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## Parametrization of approximate algebraic surfaces by lines <sup>☆</sup>

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### Abstract

In this paper we present an algorithm for parametrizing approximate algebraic surfaces by lines. The algorithm is applicable to  $\varepsilon$ -irreducible algebraic surfaces of degree  $d$  having an  $\varepsilon$ -singularity of multiplicity  $d - 1$ , and therefore it generalizes the existing approximate parametrization algorithms. In particular, given a tolerance  $\varepsilon > 0$  and an  $\varepsilon$ -irreducible algebraic surface  $V$  of degree  $d$ , the algorithm computes a new algebraic surface  $\bar{V}$ , that is rational, as well as a rational parametrization of  $\bar{V}$ . In addition, in the error analysis we show that the output surface  $\bar{V}$  and the input surface  $V$  are close. More precisely, we prove that  $\bar{V}$  lies in the offset region of  $V$  at distance, at most,  $\mathcal{O}(\varepsilon^{1/(2d)})$ .

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*Keywords:* Algebraic surfaces; Approximate parametrization;  $\varepsilon$ -points

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### Introduction

The combination of computer algebra techniques with classical theoretical results in pure mathematics has yielded to many important symbolic algorithms (i.e., algorithmic methods where input and output are assumed to be exact) to solve relevant problems, especially, in algebra and algebraic geometry (see, e.g., (Cox et al., 1997; Hoffmann et al., 1997; Winkler, 1996)). Nevertheless, in many practical applications,

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for instance in the frame of computer aided geometric design, these approaches tend to be insufficient, since in practice most of data objects are given or become approximate. As a consequence of this phenomenon, there has been an increasing interest for the development of hybrid symbolic-numerical algorithms and approximate algorithms.

Approximate algorithms deal with mathematical objects that are assumed to be given approximately, probably because they proceed from an exact data that has been perturbed under some previous measuring process or manipulation. Since the input data, say  $\mathcal{D}$ , has been perturbed, the mathematical entity  $\mathcal{E}$  that one wants to compute (for instance a gcd, a Gröbner basis, a singular locus structure, etc.) has changed and does not behave anymore as expected. Then, the problem consists in finding a new object  $\bar{\mathcal{D}}$ , “close” to  $\mathcal{D}$ , satisfying the expecting property  $\mathcal{E}$ . The notion of “closeness” depends on the particular problem that one is trying to solve, and has to be defined properly. Examples of approximate algorithms in algebra can be found in (Corless et al., 1995; Emiris et al., 1997; Pan, 1996) for computing polynomial greatest common divisors, in (Corless et al., 1995; Emiris and Pan, 2002; Fortune, 2001) for finding zeros of multivariate systems, in (Corless et al., 2001; Galligo and Rupprech, 2002; Pan, 2001; Sasaki, 2001) for factoring polynomials, in (Möller, 1998; Stetter, 1997) for numerical computation of Gröbner basis, etc.

One may illustrate this type of phenomenon by the following easy example on factorization of approximate polynomials, that has been taken from (Corless et al., 2001). Consider  $\mathcal{D}$  as the polynomial  $p(x, y) = y^2 - x^4 + 0.01x^2$ , that does not have the property  $\mathcal{E}$  of being reducible. In this situation, the problem consists in computing a new object  $\bar{\mathcal{D}}$ , in this case a new polynomial  $\bar{p}(x, y)$ , having the property  $\mathcal{E}$  and being close to  $p(x, y)$ . Applying algorithms in (Corless et al., 2001), one gets that  $\bar{p}(x, y)$  can be taken as

$$\bar{p}(x, y) = (y + x^2 - 0.0050433)(y - x^2 + 0.0049999),$$

that factors. Moreover, it holds that

$$\frac{\|p - \bar{p}\|}{\|p\|} = 0.47 \times 10^{-4}.$$

Observe that in this case, the notion of closeness, has been taken as a relative error.

For approximate algorithms in algebraic geometry we refer to (Bajaj et al., 1988; Bajaj and Xu, 1997; Demmel and Manocha, 1995; Golub and Van Loan, 1989; Hoffmann, 1993) for the computation of singularities, to (Corless et al., 2000; Dokken, 2001) for implicitization methods, to (Farouki and Rajan, 1988) for the analysis of the numerical condition of implicitly given algebraic curves and surfaces, to (Bajaj and Royappa, 2000; Gahleitner et al., 2002; Hartmann, 2000; Pérez-Díaz et al., 2004) to parametrization algorithms, etc.

In this paper, we study the problem of parametrizing approximate algebraic surfaces. In this context, and in order to be more precise, the problem can be stated as follows. Given a fixed tolerance  $\varepsilon > 0$ , that may be introduced for instance by the user or by the constrains of the application process, and given an  $\varepsilon$ -irreducible algebraic surface  $V$  (see either (Corless et al., 2001) or Section 2 for the notion of  $\varepsilon$ -irreducibility), that may be or may not be rational, one has to compute a rational surface  $\bar{V}$ , and a parametrization of  $\bar{V}$ , such that  $\bar{V}$  lies within the region limited by the external and the internal offset of  $V$ , at distance  $\delta(\varepsilon)$ , where  $|\delta(\varepsilon) - \varepsilon|$  is significantly small (see (Arrondo et al., 1997) for basic notions of offsets).

Approximate parametrization algorithms for curves and surfaces can be found in (Bajaj and Royappa, 2000; Gahleitner et al., 2002; Hartmann, 2000; Pérez-Díaz et al., 2004). In (Gahleitner et al., 2002; Hartmann, 2000) the problem is treated locally. In (Bajaj and Royappa, 2000) the problem is treated

globally and the authors present a method for conics, cubics and quadrics. In (Pérez-Díaz et al., 2004) the results in (Bajaj and Royappa, 2000) are generalized to a wider class of algebraic curves, namely those curves having “almost” a singularity of maximum multiplicity. In this paper, we show how the results presented in (Pérez-Díaz et al., 2004) can be extended to surfaces having an  $\varepsilon$ -singularity of multiplicity  $d - 1$  (see Sections 2 and 3 for the notion of  $\varepsilon$ -singularity), where  $d$  is the degree of the input surface, and  $\varepsilon > 0$  is the tolerance. Therefore in this paper we generalize the results given in (Bajaj and Royappa, 2000); note that quadrics are a particular case of the above situation. In addition, we analyze the error, and we prove that the output rational surface lies in the offset region of the input surface at distance, at most,  $\mathcal{O}(\varepsilon^{1/(2d)})$ , where  $d > 0$  is the degree of the input surface, and  $\varepsilon > 0$  is the fixed tolerance. Besides this theoretical analysis of the distance between the input and output surfaces, in the examples, we have empirically quantified the distance. The conclusion of this experimental analysis is that, in practical examples, the distance is smaller than the theoretical bound shows.

The paper is structured as follows. Section 1 is preliminary, and we briefly describe the symbolic algorithm to parametrize surfaces by lines. In Section 2 we give the approximate algorithm for quadrics, and in Section 3 we present the general algorithm for parametrizing by lines surfaces having an  $\varepsilon$ -singularity of multiplicity  $d - 1$ , where  $d$  is the degree of the input surface. Section 4 is devoted to the analysis of the error, and we prove that the output generated by our algorithm is close to the input surface.

