Some Generic Results on Algebraic Observability and Connections with Realization Theory

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Abstract. We analyze Glad/Fliess algebraic observability for polynomial control systems from a commutative algebraic/algebro-geometric point of view. Furthermore we show some relations between algebraic observability and realization theory for polynomial differential equations. Most issues are treated in a constructive framework, and we use Gröbner bases for performing elimination.

Keywords: nonlinear control systems, observability, realization theory, Gröbner bases, elimination theory, commutative algebra, differential algebra, genericity

1 Introduction

Algebraic observability has been introduced in a differential algebraic setting by Glad, Fliess and Diop [6, 7, 13]. It was also discussed for discrete time systems, from a slightly different perspective, in [10]. It should be noted that the term has also been used by Sontag et al. in a different meaning [25]. The concept applies for nonlinear systems in which all nonlinearities are polynomial and, as explained in [7], it basically agrees with the standard definitions of observability given in e.g. [14, 20]. Some basic results in this direction were also presented in [8, chapter 5].

It is the purpose of this paper to

- give some additional aspects of algebraic observability in the light of Gröbner base theory.
- indicate how the observer relations can be used to obtain information on alternative state space realizations of the same input-output equation and inequations that may have to be added to the i/o-equation.

Theorems 3.4 and 3.3 seem to be new, as do the proofs of theorems 3.1 and 3.2. The methods suggested section 4 also appear to be new, though the results of that section are still rather preliminary. Some of the results are not new, merely restated in a language different from the one of differential algebra. The author firmly believes that differential algebra is a natural and appropriate framework for dealing with polynomial control systems, both from a theoretical and constructive basis. However, according to the golden rule of science known as Occam's razor, one should never use more machinery than necessary to obtain desired results and in this spirit the reformulation might have an interest of its own.

We suppose that the reader is familiar with some basic concepts from commutative and differential algebra, such as polynomial ring, ideal, prime ideal, height and dimension of an ideal, quotient ring, field of quotients, transcendence degree and differential field. Some references are [1, 17, 24, 18, 8].

The following notation is used:

- The ideal generated by the set F is written $\langle F \rangle$.
- We use the abbreviations

$$\bar{x} = x_1, \ldots, x_n, \qquad \bar{y} = y_0, \ldots, y_{n-1} \tag{1}$$

  so that e.g. $k(\bar{y})[\bar{x}] = k(y_0, \ldots, y_{n-1})[x_1, \ldots, x_n]$.
- The field of quotients of an integral domain R is denoted $Q(R)$.
- For the variable y we use subscript to denote time-derivative, as is made in e.g. [22]:

$$y_i = \frac{d}{dt} y(t) \tag{2}$$

- For a prime ideal $\mathfrak{p} \subset k[x_1, \ldots, x_n]$ the dimension of $\mathfrak{p}$ is the number

$$\dim \mathfrak{p} = \operatorname{trdeg}_k Q(k[x_1, \ldots, x_n]/\mathfrak{p}) \tag{3}$$

  Some refer to this as the coheight of $\mathfrak{p}$. It is proved in e.g. [16] and [18] that this number is equal to the length of the longest ascending chain of prime ideals starting with $\mathfrak{p}$.

2 Elimination Theory

Elimination consists in finding the elements of an ideal that belong to a certain subring. For example we may wish to find $\mathfrak{a} \cap k[X]$ for an ideal $\mathfrak{a} \subseteq k[X, Y]$ where X, Y are sets of variables. One way of doing elimination is to compute a so called Gröbner base, or standard base, of an ideal w.r.t. the purely lexicographic term-ordering (plex). The Gröbner base (GB) for an ideal is a subset of it having some nice properties; see e.g. theorems 2.1 and 2.2 and formula (5) for some of them. For an introduction to GB see e.g. [21, 11, 2, 8]. The computer algebra programs Maple, Reduce and Macsyma all have packages for computing GB:s.

The GB for an ideal is unique, under some natural assumptions [11], so we write $GB(\mathfrak{a})$ for the GB of $\mathfrak{a}$. The following two theorems on GB are of special interest in this paper. Recall that a polynomial in $k[x_1, \ldots, x_n]$ is *regular* w.r.t. x_i if the leading term is in $k[x_i]$ for a plex term-ordering ranking x_i highest.

