_3=10$
$p_{x4}=1$	$p_{y4}=0$	$x_{x4}=5$	$x_{y4}=1$	$l_4=14$
$p_{x5}=4$	$p_{y5}=2$	$x_{x5}=0$	$x_{y5}=2$	$l_5=12$
$p_{x6}=2$	$p_{y6}=1$	$x_{x6}=4$	$x_{y6}=0$	$l_6=10$

With the unknown r_5 suppressed, the 15 polynomials, given as g_i for $i=1$, 4, 7, 9, 10, 11, 15, 16, 19, 20, 32, 34, 35, 41 and 43, can be viewed as a linear system of 15 polynomials in 15 unknowns r_2^4, $r_2^3r_4$, $r_4^2r_2^2$, $r_4^3r_2$, r_4^4, r_2^3, $r_2^2r_4$, $r_4^2r_2$, r_4^3, r_2^2, r_2r_4, r_4^2, r_2, r_4 and 1, with their coefficients expressed in terms of r_5. The determinant of the 15×15 Sylvester's matrix leads to the following 20th degree I/O polynomial.

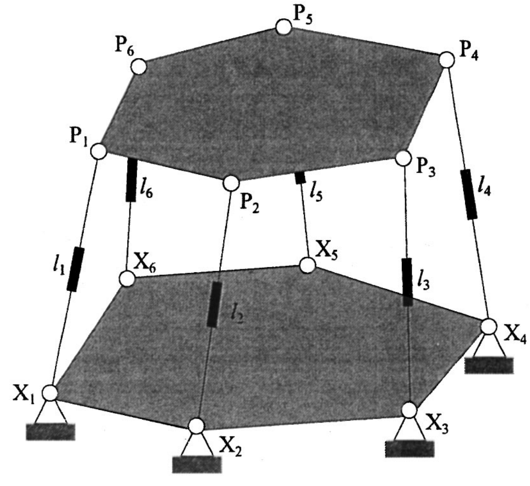

Fig. 3 A 6-DOF general Stewart platform

$$F_{I/O}=r_5^{20}+114.3340390r_5^{19}-1979.590755r_5^{18}-157556.6167r_5^{17}-2598196.663r_5^{16}+120040087.2r_5^{15}-931649530.1r_5^{14}$$
$$+0.6390674695e11r_5^{13}+0.8842245138e12r_5^{12}-0.2662428208e14r_5^{11}-0.7236977977e13r_5^{10}+0.1276963861e16r_5^{9}$$
$$-0.1434754786e16r_5^{8}+0.1317590548e17r_5^{7}-0.7865134521e17r_5^{6}+0.2257516769e18r_5^{5}-0.4413968688e18r_5^{4}$$
$$+0.5740470384e18r_5^{3}-0.4359031655e18r_5^{2}+0.1704868635e18r_5-0.2639182255e17=0 \quad (24)$$

The CPU time needed to derive this polynomial is 19.737 sec. For this problem, depending on the term order, the CPU times needed to derive G range anywhere from 30–75 seconds with an additional 13–35 seconds needed to expand the determinant to obtain the univariate resultant.

The solution for the remaining unknown variables is obtained as follows. By deleting the last row and column of Sylvester's matrix and treating the last column of Sylvester's matrix as a known column matrix, yields the following system of equations:

$$\begin{bmatrix} r_2^4 \\ \vdots \\ r_2 \\ r_4 \end{bmatrix} = \begin{bmatrix} 14\times 14 \\ a_{ij}\in k[r_5] \end{bmatrix}^{-1} \begin{bmatrix} -s_{1,15} \\ \vdots \\ -s_{13,15} \\ -s_{14,15} \end{bmatrix} \quad (25)$$