## 1. Preliminaries: symbolic parametrization of surfaces by lines

In this section, we recall how to parametrize by lines some special surfaces; for more details see (Abhyankar and Bajaj, 1989; Schicho, 1998; Wang et al., 1997). For this purpose, we consider w.l.o.g. surfaces of degree greater than 1.

Let  $V$  be an irreducible surface of degree  $d$  over an algebraically closed field  $\mathbb{K}$ ; in practice, one may think that  $\mathbb{K}$  is the field  $\mathbb{C}$  of the complex numbers. We assume that  $V$  has a point  $P$  of multiplicity  $(d - 1)$ . That is, all partial derivatives of the defining polynomial of  $V$ , till order  $d - 2$ , vanish at  $P$ . In this case,  $V$  can be parametrized by means of rational functions, i.e.,  $V$  is a rational surface. The idea for actually computing a parametrization of this type of surfaces consists in a generalization of the algorithm for quadrics presented in (Abhyankar and Bajaj, 1989; Sederberg and Snively, 1987; Wang et al., 1997). Intuitively speaking, any line through  $P$  must intersect the surface  $V$  in one additional point, by Bézout’s theorem (see Fig. 1). Thus, if one parametrizes a pencil of lines,  $H_\lambda(t, h)$ , through  $P$ , that point on  $V$  can be expressed, for each line  $H_\lambda(t, h)$  in the pencil, by an expression in  $h$  and  $t$ .

More precisely, let  $f(x, y, z)$  be the implicit equation of an irreducible affine quadric  $V$ . We consider a point  $P \in V$  not being a singularity and let  $\Pi$  be a plane not containing the point  $P$ . In these conditions, one considers the projection of  $V$  with center on  $P$  over the plane  $\Pi$ . Moreover, let  $Q(t, h)$  be a generic point of  $\Pi$  and let  $\mathcal{H}_\lambda = P + \lambda(Q(t, h) - P)$  be the pencil of lines. Thus, intersecting  $\mathcal{H}_\lambda$  and  $V$  one gets the parametrization  $\mathcal{P}(t, h)$  of  $V$ . That is, from  $f(\mathcal{H}_\lambda) = 0$  one may express rationally  $\lambda$  in terms of  $t, h$ , and afterwards by substituting it in  $\mathcal{H}_\lambda$  one obtains the parametrization of  $V$  (see Fig. 1).

In the particular case of the example illustrated in Fig. 1, i.e., where  $V$  is the sphere of equation  $f(x, y, z) = x^2 + y^2 + z^2 - 1$ , the algebraic calculation would be as follows. We consider as  $P$  the point  $P = (0, 0, 1)$  and the plane  $\Pi$  of equation  $z = 0$ ; note that  $P \notin \Pi$ . Then,  $Q(t, h) = (t, h, 0)$ , and

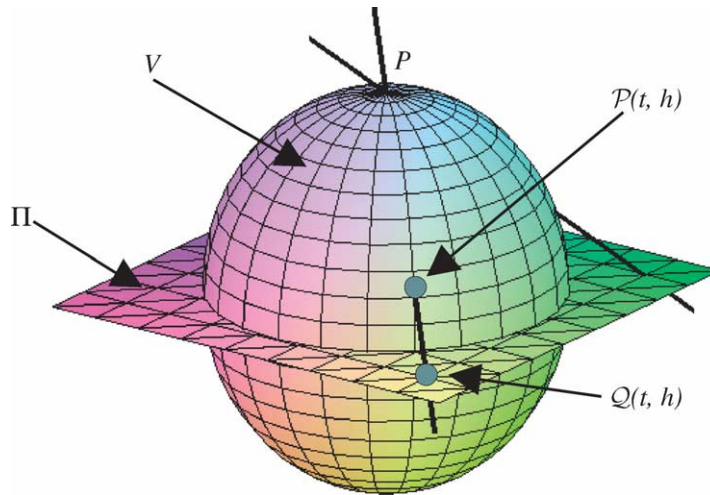


Fig. 1. Geometric idea of the parametrization of quadrics.

$\mathcal{H}_\lambda(t, h) = (\lambda t, \lambda h, 1 - \lambda)$ . Now,  $f(\mathcal{H}_\lambda(t, h)) = \lambda(\lambda t^2 + \lambda h^2 - 2 + \lambda)$ . So, for  $\lambda = 0$  one gets the point  $P$ , and for  $\lambda = 2/(t^2 + h^2 + 1)$ , one obtains the sphere rational parametrization

$$\mathcal{P}(t, h) = \left( \frac{2t}{t^2 + h^2 + 1}, \frac{2h}{t^2 + h^2 + 1}, 1 - \frac{2}{t^2 + h^2 + 1} \right).$$

The above reasoning can be generalized to irreducible surfaces of degree  $d$  having a point of multiplicity  $d - 1$ . In fact, the above geometric process can be seen algebraically as follows. Let  $f(x, y, z)$  be the implicit equation of  $V$  and we assume w.l.o.g. that the point  $P \in V$  of multiplicity  $d - 1$  is the origin. Otherwise one may consider a linear change of coordinates. In this situation, it holds that

$$f(x, y, z) = f_d(x, y, z) + f_{d-1}(x, y, z),$$

where  $f_d(x, y, z)$ ,  $f_{d-1}(x, y, z)$  are the homogeneous forms of degree  $d$  and  $d - 1$  of  $f(x, y, z)$ , respectively. In this situation, let  $Q(t, h) = (t, h, 1)$  be a generic point of the plane  $\Pi$  defined by  $z = 1$  (note that now  $P \notin \Pi$ ); hence,

$$\mathcal{H}_\lambda = (\lambda t, \lambda h, \lambda).$$

Thus, intersecting of  $V$  with  $\mathcal{H}_\lambda$  one gets that

$$f(\lambda t, \lambda h, \lambda) = \lambda^d f_d(t, h, 1) + \lambda^{d-1} f_{d-1}(t, h, 1),$$

and therefore one deduces that

$$\lambda = \frac{-f_{d-1}(t, h, 1)}{f_d(t, h, 1)}.$$

Then, a rational parametrization of  $V$  is given by

$$\mathcal{P}(t, h) = \left( -t \frac{f_{d-1}(t, h, 1)}{f_d(t, h, 1)}, -h \frac{f_{d-1}(t, h, 1)}{f_d(t, h, 1)}, -\frac{f_{d-1}(t, h, 1)}{f_d(t, h, 1)} \right).$$

More precisely, one has the following parametrization algorithm by lines:

**Algorithm.** Symbolic parametrization by lines for surfaces.

- **Given** an irreducible polynomial  $f(x, y, z) \in \mathbb{K}[x, y, z]$  defining an irreducible algebraic surface  $V$  of degree  $d > 1$  with a point of multiplicity  $d - 1$ .
- **Compute** a rational parametrization  $\mathcal{P}(t, h) = (p_1(t, h), p_2(t, h), p_3(t, h))$  of  $V$ .