Theorem 2.1 *Let $\mathfrak{a}$ be an ideal in $k[x_1, \ldots, x_n]$. If $\mathbf{G} = GB(\mathfrak{a})$ w.r.t. plex then $\dim \mathfrak{a} = 0$ iff for all i there is a $p \in \mathbf{G}$ such that p is regular w.r.t. x_i.*

Proof. See [11] or [2, page 209]. □

Theorem 2.2 *Let k be any field of characteristic zero. A plex GB for a prime zero-dimensional ideal in $k[x_1, \ldots, x_n]$ has n elements.*

Proof. See [11]. $\square$

This means that a plex-GB for a prime zero-dimensional ideal w.r.t. the variable ranking $x_1 \prec \ldots \prec x_n$ looks as follows:

$$\begin{cases} x_n^{q_n} - p_n(x_1, \ldots, x_n) \\ x_{n-1}^{q_{n-1}} - p_{n-1}(x_1, \ldots, x_{n-1}) \\ \vdots \\ x_2^{q_2} - p_2(x_1, x_2) \\ x_1^{q_1} - p_1(x_1) \end{cases} \tag{4}$$

where $\forall i: \deg(p_i, x_i) < q_i$. This is particularly interesting in the case when k is not a number field and the ideal in question is prime, because in this case we can use lemma 3.2 below to adjoin variables to the coefficient ring until the ideal becomes zero-dimensional. We can compare the GB to a Ritt characteristic set and easily see that the GB is actually a characteristic set after such simple manipulations. This observation is also made in [4].

We say that a property is *generic* on a space if it holds on a dense subset of the space in question.

Theorem 2.3 *Let $\mathbf{G}$ be a plex GB for a prime zero-dimensional ideal in $k[x_1, \ldots, x_n]$. Generically the $n-1$ first elements of $\mathbf{G}$ are linear in the leading variable.*

Proof. Proved in [11]. $\square$

Here the term generic refers to the coefficients of the polynomials that generate the ideal. Thus the *generic* look of a plex-GB for a zero-dimensional prime ideal w.r.t. the variable ranking $x_1 \prec \ldots \prec x_n$ is

$$\begin{cases} x_n - p_n(x_1) \\ \vdots \\ x_2 - p_2(x_1) \\ p_1(x_1) \end{cases} \tag{5}$$

where $\forall i: \ p_i \in k[x_1]$ and $\deg p_1 > \deg p_i$ for $i = 2, \ldots, n$ (note that this is not entirely correctly stated in [11]).

3 Algebraic Observability

Consider a polynomial system in state space form:

$$\dot{x}(t) = f(x(t)), \quad y(t) = h(x(t)) \tag{6}$$

where f is a polynomial mapping $\mathbb{R}^n \to \mathbb{R}^n$ and h a polynomial output map $\mathbb{R}^n \to \mathbb{R}$; i.e.

$$h \in k[x_1, \ldots, x_n], \qquad \forall i: \quad f_i \in k[x_1, \ldots, x_n] \tag{7}$$

Thus there is no input signal so far. Most of the analysis below goes through in the case of an input signal as well, though; just change the coefficient field k to include $u, \dot{u}, \ldots, u^{(\nu)}$ for some ν large enough. Some of the additional difficulties arising when inputs are added are treated in [7].

Definition 3.1 The *Lie derivative* w.r.t. f is the differential operator $L_f = \sum_{i=1}^{n} f_i \frac{\partial}{\partial x_i}$. □

One of our main objects of study in this paper will be the ideal

$$\mathbf{L}_f^j h = \langle\, y_0 - h,\, y_1 - L_f h,\, \ldots,\, y_j - L_f^j h\,\rangle \tag{8}$$

for different values of j. The generators of the ideal are thus some j polynomials $L_f^i h$, each one associated to a "tag-variable" y_i. The technique of tag-variables is commonly used in constructive commutative algebra; see e.g. [23, 11]. In the current context, we of course think of the *subscript of* y *as denoting time-derivative*:

$$y_i = \frac{d}{dt} y(t)$$

Let us state some important properties of the ideal $\mathbf{L}_f^j h$. Recall the abbreviations (1).