The unknown variables r_2 and r_4 appear linearly in the left-hand side of Eq. (25). The right-hand side of Eq. (25) only contains variable r_5. Upon expanding Eq. (25), explicit expressions for r_2 and r_4 can be derived in terms of r_5. With the unknown variables r_2 and r_4 uniquely defined for each of the 20 solutions of r_5 in Eq. (24), the polynomials $g_{12}\in[r_1,r_2,r_4,r_5]$, $g_{25}\in k[r_2,r_4,r_5,l_{x1}]$, $g_{26}\in k[r_2,r_4,r_5,h_2]$, $g_{27}\in k[r_2,r_4,r_5,l_{y1}]$ and $g_{28}\in k[r_2,r_4,r_5,h_1]$ are linear in r_1, l_{x1}, h_2, l_{y1} and h_1 respectively. Hence, each of these polynomials yield a linear expression for r_1, l_{x1}, h_2, l_{y1} and h_1 respectively, in terms of r_2, r_4 and r_5. Similarly the polynomials $g_{18}\in k[r_2,r_5,r_8]$, $g_{24}\in k[r_2,r_4,r_5,r_7]$ and $g_{30}\in k[r_2,r_4,r_5,l_{z1}]$ are of 2nd order in r_8, r_7 and l_{z1} respectively.

$$g_{18}=a_{18}r_8^2+b_{18},\quad a_{18}\in k,\quad b_{18}\in k[r_2,r_5]$$
$$g_{24}=a_{24}r_7^2+b_{24},\quad a_{24}\in k,\quad b_{24}\in k[r_2,r_4,r_5] \quad (26)$$
$$g_{30}=a_{30}l_{z1}^2+b_{30},\quad a_{30}\in k,\quad b_{30}\in k[r_2,r_4,r_5]$$

Here any one of the above generators can be used to solve for one unknown. For instance if g_{30} is used to solve for l_{z1} with $l_{z1}=\pm\sqrt{-b_{30}/a_{30}}$, we can solve for r_8 and r_7 using generators g_{14} and g_{21} which are linear in r_8 and r_7 respectively.

$$g_{14}=r_7r_8-4.9167r_2^2-6.9167r_2r_4-9.75r_2r_5+15.5r_2+r_4r_5,$$
$$g_{21}=l_{z1}r_7-32.778r_2^2-107.222r_2r_4-136.29167r_2r_5+208.958r_2$$
$$-90.444r_4^2-226.2917r_4r_5+322.4583r_4-141.375r_5^2$$
$$+432.625r_5-327.25,$$

Since Eqs. (24) and (25) yield a total of 20 sets of solutions for r_2, r_4 and r_5, g_{12}, g_{25}, g_{26}, g_{27}, g_{28}, yield unique values for r_1, l_{x1}, h_2, l_{y1}, h_1 respectively, g_{30} leads to two values for l_{z1}, and g_{14}, g_{21} yield unique values for r_8 and r_7 respectively, the Stewart platform has a total of 40 (real and complex) solutions. This result is in agreement with the 40th degree I/O polynomial for the Stewart platform predicted by Raghavan [14] using polynomial continuation and a closed-form solution derived by Zhang and Song [15]. However, unlike Zhang and Song's approach where 21 equations were used to derive the 20th degree I/O polynomial, it is shown herein that this polynomial can be obtained using only 15 equations. Eq. (24) can also be derived by using an even smaller 13×13 Sylvester's matrix obtained by using the term order $h_1<_T h_2<_T r_8<_T r_7<_T r_5<_T r_4<_T r_2<_T r_1<_T l_{z1}<_T l_{y1}<_T l_{x1}$.