- (1) If  $d = 2$  take a regular point  $P$  on  $V$ , else determine a point  $P$  of multiplicity  $(d - 1)$  on  $V$ .
- (2) If  $P$  is at infinity, consider a linear change of variables such that  $P$  is transformed into an affine point. Let  $P = (a, b, c)$ .
- (3) Compute

$$A(x, y, z, t, h) = \frac{\sum_{r+s=0}^{d-1} \frac{\partial^{d-1} f}{\partial^r x \partial^s y \partial^{d-1-r-s} z} t^r h^s}{(d-1-r-s)! r! s!} \cdot \frac{\sum_{r+s=0}^d \frac{\partial^d f}{\partial^r x \partial^s y \partial^{d-r-s} z} t^r h^s}{(d-r-s)! r! s!}.$$

- (4) Consider

$$\mathcal{P}(t, h) = (-tA(P, t, h) + a, -hA(P, t, h) + b, -A(P, t, h) + c).$$

- (5) Return the parametrization obtained when one applies to  $\mathcal{P}(t, h)$  the inverse of the change considered in (2).

**Remark.** The parametrization  $\mathcal{P}(t, h)$  in step (4) can also be obtained as follows: compute  $g(x, y, z) = f(x + a, y + b, z + c)$ , and return

$$\mathcal{P}(t, h) = \left( \frac{-tg_{d-1}(t, h, 1)}{g_d(t, h, 1)} + a, \frac{-hg_{d-1}(t, h, 1)}{g_d(t, h, 1)} + b, \frac{-g_{d-1}(t, h, 1)}{g_d(t, h, 1)} + c \right),$$

where  $g_d(x, y, z)$  and  $g_{d-1}(x, y, z)$  are the homogeneous components of  $g(x, y, z)$  of degree  $d$  and  $d - 1$ , respectively.

The following example illustrates the above parametrization algorithm for a surface of degree 5 defined over  $\mathbb{C}$ .

**Example 1.** Let  $V$  be the irreducible surface over  $\mathbb{C}$  defined by the implicit equation

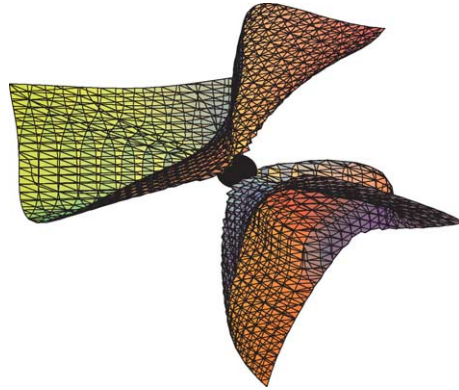
$$f(x, y, z) = y^5 + x^5 + x^4 - 2y^4 + 2z^4y + 3z^3x.$$

Note that  $V$  has a singularity of multiplicity 4 in  $P = (0, 0, 0)$  (see Fig. 2). Applying the above algorithm one computes

$$A(x, y, z, t, h) = \frac{\sum_{r+s=0}^4 \frac{\partial^4 f}{\partial^r x \partial^s y \partial^{4-r-s} z} t^r h^s}{(4-r-s)! r! s!}, \frac{\sum_{r+s=0}^5 \frac{\partial^5 f}{\partial^r x \partial^s y \partial^{5-r-s} z} t^r h^s}{(5-r-s)! r! s!},$$

and hence we obtain the rational parametrization of  $V$  defined as

$$\begin{aligned} \mathcal{P}(t, h) &= (-tA(P, t) + a, -hA(P, t) + b, -A(P, t) + c) \\ &= \left( \frac{-(3t + t^4 - 2h^4)t}{h^5 + 2h + t^5}, \frac{-(3t + t^4 - 2h^4)h}{h^5 + 2h + t^5}, \frac{-(3t + t^4 - 2h^4)}{h^5 + 2h + t^5} \right). \end{aligned}$$

Fig. 2. Rational surface  $V$ .

## 2. Parametrization of approximate quadrics

In this section we study the problem of parametrizing approximate quadrics. We consider a fixed tolerance  $\varepsilon > 0$ , that may be introduced for instance by the user or by the constraints of the application process, and a quadric  $V$ . Then, we want to compute the parametrization of a new quadric  $\bar{V}$  that lies within the offset region of  $V$  at some small distance. More precisely, we present an algorithm that determines a rational parametrization of  $\bar{V}$ , that lies within the offset region of  $V$  at distance at most  $\mathcal{O}(\sqrt[4]{\varepsilon})$ . In fact, in Section 4, we will see that the offset region of  $V$ , where  $\bar{V}$  is contained, can be taken as distance, at most,  $3e^3 \sqrt{3} \sqrt[4]{\varepsilon}$ , where  $e = \exp(1)$ .

The results obtained in this case are similar to those presented in (Bajaj and Royappa, 2000). However, the method that we present will be generalized to surfaces of degree  $d$  with “almost” a point of multiplicity  $d - 1$  (see Section 3). Therefore, the results in (Bajaj and Royappa, 2000) will be extended.

Throughout this paper, we consider that a fixed tolerance  $\varepsilon > 0$  is given, and we will use the polynomial  $\infty$ -norm; i.e., if

$$p(x, y, z) = \sum_{i,j,k \in I} a_{i,j,k} x^i y^j z^k \in \mathbb{C}[x, y, z]$$

then  $\|p(x, y, z)\|$  is defined as

$$\|p(x, y, z)\| = \max\{|a_{i,j,k}|/i, j, k \in I\}.$$

In particular if  $p(x, y, z)$  is a constant coefficient,  $\|p(x, y, z)\|$  will denote its module. Furthermore, we consider a real quadric  $V$  defined by an  $\varepsilon$ -irreducible polynomial (see for instance (Corless et al., 2001; Sasaki, 2001)),  $f(x, y, z) \in \mathbb{R}[x, y, z]$ ; that is,  $f(x, y, z)$  cannot be expressed as

$$f(x, y, z) = g(x, y, z)h(x, y, z) + \mathcal{E}(x, y, z)$$

where  $g, h, \mathcal{E} \in \mathbb{C}[x, y, z]$  and

$$\|\mathcal{E}(x, y, z)\| < \varepsilon \|f(x, y, z)\|.$$

We observe that the notion of  $\varepsilon$ -irreducibility implies the notion of “exact” irreducibility. Therefore, in our case, one has that the polynomial  $f(x, y, z)$  is irreducible. On the other hand, since irreducible quadrics are rational, one deduces that our quadric  $V$  can be parametrized. In order to compute a

parametrization of  $V$ , one may apply the symbolic parametrization algorithm to  $V$  (see Section 1). However, the symbolic parametrization algorithm requires the computation of a simple point on the quadric; in fact, once the simple point is determined, the remaining steps of the algorithm can be executed symbolically without further difficulties. Note that, since the surface  $V$  is real, this point can be taken over  $\mathbb{R}$ . The computation of this simple point can be performed either symbolically, for instance introducing algebraic numbers, or numerically by root finding methods (see (Fortune, 2001; Golub and Van Loan, 1989)).

If one works symbolically then the direct application of the algorithm will provide an exact answer. However, in the frame of this paper, we are interested in the approximate approach. Thus, we assume that the simple point is approximated.