Lemma 3.1 *We have that* $(\mathbf{L}_f^j h) \cap k[\bar{x}] = 0$ *where* $\mathbf{L}_f^j h$ *is defined by formula (8).*

Proof. Write $a_i = y_i - L_f^i h$. Suppose that $s \in k[\bar{x}] \cap \mathbf{L}_f^j h$. By definition

$$s = \sum_i \alpha_i\, a_i, \qquad \alpha_i \in k[\bar{x}] \tag{9}$$

If $s \neq 0$ then we choose a value $\xi \in k^n$ of $\bar{x}$ such that $s(\xi) \neq 0$. Then, for each i, we choose $y_i = f_i(\xi)$. This implies that the right hand side of (9) is zero, and we get a contradiction. □

Theorem 3.1 *The ideal* $\mathbf{L}_f^j h$ *is prime for all* j.

Proof. We prove that $A = k[\bar{x}, y_0, \ldots, y_j]/(\,\mathbf{L}_f^j h)$ is an integral domain. Suppose that $pq = 0$ in A. According to lemma 3.1 it is impossible that both p and q are in $k[\bar{x}]$. Suppose p contains some y_i. Then we use the relation $a_i = 0$ to replace y_i with an expression in $\bar{x}$. In this way we can eliminate all y_i from p, and get a contradiction. □

The above theorem can also be proved using Ritt's characterization of prime ideals via characteristic sets [22, page 89], as is done in [12].

Now we only need the following theorem in order to state some facts about algebraic observability (the theorem is also used in e.g. [11]):

Lemma 3.2 *Let* R *be an integral domain and* x *an indeterminate. There is a one-to-one correspondence between the prime ideals* $\mathfrak{p}$ *of* $R[x]$ *such that* $R \cap \mathfrak{p} = 0$ *and the prime ideals of* $Q(R)[x]$.

Proof. See [15, Theorem 36]. Corollary 1 to theorem 16 in chapter IV of [26] is a slightly more general formulation of this theorem. □

Lemma 3.3 *We have that* $\dim(\mathbf{L}_f^{n-1}h) = n$.

Proof. Since, for any prime ideal $\mathfrak{p}$ in $k[\bar{x},\bar{y}]$ we have

$$\dim \mathfrak{p} + \operatorname{ht} \mathfrak{p} = \dim k[\bar{x},\bar{y}] = 2n \tag{10}$$

where $\operatorname{ht}\mathfrak{p}$ denotes the height of $\mathfrak{p}$ it is necessary and sufficient that $\operatorname{ht}(\mathbf{L}_f^{n-1}h) = n$. Now

$$\mathbf{L}_f^{n-1}h \supset \ldots \supset \mathbf{L}_f^1 h \supset \mathbf{L}_f^0 h \supset 0 \tag{11}$$

is a strictly descending chain of prime ideals according to theorem 3.1. Thus $\operatorname{ht}(\mathbf{L}_f^{n-1}h) \geq n$. The generalized Krull's Hauptidealsatz (see [16, theorem 3.4] or [18, theorem 13.5]) states that the height of a prime ideal is less than or equal to the minimal number of generators of the ideal, so $\operatorname{ht}(\mathbf{L}_f^{n-1}h) \leq n$ and we have equality. □

Definition 3.2 The system (6) is algebraically observable if each x_i is algebraic over the differential field $k\langle y\rangle$. □

This just means that the system is algebraically observable if for each x_i there is an r such that there is a polynomial

$$a_i \in k[x_i, y_0, \ldots, y_r], \quad \deg(a_i, x_i) > 0 \tag{12}$$

We call this a_i, if it exists, an *observer relation* for x_i.

The lemma below serves to simplify the concept of algebraic observability a little. To prove it we need the elementary fact

$$\operatorname{trdeg} L/k = \operatorname{trdeg} L/K + \operatorname{trdeg} K/k \tag{13}$$

for an arbitrary tower of field extensions $L \supset K \supset k$ (see e.g. [24, theorem 12.56]).

Lemma 3.4 *The system (6) is algebraically observable iff each* x_i *is algebraic over* $k(\bar{y})$.

Proof. Let $r \in \mathbb{N}$ be arbitrary. It is easy to prove (done in e.g. [8]) that the extension $k(y_0,\ldots,y_r)/k(\bar{y})$ is algebraic. It is a direct consequence of formula (13) the extension $k(x_i, y_0,\ldots,y_r)/k(y_0,\ldots,y_r)$ is algebraic iff $k(x_i,y_0,\ldots,y_{n-1})/k(\bar{y})$ is. □

The following theorem was first proved by Glad in [13]. The proof given here differs substantially from the original one.