6 8-Link 1-DOF Planar Mechanism

Consider the 8-link mechanism shown in Fig. 4 with link 5 as the ground. This mechanism does not contain any 4-link loops. The input is provided to link 3 and link 1 is the output link. For this mechanism with three independent loops, namely OABCD, OEFGCD and OEHID, the loop-closure equations are

$$r_3\cos(\theta_3)+r_6\cos(\theta_6)=r_5+r_{1a}\cos(\theta_1+\beta)+r_{2a}\cos(\theta_2+\delta) \quad (27)$$
$$r_3\sin(\theta_3)+r_6\sin(\theta_6)=r_{1a}\sin(\theta_1+\beta)+r_{2a}\sin(\theta_2+\delta)$$

$$r_{3a}\cos(\theta_3+\alpha)+r_4\cos(\theta_4)+r_8\cos(\theta_8)$$
$$=r_5+r_{1a}\cos(\theta_1+\beta)+r_2\cos(\theta_2) \quad (28)$$
$$r_{3a}\sin(\theta_3+\alpha)+r_4\sin(\theta_4)+r_8\sin(\theta_8)$$
$$=r_{1a}\sin(\theta_1+\beta)+r_2\sin(\theta_2)$$

$$r_{3a}\cos(\theta_3+\alpha)+r_{4a}\cos(\theta_4+\gamma)+r_7\cos(\theta_7)=r_5+r_1\cos(\theta_1) \quad (29)$$
$$r_{3a}\sin(\theta_3+\alpha)+r_{4a}\sin(\theta_4+\gamma)+r_7\sin(\theta_7)=r_1\sin(\theta_1)$$

In Eqs. (27)–(29), r_i, θ_3, α, β, γ and δ are known and θ_1, θ_2, θ_4, θ_6, θ_7 and θ_8 are unknown. If $\cos(\theta_j)$ and $\sin(\theta_j)$ are treated as algebraic variables with $x_j=\cos(\theta_j)$ and $y_j=\sin(\theta_j)$, then Eqs. (27)–(29) and the trigonometric identity $\cos^2(\theta_j)+\sin^2(\theta_j)=1$ for $j=1, 2, 4, 6, 7, 8$, yield a system of 12 algebraic equations in 12 unknowns. Using the *dlex* term ordering with $x_j<_T y_j<_T \ldots$ for $j=1, 8, 7, 6, 4, 2$ and the method of Gröbner bases, the above system of 12 equations leads to the reduced Gröbner basis $G=\{g_1,\ldots,g_{29}\}$. Due to space limitations, these polynomials are not reported here, but for the numerical data given below, the 29 polynomials are given in Appendix G of Almadi [9].

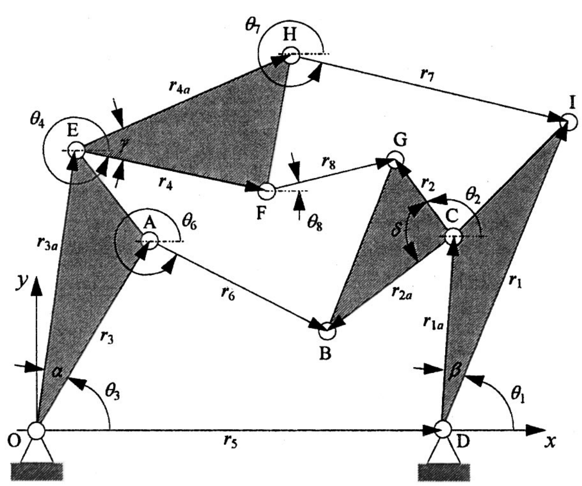

Fig. 4 A general 8-link 1-DOF mechanism

$r_1=13.00$	$r_2=6.90$	$r_3=5.50$	$r_4=7.00$	$r_5=15.00$
$r_6=4.70$	$r_7=23.50$	$r_8=6.00$	$r_{1a}=3.50$	$r_{2a}=2.50$
$r_{3a}=10.00$	$r_{4a}=7.50$	$\alpha=\pi/2$	$\beta=\pi/2$	$\delta=\pi/2$
$\gamma=\pi/2$	$\theta_3=21.00°$			