In order to deal with the approximate simple point, we will introduce the notion of  $\varepsilon$ -point of a surface. This concept essentially consists of a point that almost lies on the surface. Algebraically, if  $f(x, y, z)$  is a polynomial defining the surface  $V$  and  $P \in \mathbb{C}^3$  is the point, the notion of  $\varepsilon$ -point may be approached asking that  $|f(P)|$  is small, let us say smaller than the tolerance. However, since for every non-zero  $\lambda \in \mathbb{C}$ , the polynomial  $\lambda f(x, y, z)$  also defines the same surface, the above condition is not enough. For controlling this phenomenon one may consider relative errors, and one may ask that  $|f(P)|/\|f\|$  is small. More precisely, one has the following definition.

**Definition 1.** We say that  $\bar{P} = (\bar{a}, \bar{b}, \bar{c}) \in \mathbb{C}^3$  is an (affine)  $\varepsilon$ -point of an algebraic surface  $V$  defined by an  $\varepsilon$ -irreducible polynomial  $f(x, y, z) \in \mathbb{R}[x, y, z]$  if it holds that

$$\frac{|f(\bar{P})|}{\|f(x, y, z)\|} < \varepsilon,$$

and for any  $i_0, j_0, k_0 \in \mathbb{N}$  with  $i_0 + j_0 + k_0 = 1$ ,

$$\frac{\left| \frac{\partial f}{\partial^{i_0} x \partial^{j_0} y \partial^{k_0} z}(\bar{P}) \right|}{\|f(x, y, z)\|} \geq \varepsilon;$$

that is,  $\bar{P}$  is a simple point on  $\mathcal{C}$  computed under fixed precision  $\varepsilon\|f(x, y, z)\|$ .

Now, we proceed to describe the method for parametrizing by lines approximate quadrics. For this purpose, let  $\bar{P} = (\bar{a}, \bar{b}, \bar{c})$  be an affine  $\varepsilon$ -point of the quadric  $V$ , and let us consider the quadric  $\bar{V}$  defined by the polynomial

$$\bar{f}(x, y, z) = f(x, y, z) - f(\bar{P}).$$

We observe that the point  $\bar{P}$  is an exact simple point of  $\bar{V}$  (at least a partial derivative of  $f$  of order 1 does not vanish at  $\bar{P}$ ). In these conditions, the following lemma holds.

**Lemma 1.** The quadric  $\bar{V}$  is irreducible over  $\mathbb{C}$ .

**Proof.** If  $\bar{f}$  factors as  $\bar{f} = \bar{g}\bar{h}$  then  $f = \bar{g}\bar{h} + f(\bar{P})$  and since  $\bar{P}$  is an  $\varepsilon$ -point, it holds that  $|f(\bar{P})| < \varepsilon\|f\|$ . Then,  $f$  is not  $\varepsilon$ -irreducible over  $\mathbb{C}$ , which is impossible.  $\square$

Therefore, we have constructed a rational irreducible quadric, namely  $\bar{V}$  on which we know a simple point, namely  $\bar{P}$ . Hence, taking into account the above remarks, we may directly apply the symbolic algorithm to  $\bar{V}$  (see Section 1) to get the rational parametrization

$$\bar{\mathcal{P}}(t, h) = (-tA(\bar{P}, t, h) + \bar{a}, -hA(\bar{P}, t, h) + \bar{b}, -A(\bar{P}, t, h) + \bar{c}),$$

where

$$A(x, y, z, t, h) = \frac{\frac{\partial f}{\partial x}t + \frac{\partial f}{\partial y}h + \frac{\partial f}{\partial z}}{\frac{\partial^2 f}{\partial x^2} \frac{t^2}{2} + \frac{\partial^2 f}{\partial y^2} \frac{h^2}{2} + \frac{\partial^2 f}{\partial z^2} \frac{1}{2} + \frac{\partial^2 f}{\partial x \partial y} th + \frac{\partial^2 f}{\partial x \partial z} t + \frac{\partial^2 f}{\partial y \partial z} h}.$$

From the reasoning presented above, one gets the following theorem.

**Theorem 1.** *Let  $f(x, y, z)$  be the implicit equation of the quadric  $\varepsilon$ -irreducible  $V$ , and let  $\bar{P} = (\bar{a}, \bar{b}, \bar{c})$  be an  $\varepsilon$ -point of  $V$ . We consider the parametrization*

$$\bar{\mathcal{P}}(t, h) = (-tA(\bar{P}, t, h) + \bar{a}, -hA(\bar{P}, t, h) + \bar{b}, -A(\bar{P}, t, h) + \bar{c}),$$

where

$$A(x, y, z, t, h) = \frac{\frac{\partial f}{\partial x}t + \frac{\partial f}{\partial y}h + \frac{\partial f}{\partial z}}{\frac{\partial^2 f}{\partial x^2} \frac{t^2}{2} + \frac{\partial^2 f}{\partial y^2} \frac{h^2}{2} + \frac{\partial^2 f}{\partial z^2} \frac{1}{2} + \frac{\partial^2 f}{\partial x \partial y} th + \frac{\partial^2 f}{\partial x \partial z} t + \frac{\partial^2 f}{\partial y \partial z} h}.$$

Then, the implicit equation of the quadric  $\bar{V}$  defined by the parametrization  $\bar{\mathcal{P}}(t, h)$  is

$$\bar{f}(x, y, z) = f(x, y, z) - f(\bar{P}).$$

In (Bajaj and Royappa, 2000), the authors present similar results to Theorem 1. However, as we will see in Section 3, the formulation given in Theorem 1 shows how to generalize these ideas to the case of surfaces of arbitrary degree  $d$  with the property of having an  $\varepsilon$ -point of multiplicity  $d - 1$ . Theorem 1 provides the following algorithm for parametrizing approximate quadrics.

**Algorithm 1.** Approximate parametrization by lines for quadrics.

- **Given** a tolerance  $\varepsilon > 0$  and an  $\varepsilon$ -irreducible polynomial  $f(x, y, z) \in \mathbb{R}[x, y, z]$ , defining a quadric  $V$ .
  - **Compute** a rational parametrization  $\bar{\mathcal{P}}(t, h)$  of a quadric  $\bar{V}$  close to  $V$ .
- (1) Compute an affine  $\varepsilon$ -point  $\bar{P} = (\bar{a}, \bar{b}, \bar{c})$  of  $V$ .
  - (2) Determine  $\bar{f}(x, y, z) = f(x, y, z) - f(\bar{P})$ .
  - (3) Compute

$$A(x, y, z, t, h) = \frac{\frac{\partial \bar{f}}{\partial x}t + \frac{\partial \bar{f}}{\partial y}h + \frac{\partial \bar{f}}{\partial z}}{\frac{\partial^2 \bar{f}}{\partial x^2} \frac{t^2}{2} + \frac{\partial^2 \bar{f}}{\partial y^2} \frac{h^2}{2} + \frac{\partial^2 \bar{f}}{\partial z^2} \frac{1}{2} + \frac{\partial^2 \bar{f}}{\partial x \partial y} th + \frac{\partial^2 \bar{f}}{\partial x \partial z} t + \frac{\partial^2 \bar{f}}{\partial y \partial z} h},$$

and return

$$\bar{\mathcal{P}}(t, h) = (-tA(\bar{P}, t, h) + \bar{a}, -hA(\bar{P}, t, h) + \bar{b}, -A(\bar{P}, t, h) + \bar{c}).$$



Note that for computing  $\bar{P}$  in step (1) one may use numerical techniques, for instance (Fortune, 2001) or (Golub and Van Loan, 1989). Taking into account Theorem 1 and the above algorithm, one deduces the following result.