Theorem 3.2 *The system (6) is algebraically observable iff*

$$(\mathbf{L}_f^{n-1}h) \cap k[\bar{y}] = 0$$

Proof. Consider the two fields

$$k(\bar{x},\bar{y}) = Q(k[\bar{x},\bar{y}]/(\mathbf{L}_f^{n-1}h)) \tag{14}$$

and

$$k(\bar{y}) = Q(k[\bar{y}]/(k[\bar{y}] \cap (\mathbf{L}_f^{n-1}h))) \tag{15}$$

If $(\mathbf{L}_f^{n-1}h) \cap k[\bar{y}] = 0$ then $k(\bar{y})$ is a purely transcendental extension of k, of course. The system is algebraically observable iff $\operatorname{trdeg} k(\bar{x}, \bar{y})/k(\bar{y}) = 0$ according to lemma 3.4. Formula (13) gives that

$$\operatorname{trdeg} k(\bar{x}, \bar{y})/k \;=\; \operatorname{trdeg} k(\bar{x}, \bar{y})/k(\bar{y}) + \operatorname{trdeg} k(\bar{y})/k \tag{16}$$

Lemma 3.3 implies that $\operatorname{trdeg} k(\bar{x}, \bar{y})/k = n$. But this means that we have algebraic observability iff $\operatorname{trdeg} k(\bar{y})/k = n$. □

Thus we have a simple test for algebraic observability: we just check if there is a relation between $y_0, \ldots, y_{n-1}$ using elimination theory.

To be able to use the results on zero-dimensional GB:s we will now change the ground field using theorem 3.2. In other words we will make embedddings of the type

$$\varphi : \quad k[\bar{x}, \bar{y}] \;\hookrightarrow\; k(\bar{y})[\bar{x}] \tag{17}$$

The extension of a prime ideal $\mathfrak{p}$ in $k[\bar{x}, \bar{y}]$ to $k(\bar{y})[\bar{x}]$ is

$$\mathfrak{p}^e \;=\; \varphi(\mathfrak{p})\; k(\bar{y})[\bar{x}] \tag{18}$$

For an arbitrary $\mathfrak{p}$ the extension $\mathfrak{p}^e$ is not necessarily prime (see e.g. [1]), but the ideals that are of interest to us turn out to have prime extensions.

Theorem 3.3 *If the system (6) is algebraically observable, then $(\mathbf{L}_f^{n-1}h)^e$ a zero-dimensional prime ideal in $k(\bar{y})[\bar{x}]$ and $(\mathbf{L}_f^{n}h)^e$ is zero-dimensional prime in $k(\bar{y})[\bar{x}, y_n]$.*

Proof. According to theorem 3.1 $\mathbf{L}_f^{n-1}h$ is a prime ideal in $k[\bar{x}, \bar{y}]$. Theorem 3.2 assures that $\mathbf{L}_f^{n-1}h$ contracts to zero in $k[\bar{y}]$, so lemma 3.2 implies that $(\mathbf{L}_f^{n-1}h)^e$ is prime in $k(\bar{y})[\bar{x}]$. Lemma 3.3 implies that the dimension is zero, since we have decreased the dimension of the ring to n by adjoining variables to the ground field. The situation is completely analogous for $\mathbf{L}_f^{n}h$ and $(\mathbf{L}_f^{n}h)^e$. □

Theorem 3.3 is not only of theoretical interest, it also has practical consequences if we wish to compute with $\mathbf{L}_f^{n}h$ in a computer algebra system such as Maple [3]: computations are more efficient when made in a ring of fewer variables ($K[\bar{x}, y_n]$ compared to $k[y_0, \ldots, y_n, \bar{x}]$). This observation was used by the author in the Maple package POLYCON [9].

It now follows from theorem 2.2 that for an algebraically observable system there is only one observer relation for each variable, in principle.

Theorem 3.4 *Suppose that (6) is a generic algebraically observable system. Then, for all i, x_i is a rational function of $y_0, \ldots, y_n$ where subindices denote time derivatives of y.*

Proof. According to theorem 3.3 the result follows as we apply theorem 2.3 to $(\mathbf{L}_f^{n-1}h)^e$. □

So, even if $\mathbf{L}_f^{n-1}h$ always contains observer relations for all state variables if the system is algebraically observable, the chance of getting a relation linear in x_i is much greater if we compute in $\mathbf{L}_f^{n}h$ instead.