The polynomials in G can be viewed as a system of 29 equations in 29 unknowns x_8^3, $x_8^2 y_1$, y_7^2, $x_7 y_7$, x_7^2, $y_8 y_7$, $x_7 y_8$, y_8^2, $y_7 x_8$, $x_8 x_7$, $y_8 x_8$, x_8^2, $y_7 y_1$, $y_1 x_7$, $y_1 y_8$, $x_8 y_1$, y_1^2, y_2, x_2, y_4, x_4, y_6, x_6, y_7, x_7, y_8, x_8, y_1 and 1, with coefficients $a_{ij} \in k[x_1]$. The determinant of 29×29 Sylvester's (coefficient) matrix yields the 16th order I/O polynomial in $x_1(=\cos(\theta_1))$:

$$
\begin{aligned}
F_{\mathrm{I/O}} = {} & x_1^{16} - 2.300807210x_1^{15} + 9.154401868x_1^{14} + 0.1203191792x_1^{13} \\
& - 10.46924935x_1^{12} + 31.75664714x_1^{11} - 11.06912500x_1^{10} \\
& - 8.465327682x_1^{9} + 50.66174232x_1^{8} - 26.34720696x_1^{7} \\
& - 2.852887181x_1^{6} + 30.78856558x_1^{5} - 25.56451701x_1^{4} \\
& + 3.960569625x_1^{3} + 10.28770088x_1^{2} - 7.132207031x_1 \\
& + 1.232747445 = 0
\end{aligned}
$$

Here the entire set of generators (polynomials) in G are needed to set up the Sylvester's matrix because all links fully participate in the mechanism motion. Hence there exists no subset of G that can be used to derive the I/O polynomial since no decoupling in the generators of G is possible. The solution for the remaining unknowns proceeds as outlined for Stewart's mechanism and platform. Since all unknown variables y_1, x_j, y_j, for $j=2, 4, 6, 7, 8$, appear linearly in the monomials column matrix, each of the 11 unknown variables can be expressed linearly in terms of x_1. Hence, this eight-link mechanism has a total of 16 possible assembly configurations (real and complex). This result is in agreement with the 16th degree I/O polynomial obtained by Innocenti [16] and Almadi et al. [17]. The formulation by Innocenti results in an 18th degree polynomial containing two spurious roots which are factored out giving the correct 16th degree polynomial. The successive elimination approach used by Almadi et al. [17] obtains the 16th degree I/O polynomial (directly) devoid of any spurious roots.

7 Conclusions

For synthesis and analysis problems arising in kinematics, the Sylvester's matrix is normally constructed by generating as many equations as there are monomials. This is accomplished by multiplying the polynomials in the given system of equations F by certain classes of monomials until the number of equations in the new system of equations H is equal to the number of monomials in H. This allows H to be treated as a system of linear equations in the monomials in H. However, there is no guarantee that the monomial ideal of F is uniquely defined by the system of polynomials in H. If ideal $(F) \neq \mathrm{ideal}(H)$, spurious factors may be present in the resultant. This has been a major problem with Sylvester's method as discussed by Roth [6].

The Gröbner-Sylvester hybrid method provides an alternate approach for solving a system of algebraic equations, which appears to overcome this difficulty. It combines the method of Gröbner bases under dlex term ordering and Sylvester's Dialytic elimination method. The method of Gröbner bases, through the S-polynomial algorithm, allows one to generate new polynomials and the uniqueness of the newly generated polynomial w.r.t. to the monomial ideal (of F) is guaranteed through the normal form reduction algorithm, i.e., whenever a new polynomial is generated using the S-polynomial algorithm, the normal form reduction algorithm allows one to determine whether or not the new polynomial is reducible to zero using other polynomials. Since ideal $(F)=\mathrm{ideal}(G)$ throughout the solution process, this insures that vanishing of the resultant, given as the determinant of Sylvester's matrix, yields the necessary condition for the polynomials in G (as well as F) to have a common factor.