**Theorem 2.** *The input surface and the output surface given by Algorithm 1 have the same degree. Furthermore, all the coefficients, but the independent ones, of the defining polynomials of the input and the output surfaces of Algorithm 1 are the same; i.e., its difference is a constant.*

The following example illustrates Algorithm 1. In the description of the example we will remark the mentioned particular properties of the parametrization method, and we will also estimate, for this particular case, the distance of the input and output surface; for a theoretical treatment of this fact, we refer to Section 4. In order to estimate the distance, we particularize the theoretical reasoning in Section 4 to this example, proceeding as follows. We randomly generated 10000 points on the output surface. Since this surface is rational, and since we know a parametrization of it, the points on the surface are obtained by giving random values to the parameters. Once the points on the output surface are determined, we compute, by intersecting with a pencil of lines (see details in the example), points on the input surface, and we measure the corresponding distances. For the set of all the obtained distances we get the mean value as well as the statistical standard error, that turn to be very small.

**Example 2.** We consider  $\varepsilon = 0.001$  and the quadric  $V$  defined by the  $\varepsilon$ -irreducible polynomial

$$f(x, y, z) = 97.00100000y + 50.00300000x^2 + 79xy + 56xz + 49.00100000yz + 63z^2 + 0.001000000000 + 0.001000000000x.$$

First of all, we determine an affine  $\varepsilon$ -point  $\bar{P}$ . For this purpose, using numerical techniques (see for instance (Fortune, 2001) and (Golub and Van Loan, 1989)) we get  $\bar{P} = (0, 0, 0)$ .

Applying step (2) of Algorithm 1 we obtain the quadric  $\bar{V}$  defined by the irreducible polynomial

$$\bar{f}(x, y, z) = f(x, y, z) - f(\bar{P}) = 97.00100000y + 50.00300000x^2 + 79xy + 56xz + 49.00100000yz + 63z^2 + 0.001000000000x.$$

Note that  $\bar{f}(x, y, z)$  is irreducible (see Lemma 1), and that  $\bar{f}(\bar{P}) = 0$ , and hence  $\bar{P}$  is an exact point on the irreducible quadric  $\bar{V}$ . Finally, applying step (3) of Algorithm 1, we obtain a rational parametrization of  $\bar{V}$  defined by

$$\bar{P}(t, h) = (\bar{p}_1(t, h), \bar{p}_2(t, h), \bar{p}_3(t, h)),$$

where

$$\bar{p}_1(t, h) = \frac{-97001.00001th - .9999999999t^2}{50002.99999t^2 + 79000.00002th + 56000.00000t + 49000.99999h + 62999.99997},$$

$$\bar{p}_2(t, h) = \frac{-97001.00001h^2 - .9999999999th}{50002.99999t^2 + 79000.00002th + 56000.00000t + 49000.99999h + 62999.99997},$$

and

$$\bar{p}_3(t, h) = \frac{-97001.00001h - .9999999999t}{50002.99999t^2 + 79000.00002th + 56000.00000t + 49000.99999h + 62999.99997}.$$

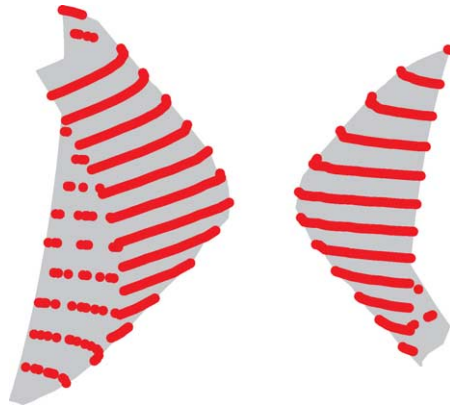


Fig. 3. Input surface  $V$  (grey) and contour of the output surface  $\bar{V}$  (red).

Note that, as stated in Theorem 2,  $\deg(f(x, y, z)) = \deg(\bar{f}(x, y, z))$  and  $\bar{f}(x, y, z) - f(x, y, z)$  is a constant, namely  $f(\bar{P})$ .

In Fig. 3 one may compare the input quadric and the output rational quadric. Note that the surface  $\bar{V}$  is close to the input surface  $V$ . This behavior will be studied in Section 4. However, as we have mentioned above, we will statistically estimate the distance between  $\bar{V}$  and  $V$ .

For this purpose, we particularize the theoretical reasoning in Section 4 to this example. More precisely, giving random values to the parameters  $t, h$  in  $\bar{\mathcal{P}}(t, h)$  we generate 10000 points on  $\bar{V}$ . Let  $\mathcal{A}$  denote the set of all these points. Now, for each point  $Q \in \mathcal{A}$  we consider the line  $\mathcal{L}$  of parametric equation  $Q + \lambda(Q - \bar{P})$ . Afterwards, we compute the intersection of the line  $\mathcal{L}$  and the surface  $V$ , i.e., we approximate the roots of  $f(Q + \lambda(Q - \bar{P}))$ . This computation yields to a finite set  $\mathcal{B}_Q$  of points on  $V$ , and we take the minimum of the Euclidean distances of  $Q$  to the points in  $\mathcal{B}_Q$ . After repeating this construction for all points in  $\mathcal{A}$ , we have a set of distances, and we compute their mean value  $\mu$  as well as the statistical standard error  $\rho$ . In this case, we have obtained

$$\mu = 0.003643288980, \quad \rho = 0.0005411567185,$$

from where one can statistically deduce that the distance is, in average, in the interval  $[\mu - 1.96\rho, \mu + 1.96\rho] = [0.002582621812, 0.004703956148]$ .

### 3. Parametrization by lines of approximate surfaces

In Section 2 we have presented an approximate parametrization algorithm for quadrics. In this section we generalize these results for surfaces of degree  $d$  with the property of having “almost” a point of multiplicity  $d - 1$ . Therefore, we extend the results presented in (Bajaj and Royappa, 2000).

The main difference with the quadric case (see Section 2) is that the given approximate algebraic surface is, in general, non-rational even though it might correspond to the perturbation of a rational surface. The basic idea to approach the problem consists in generalizing the construction done for quadrics in such a way that a rational surface is constructed.

For this purpose, we observe that the output surface in the 2-degree case is the original polynomial minus its Taylor expansion up to order 1 at the  $\varepsilon$ -point, i.e., the evaluation of the polynomial at the point

(see Theorem 2). For surfaces of degree  $d$  having “almost” a singularity of multiplicity  $d - 1$ , we see that one may generalize properly the process by subtracting to the original polynomial its Taylor expansion up to order  $d - 1$  at the “quasi” singularity, to get a rational surface close to the given one.

In order to be precise, we first introduce the notion of  $\varepsilon$ -singularity, that is the natural generalization of the concept of  $\varepsilon$ -point given in Definition 1.