We conclude this section by restating the main results in terms of morphisms, as defined in algebraic geometry (see e.g. [19]).

Define the observability morphism $H : \mathbb{C}^n \to \mathbb{C}^n$ to be the mapping

$$H: \quad (x_1, \dots, x_n) \mapsto (h(\bar{x}), (L_f h)(\bar{x}), \dots, (L_f^{n-1} h)(\bar{x})) \tag{19}$$

Now theorem 3.2 can be reformulated as:

The system (6) is algebraically observable iff H is dominant.

According to [19, page 71] the fibres of a dominant morphism are generically finite, i.e. there is a dense set $U \subseteq \mathbb{C}^n$ such that

$$\forall y \in U: \quad \dim H^{-1}(y) = 0 \tag{20}$$

4 Connections with Realization Theory

This section contains some preliminary results on how the observer relations described in the previous section can be used to obtain some information on

- alternative state space realizations of the same input-output equation and observer relations.
- inequations that may have to be added to the i/o-equation in order to get equivalence between solutions.

The mathematical rigor of this section is not yet as satisfactory as that of the preceding one, but the results seem interesting enough to deserve being mentioned in this context.

The basic idea is to differentiate the observer relations in order to retrieve the state equations by elimination. This is not interesting in the case where the elimination only renders a single equation that tells us that $\dot{x}_i = f_i$, which we already knew.

It becomes more interesting when the ideal considered is non-prime, for then we may get several suggestions for state equations and also some expressions in $\bar{x}$ only that give conditions on the states. Let us first describe the method suggested in general terms and then consider some simple examples:

Suppose that the system considered is algebraically observable. Starting with $\mathbf{L}_f^{n-1} h$ we

1. Find a differential equation $p = 0$ for y, for example by elimination in $(\mathbf{L}_f^n h)^e$ as described in [8]
2. Compute a plex Gröbner base $\mathbf{G}$ for $(\mathbf{L}_f^{n-1} h)^e$ ranking x_i, say, lowest. This gives an observer relation $a_i(x_i, \bar{y}) = 0$ for x_i according to the previous section. and implemented in [9].
3. Differentiate a_i w.r.t. time, to get $\mathrm{d}a_i \in k[\bar{y}, y_n, x_i, \mathrm{d}x_i]$.
4. Repeat steps 2 and 3 for each i in order to get $a_1, \dots, a_n, \mathrm{d}a_1, \dots, \mathrm{d}a_n$
5. Compute Gröbner bases for the ideal

$$\mathfrak{a} = (\mathbf{L}_f^{n-1} h) + \langle p, \mathrm{d}a_1, \dots, \mathrm{d}a_n \rangle \tag{21}$$

   in $k[\bar{y}, y_n, \bar{x}, \mathrm{d}x_1, \dots, \mathrm{d}x_n]$ w.r.t. rankings such that the y_i and all $\mathrm{d}x_i$ except one are eliminated. This gives one or several expressions for $\mathrm{d}x_i$ in terms of $\bar{x}$.

Summarizing we get an answer to the question: "Which state space realizations have the prescribed input-output equation and observer relations?" and "Do we have to put constraints on the state variables for equivalence to hold?"

The reader should please keep in mind that it has not yet been proved that the procedure always gives all the information wanted. The weak spot is clearly step 5: Is it always possible to eliminate the variables mentioned? This question is subject to ongoing research.

The following examples illustrate on simple systems how the method outlined is intended to work.

Example 4.1 (Inspired by example 2 in [5])
Consider the simple system

$$\dot{x} = x, \quad y = x^2 + x \tag{22}$$

If we compute a Gröbner base for the ideal

$$\mathbf{L}_f^1(x^2 + x) = \langle\, y_0 - x^2 - x,\, y_1 - 2x^2 - x \,\rangle \tag{23}$$

w.r.t. the ranking $x \succ y_1 \succ y_0$ we get

$$\mathbf{G} = \{\, x - 2y_0 + y_1,\, p \,\} \tag{24}$$

where

$$p = y_0 + 4y_0^2 - (4y_0 + 1)y_1 + y_1^2 \tag{25}$$

So $p = 0$ is a differential equation for y but, as is noted in [5], whereas $y(t) \equiv -1/4$ is a zero of p there is no corresponding solution to (22). Thus we have to add the *inequation* $y \not\equiv -1/4$ to get equivalence between solutions. In [5] a method from differential algebra due to Seidenberg is used to keep track of inequations.