The application of the proposed method is rather straightforward and it is demonstrated through kinematic analysis of a number of mechanisms. Using the Gröbner-Sylvester hybrid method, a 40th degree I/O polynomial is derived for the forward kinematic analysis of the general Stewart mechanism. In the case of Stewart platform, the forward kinematics problem resulted in the 20th degree (40th degree counting all solutions including mirror images) I/O polynomial. The applicability of the hybrid approach is also demonstrated through the derivation of the 16th degree I/O polynomial for an 8-link 1-DOF mechanism that does not contain any 4-link loop. These three examples demonstrate that the proposed approach can be successfully used to perform closed-form displacement analysis of planar and spatial mechanisms devoid of any extraneous roots.

References

[1] Morgan, A. P., 1987, *Solving Polynomial Systems Using Continuation for Engineering and Scientific Problems*, Prentice Hall, New Jersey.
[2] van Waerden, B. L., 1991, *Algebra*, Vols. 1 and 2, Springer Verlag, New York.
[3] Buchberger, B., 1985, ''Gröbner Bases: An Algorithmic Methods in Polynomial Ideal Theory,'' in Multidimensional Systems Theory, N. K. Bose (ed), D. Reidel Publishing Company, Holland, pp. 184–232.
[4] Macaulay, F. S., 1994, *The Algebraic Theory of Modular Systems*, Cambridge University Press, New York.
[5] Mishra, B., 1993, *Algorithmic Algebra*, Springer Verlag, New York.
[6] Roth, B., 1994, ''Computational Advances in Robot Kinematics,'' in Advances in *Robot Kinematics and Computational Geometry*, J. Lenarcic and B. Ravani (eds), Kluwer Academic Publishers, pp. 7–16.
[7] Raghavan, M., and Roth, B., 1995, ''Solving Polynomial Systems for the Kinematic Analysis and Synthesis of Mechanisms and Robot Manipulators,'' Special 50th Anniv. Issue of the ASME J. Mech. Des., **117**, pp. 71–79.
[8] Dhingra, A. K., Kohli, D., and Xu, Y. X., 1992, ''Direct Kinematics of General Stewart Platforms,'' in *Robotics, Spatial Mechanisms and Mechanical Systems*, DE-Vol. 45, pp. 107–112.
[9] Almadi, A. N., 1996, ''On New Foundations of Kinematics Using Classical and Modern Algebraic Theory and Homotopy,'' Ph.D. Thesis, University of Wisconsin-Milwaukee.
[10] Husty, M. L., 1996, ''An Algorithm for Solving the Direct Kinematics of General Stewart-Gough Platforms,'' Mech. Mach. Theory, **31**, pp. 365–380.
[11] Mourrain, B., 1993, ''The 40 Generic Positions of a Parallel Robot,'' in Proc. of the ACM-Intl. Symp. on Symbolic and Algebraic Computation, Kiev, Ukraine, pp. 173–182.
[12] Lazard, D., 1993, ''On the Representation of Rigid-Body Motions and its Applications to Generalized Platform Manipulators,'' in *Computational Kinematics*, J. Angeles et al. (eds), pp. 175–181.
[13] Wampler, C. W., 1996, ''Forward Displacement Analysis of General Six-in-Parallel SPS (Stewart) Platform Manipulators Using Soma Coordinates,'' Mech. Mach. Theory, **31**, pp. 331–337.
[14] Raghavan, M., 1993, ''The Stewart Platform of General Geometry Has 40 Configurations,'' ASME J. Mech. Des., **115**, pp. 277–282.
[15] Zhang, C., and Song, S. M., 1994, ''Forward Displacement Analysis of Nearly General Stewart Platforms,'' ASME J. Mech. Des., **116**, pp. 54–60.
[16] Innocenti, C., 1994, ''Analytical-Form Position Analysis of the 7-link Assur Kinematic Chain with Four Serially Connected Ternary Links,'' ASME J. Mech. Des., **116**, pp. 622–628.
[17] Almadi, A. N., Dhingra, A. K., and Kohli, D., 1996, ''Closed-Form Displacement Analysis of SDOF 8 Link Mechanisms,'' in Proc. of ASME 1996 Design Engineering Technical Conf., Paper No. 96-DETC/MECH-1206.