**Definition 2.** We say that  $\bar{P} = (\bar{a}, \bar{b}, \bar{c}) \in \mathbb{C}^3$  is an (affine)  $\varepsilon$ -singularity of multiplicity  $r$  of a surface defined by an  $\varepsilon$ -irreducible polynomial  $f(x, y, z) \in \mathbb{R}[x, y, z]$  if for  $0 \leq i + j + k \leq r - 1$ , it holds that

$$\frac{\left| \frac{\partial^{i+j+k} f}{\partial^i x \partial^j y \partial^k z}(\bar{P}) \right|}{\|f(x, y, z)\|} < \varepsilon,$$

and for some  $i_0, j_0, k_0 \in \mathbb{N}$  with  $i_0 + j_0 + k_0 = r$

$$\frac{\left| \frac{\partial^r f}{\partial^{i_0} x \partial^{j_0} y \partial^{k_0} z}(\bar{P}) \right|}{\|f(x, y, z)\|} > \varepsilon.$$

Note that an  $\varepsilon$ -singularity of multiplicity 1 is an  $\varepsilon$ -point on the surface. In this situation, we introduce the set  $\mathcal{S}_\varepsilon^d$  as follows.

**Definition 3.** We denote by  $\mathcal{S}_\varepsilon^d$  the set of all the  $\varepsilon$ -irreducible algebraic real surfaces of degree  $d$  having a real  $\varepsilon$ -singularity of multiplicity  $d - 1$ .

In the following, we assume that  $d > 2$  and we prove that the elements in  $\mathcal{S}_\varepsilon^d$  can be parametrized by lines. Note that the case  $d = 1$ , i.e., planes, is trivial, and that the case  $d = 2$  has been analyzed in Section 2. But first, we deal with the problem of checking whether a given real algebraic surface  $V$  belongs or not to  $\mathcal{S}_\varepsilon^d$ .

In order to check whether a given real surface  $V$  of degree  $d$ , defined by a polynomial  $f(x, y, z)$ , belongs to  $\mathcal{S}_\varepsilon^d$ , one has to check the  $\varepsilon$ -irreducibility of  $f(x, y, z)$  as well as the existence of an  $\varepsilon$ -singularity of multiplicity  $d - 1$ . To analyze the  $\varepsilon$ -irreducibility, one may use any of the existing algorithms (see, e.g., (Corless et al., 2001; Pérez-Díaz et al., 2004; Sasaki, 2001)). For checking the existence and actual computation of  $\varepsilon$ -singularities of multiplicity  $d - 1$  one has to solve the system of algebraic equations

$$\mathcal{A} = \left\{ \frac{\partial^{i+j+k} f}{\partial^i x \partial^j y \partial^k z}(x, y, z) = 0 \right\}_{i+j+k=0, \dots, d-2}$$

under fixed precision  $\varepsilon \cdot \|f(x, y, z)\|$ , by applying root finding techniques (see, e.g., (Corless et al., 1995; Fortune, 2001; Golub and Van Loan, 1989; Hoffmann, 1993; Krishnan and Manocha, 1996)). The system  $\mathcal{A}$  may be simplified by reducing the number of equations and their degrees. More precisely, first, we choose three triples  $(i_\ell, j_\ell, k_\ell)$ , with  $\ell = 1, 2, 3$ , such that  $i_\ell + j_\ell + k_\ell = d - 2$ , and we consider the new system

$$\mathcal{B} = \left\{ \frac{\partial^{d-2} f}{\partial^{i_1} x \partial^{j_1} y \partial^{k_1} z}(x, y, z) = \frac{\partial^{d-2} f}{\partial^{i_2} x \partial^{j_2} y \partial^{k_2} z}(x, y, z) = \frac{\partial^{d-2} f}{\partial^{i_3} x \partial^{j_3} y \partial^{k_3} z}(x, y, z) = 0 \right\}$$

under fixed precision  $\varepsilon \|f(x, y, z)\|$ . Note that now the three involved equations are quadratic. One also has to observe that it may happen that all partial derivatives of order  $d - 2$  are in fact zero, but this

is quite unlikely since we work with approximations. Nevertheless, if this is the case, one would work with the previous non-zero derivatives, and the degree would have increased slightly. In this situation, one computes the solutions of  $\mathcal{B}$  by using any of the existing methods (see, e.g., (Corless et al., 1995; Fortune, 2001; Golub and Van Loan, 1989; Hoffmann, 1993; Krishnan and Manocha, 1996)). Once these solutions have been approximated, one simply has to check if any of the solutions,  $\bar{P}$ , of  $\mathcal{B}$  satisfies that

$$\left\| \frac{\partial^{i+j+k} f}{\partial^i x \partial^j y \partial^k z}(\bar{P}) \right\| \leq \varepsilon \|f(x, y, z)\|, \quad i + j + k = 0, \dots, d - 2.$$

These ideas are illustrated in Examples 3 and 4 in this section.

Numerical methods for solving systems of algebraic equations may fail when the set of solutions is not zero-dimensional, i.e., when there exist infinitely many solutions. In our case, this phenomenon may appear if the surface has infinitely many  $\varepsilon$ -singularities. Geometrically, this means that the surface might contain a whole curve whose points are  $\varepsilon$ -singularities, i.e., the surface has an  $\varepsilon$ -singular curve. Therefore, when applying the process described above, numerical methods may not compute all the possibilities, and hence one might not guarantee whether  $V$  is in  $\mathcal{S}_\varepsilon^d$ . That is, if applying these methods we may compute an  $\varepsilon$ -singularity of multiplicity  $d - 1$ , then we conclude that  $V \in \mathcal{S}_\varepsilon^d$ ; otherwise, we cannot conclude whether  $V$  is not in  $\mathcal{S}_\varepsilon^d$  and the parametrization algorithm could not be applied. We illustrate this difficulty with the following example. Let  $V$  be the surface of degree 3 defined by the irreducible polynomial

$$f(x, y, z) = x_1^3 + x_3 x_1^2 - x_2^2 - \frac{1}{1000},$$

and let  $\varepsilon = 0.001$ . This surface corresponds to a small perturbation of a generalization of “Cartan umbrella” (see p. 60 in (Bochnak et al., 1998)). It is easy to check that all points, in the line of equations  $x = y = 0$ , are  $\varepsilon$ -singularities of multiplicity 2 of  $V$  (see Fig. 4). However, applying numerical methods one does not reach the expected answer.

We do not know how to solve this problem, and therefore we do not have a deterministic algorithmic criterion for deciding whether  $V \in \mathcal{S}_\varepsilon^d$ . However, a possible idea to approach this problem might be to apply the approximate parametrization algorithm for quadrics (see Section 2) to find a suitable parametrization of one of the polynomials in  $\mathcal{B}$  to afterwards substitute this parametrization in the other polynomials involved in  $\mathcal{B}$ . In this way the problem is reduced to the bivariate case, and computing  $\varepsilon$ -gcds and approximately crossing it out, we arrive at a zero-dimensional system and hence current numerical methods would compute all solutions. Nevertheless we have been unable to guarantee the numerical stability of the process, and therefore we prefer to leave it open.

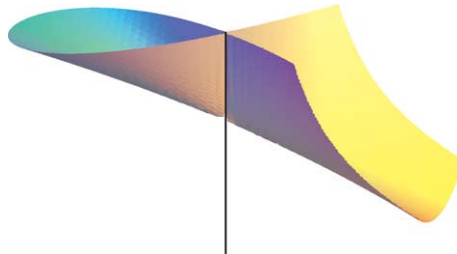


Fig. 4. Surface  $V$ .