Now, the method described above will work as follows in this example: the observer relation for x in $\mathbf{L}_f^{n-1}h$ is simply the generator $y_0 - x^2 - x$. Differentiate it and add its derivative and p to $\mathbf{L}_f^0 h$ to get ideal

$$\mathfrak{a} = \langle\, y_0 - x^2 - x,\, (2x - 1)\mathrm{d}x - y_1,\, y_0 + 4y_0^2 - (4y_0 + 1)y_1 + y_1^2 \,\rangle \tag{26}$$

A GB for $\mathfrak{a}$ w.r.t. the ranking $y_1 \succ y_0 \succ \mathrm{d}x \succ x$ is

$$\widehat{G} = \{\, y_1 - (2x + 1)\mathrm{d}x,\, y_0 - x^2 - x,\, (1 + 2x)^2(x + 1 - \mathrm{d}x)(x - \mathrm{d}x)\} \tag{27}$$

The last element of $\widehat{G}$ is the only non-trivial one. It shows that $\mathfrak{a}$ is not prime, and each of the factors tells us something: we have that

$$\begin{array}{ll} \textit{either} & x = -1/2 \\ \textit{or} & \dot{x} = x \\ \textit{or} & \dot{x} = x + 1 \end{array}$$

The interpretation is clear: if $x(t) \not\equiv -1/2$ (i.e. $y(t) \not\equiv -1/4$) and $y = x^2 + x$ then the equation $p = 0$ can be realized either as $\dot{x} = x$ or as $\dot{x} = x + 1$.

To illustrate that the method is completely algorithmical we show a Maple session for doing the above computations (by the way also using POLYCON):

```
> with(grobner): read(`polycon.m`):
> f:=vector([x]): h:=x^2+x:
> p:=ss2ioc(f,h,[x]);
                        2                          2
               p := y1  + (- 4 y0 - 1) y1 + 4 y0  + y0

> a:=[y0-x^2-x,y1-(2*x+1)*dx,p]:
> ghat:=gbasis(a,[y1,y0,dx,x],plex):
> nops(ghat);
                                  3
> factor(ghat[3]);
                            2
                 (1 + 2 x)  (x - dx + 1) (x - dx)
```

□

Example 4.2 For the realization $\dot{x} = 1/2$, $y = x^2$ of the differential equation $y_1^2 - y_0 = 0$ we get the ideal

$$\mathfrak{a} = \langle y_0 - x^2,\, y_1^2 - y_0,\, y_1 - 2x\mathrm{d}x \rangle \tag{28}$$

Now we see quite easily that eliminating y_1, y_0 we get

$$4x^2\,\mathrm{d}x^2 - x^2 = x^2(2\,\mathrm{d}x - 1)(2\,\mathrm{d}x + 1) \tag{29}$$

In other words: the realization only covers solutions with $x(t) \not\equiv 0$ and an equivalent realization is provided by $\dot{x} = -1/2$, $y = x^2$. □

Example 4.3 Consider the second order system

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = x_1x_2, \quad y = x_1^2 \tag{30}$$

If we proceed as described above we find that e.g.

$$a_1 = y0 - x_1^2, \quad a_2 = 4x_2^2y_0 - y_1^2 \tag{31}$$

etc. The GB for $\mathfrak{a}$ w.r.t. the ranking

$$y_2 \succ y_1 \succ y_0 \succ \mathrm{d}x_2 \succ \mathrm{d}x_1 \succ x_2 \succ x_1$$

contains 6 elements, among which are

$$x_1^6x_2\,(x_1x_2 - \mathrm{d}x_2)\,(x_1x_2 + \mathrm{d}x_2) \tag{32}$$

and

$$-\,x_1\,(x_2 - \mathrm{d}x_1) \tag{33}$$

So, for example, we can conclude that $\dot{x}_2 = \pm x_1x_2$ follows from the "input-output equation" and $a_1 = a_2 = 0$ together, as long as $x_1x_2 \neq 0$. □

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