In the following, we assume that  $V \in \mathcal{S}_\varepsilon^d$  and that  $f(x, y, z) \in \mathbb{R}[x, y, z]$  is the implicit equation defining  $V$ . In addition, we consider that  $\bar{P} = (\bar{a}, \bar{b}, \bar{c}) \in \mathbb{R}^3$  is an  $\varepsilon$ -singularity of multiplicity  $d - 1$  of the surface  $V$ , and we parametrize by lines, approximately, the surface  $V$ . For this purpose, if  $\mathcal{Q}(t, h)$  represents a generic point of a plane not containing the point  $\bar{P}$ , we consider the pencil of lines defined by the parametrization

$$\mathcal{H}_\lambda(t, h) = \bar{P} + \lambda(\mathcal{Q}(t, h) - \bar{P}).$$

If  $\bar{P}$  would be an exact singularity, then the symbolic algorithm (see Section 1) would output the parametrization

$$\bar{\mathcal{P}}(t, h) = \bar{P} + \mu(t, h)(\mathcal{Q}(t, h) - \bar{P}) \in \mathbb{R}(t, h)^3,$$

where  $\mu(t, h) \in \mathbb{R}(t, h)$  is the root w.r.t.  $\lambda$  of the linear polynomial

$$\frac{f(\mathcal{H}_\lambda(t, h))}{\lambda^{d-1}}.$$

However, in our case  $\bar{P}$  is not a singularity but an  $\varepsilon$ -singularity. Then, the idea consists in computing the root in  $\mathbb{R}(t, h)$  of the quotient of  $f(\mathcal{H}_\lambda(t, h))$  and  $\lambda^{d-1}$  w.r.t.  $\lambda$ . Note that  $\deg_\lambda(f(\mathcal{H}_\lambda(t, h))) = d$ , and therefore the quotient is linear in  $\lambda$ . Let  $\mu(t, h)$  be this root. Then, we will see that

$$\bar{\mathcal{P}}(t, h) = \bar{P} + \mu(t, h)(\mathcal{Q}(t, h) - \bar{P}),$$

is an approximate parametrization of  $V$ .

For this purpose, first we prove that  $\bar{\mathcal{P}}(t, h)$  is a rational parametrization. Furthermore, we show that if  $\mathcal{Q}(t, h)$  is proper, then the parametrization  $\bar{\mathcal{P}}(t, h)$  is also proper.

**Lemma 2.** *Let  $f(x, y, z)$  be the implicit equation of the surface  $V \in \mathcal{S}_\varepsilon^d$  and let  $\bar{P} = (\bar{a}, \bar{b}, \bar{c}) \in \mathbb{R}^3$  an affine  $\varepsilon$ -singularity of multiplicity  $d - 1$  of  $V$ . We consider  $\mu(t, h)$  the root in  $\mathbb{R}(t, h)$  of the quotient of the polynomial  $f(\mathcal{H}_\lambda(t, h))$  and  $\lambda^{d-1}$  w.r.t.  $\lambda$ . Then,*

$$\bar{\mathcal{P}}(t, h) = \bar{P} + \mu(t, h)(\mathcal{Q}(t, h) - \bar{P})$$

is a rational parametrization.

**Proof.** To prove the lemma we show that at least one of the components of  $\bar{\mathcal{P}}(t, h)$  is not a constant. Let us assume that all the components of  $\bar{\mathcal{P}}(t, h)$  are constant. In this situation, since

$$\bar{\mathcal{P}}(t, h) = (\bar{a} + \mu(t, h)(q_1(t, h) - \bar{a}), \bar{b} + \mu(t, h)(q_2(t, h) - \bar{b}), \bar{c} + \mu(t, h)(q_3(t, h) - \bar{c})),$$

where  $\mathcal{Q}(t, h) = (q_1(t, h), q_2(t, h), q_3(t, h))$  parametrizes a plane that does not contain the point  $\bar{P}$ , one deduces that  $\mu(t, h) = 0$ . Now, we consider the Taylor expansion of  $f(x, y, z)$  at  $\bar{P}$ ; that is,

$$f(x, y, z) = f(\bar{P}) + \sum_{j_1+j_2+j_3=1}^d \frac{\partial^{j_1+j_2+j_3} f}{\partial^{j_1} x \partial^{j_2} y \partial^{j_3} z}(\bar{P})(x - \bar{a})^{j_1} (y - \bar{b})^{j_2} (z - \bar{c})^{j_3} \frac{1}{j_1! j_2! j_3!}.$$

Thus,

$$f(\mathcal{H}_\lambda(t, h)) = f(\bar{a} + \lambda(q_1 - \bar{a}), \bar{b} + \lambda(q_2 - \bar{b}), \bar{c} + \lambda(q_3 - \bar{c}))$$

$$\begin{aligned}
 &= f(\bar{P}) + \sum_{j_1+j_2+j_3=1}^d \frac{\partial^{j_1+j_2+j_3} f}{\partial^{j_1} x \partial^{j_2} y \partial^{j_3} z}(\bar{P}) \frac{\lambda^{j_1+j_2+j_3} (q_1 - \bar{a})^{j_1} (q_2 - \bar{b})^{j_2} (q_3 - \bar{c})^{j_3}}{j_1! j_2! j_3!} \\
 &= \lambda^{d-1} \left( \sum_{j_1+j_2+j_3=d-1}^d \frac{\partial^{j_1+j_2+j_3} f}{\partial^{j_1} x \partial^{j_2} y \partial^{j_3} z}(\bar{P}) \frac{\lambda^{j_1+j_2+j_3-d+1} (q_1 - \bar{a})^{j_1} (q_2 - \bar{b})^{j_2} (q_3 - \bar{c})^{j_3}}{j_1! j_2! j_3!} \right) \\
 &\quad + \left( f(\bar{P}) + \sum_{j_1+j_2+j_3=1}^{d-2} \frac{\partial^{j_1+j_2+j_3} f}{\partial^{j_1} x \partial^{j_2} y \partial^{j_3} z}(\bar{P}) \frac{\lambda^{j_1+j_2+j_3} (q_1 - \bar{a})^{j_1} (q_2 - \bar{b})^{j_2} (q_3 - \bar{c})^{j_3}}{j_1! j_2! j_3!} \right).
 \end{aligned}$$

Therefore,  $f(\mathcal{H}_\lambda(t, h))$  can be expressed as

$$f(\mathcal{H}_\lambda(t, h)) = \lambda^{d-1} M(t, h, \lambda) + N(t, h, \lambda),$$

where

$$N(t, h, \lambda) = T(\mathcal{H}_\lambda(t, h)), \quad M(t, h, \lambda) = \frac{S(\mathcal{H}_\lambda(t, h))}{\lambda^{d-1}},$$

$S(x, y, z)$  is the Taylor expansion of  $f(x, y, z)$  from order  $d - 1$  up to order  $d$  at  $\bar{P}$ , and  $T(x, y, z)$  is the Taylor expansion of order  $d - 1$  of the polynomial  $f(x, y, z)$  at  $\bar{P}$ . Note, that  $\deg_\lambda(M) = 1$  and  $\deg_\lambda(N) \leq d - 2$ . On the other hand, let  $U(t, h, \lambda)$  and  $V(t, h, \lambda)$  be the quotient and the remainder of  $f(\mathcal{H}_\lambda(t, h))$  and  $\lambda^{d-1}$  w.r.t.  $\lambda$ , respectively. Then, one has that

$$f(\mathcal{H}_\lambda(t, h)) = \lambda^{d-1} U(t, h, \lambda) + V(t, h, \lambda)$$

with  $\deg_\lambda(V) \leq d - 2$ . Thus,

$$\lambda^{d-1} (M(t, h, \lambda) - U(t, h, \lambda)) = V(t, h, \lambda) - N(t, h, \lambda).$$

Since the degree w.r.t.  $\lambda$  of  $V - N$  is less or equal than  $d - 2$  and  $\lambda^{d-1}$  divides to  $V - N$ , one deduces that  $M = U$  and  $V = N$ . Therefore, since  $\mu(t, h) = 0$  is the root in  $\lambda$  of

$$U(t, h, \lambda) = \sum_{j_1+j_2+j_3=d-1}^d \frac{\partial^{j_1+j_2+j_3} f}{\partial^{j_1} x \partial^{j_2} y \partial^{j_3} z}(\bar{P}) \frac{\lambda^{j_1+j_2+j_3-d+1} (q_1 - \bar{a})^{j_1} (q_2 - \bar{b})^{j_2} (q_3 - \bar{c})^{j_3}}{j_1! j_2! j_3!},$$

and since  $\mathcal{Q}(t, h)$  is a plane not containing the point  $\bar{P}$ , we deduce that for every  $i, j, k \in \mathbb{N}$  with  $i + j + k = d - 1$ ,

$$\frac{\partial^{d-1} f}{\partial^i x \partial^j y \partial^k z}(\bar{P}) = 0.$$

Finally, since the multiplicity of the  $\varepsilon$ -singularity  $\bar{P}$  is  $d - 1$ , there exist  $i_0, j_0, k_0 \in \mathbb{N}$  such that  $i_0 + j_0 + k_0 = d - 1$  and

$$\left| \frac{\partial^{d-1} f}{\partial^{i_0} x \partial^{j_0} y \partial^{k_0} z}(\bar{P}) \right| \geq \varepsilon \cdot \|f(x, y, z)\| > 0.$$

Thus  $\frac{\partial^{d-1} f}{\partial^{i_0} x \partial^{j_0} y \partial^{k_0} z}(\bar{P}) \neq 0$ , which contradicts the hypothesis.  $\square$

The following lemma shows that if  $\mathcal{Q}(t, h)$  is a proper parametrization, then  $\bar{P}(t, h)$  is also a proper parametrization. For this purpose, we assume w.l.o.g. that the plane not containing the point  $\bar{P}$  is  $z =$

$\bar{c} + 1$ , and that  $\varepsilon < 1$  (which ensures that  $\bar{P}$  is not on the plane). Otherwise, we would consider the plane  $z = \bar{c} + \rho$  with  $\rho > \varepsilon$ . In this situation, we consider the proper parametrization of the plane  $Q(t, h) = (t, h, \bar{c} + 1)$ . Then, the parametrization  $\bar{P}(t, h)$  is given by

$$\bar{P}(t, h) = (\bar{a} + \mu(t, h)(t - \bar{a}), \bar{b} + \mu(t, h)(h - \bar{b}), \bar{c} + \mu(t, h)).$$

**Lemma 3.** *The parametrization*

$$\bar{P}(t, h) = (\bar{a} + \mu(t, h)(t - \bar{a}), \bar{b} + \mu(t, h)(h - \bar{b}), \bar{c} + \mu(t, h))$$

is proper.

**Proof.** We denote by  $\bar{p}_i(t, h)$  the  $i$ th component of  $\bar{P}(t, h)$ . Thus, it holds that

$$t = \frac{(\bar{p}_1(t, h) - \bar{a})}{\bar{p}_3(t, h) - \bar{c}} + \bar{a}, \quad h = \frac{(\bar{p}_2(t, h) - \bar{b})}{\bar{p}_3(t, h) - \bar{c}} + \bar{b}.$$

Therefore,  $\bar{P}(t, h)$  is a proper parametrization and its inverse is given by

$$\left( \frac{(x - \bar{a})}{z - \bar{c}} + \bar{a}, \frac{(y - \bar{b})}{z - \bar{c}} + \bar{b} \right). \quad \square$$

Now we prove that if there exists an  $\varepsilon$ -singularity, then there exists infinitely many singularities close to the original one that can be considered equivalent. For this purpose, for  $\bar{P} \in \mathbb{R}^3$  and  $\delta > 0$ , we denote by  $B(\bar{P}, \delta)$  the Euclidean ball

$$B(\bar{P}, \delta) = \{(x, y, z) \in \mathbb{R}^3 \mid \|(x, y, z) - \bar{P}\|_2 < \delta\}.$$

**Lemma 4.** *Let  $V$  be an affine algebraic surface defined by the polynomial  $f(x, y, z) \in \mathbb{R}[x, y, z]$  with a real  $\varepsilon$ -singularity  $\bar{P}$  of multiplicity  $r$ . Then, there exists  $\delta > 0$  such that any point  $Q \in B(\bar{P}, \delta)$  is an  $\varepsilon$ -singularity of multiplicity at least  $r$  of  $V$ . Furthermore, there exist at least one partial derivative of  $f(x, y, z)$  of order  $r$  that does not vanish at  $Q$ .*

**Proof.** We represent by  $f_{i,j,k}$  the partial derivative  $\frac{\partial^{i+j+k} f}{\partial x^i \partial y^j \partial z^k}$ . Since  $\bar{P}$  is an  $\varepsilon$ -singularity of multiplicity  $r$ , for  $i + j + k = 0, \dots, r - 1$  it holds that  $|f_{i,j,k}(\bar{P})| < \varepsilon \|f\|$ . Let us denote  $|f_{i,j,k}(\bar{P})| = \varepsilon_{i,j,k}$  for  $i + j + k = 0, \dots, r - 1$ . Then, for each  $\varepsilon_{i,j,k}$  there exists  $\lambda_{i,j,k} > 0$  such that

$$\varepsilon_{i,j,k} = \varepsilon \|f\| - \lambda_{i,j,k} < \varepsilon \|f\|.$$

We consider  $\lambda = \min\{\lambda_{i,j,k}, i + j + k = 0, \dots, r - 1\}$  (note that  $\lambda > 0$ ). On the other hand, since all partial derivatives,  $|f_{i,j,k}|$ , are continuous, let  $M$  bound all partial derivatives up to order  $r$  in the compact set  $B(\bar{P}, \varepsilon)$ , and let  $\delta_1$  be strictly smaller than  $\min\{\lambda/(2M), \varepsilon\}$ ; note that  $M > 0$  since otherwise it would imply that  $V$  contains a 3-dimensional ball of points which is impossible. Now, take  $Q \in B(\bar{P}, \delta_1)$ . Then, by applying the Mean Value Theorem, we have that for  $i + j + k = 0, \dots, r - 1$

$$|f_{i,j,k}(Q)| \leq |f_{i,j,k}(\bar{P})| + |f_{i,j,k}(\bar{P}) - f_{i,j,k}(Q)| \leq \varepsilon_{i,j,k} + |\nabla f_{i,j,k}(\xi_{i,j,k}) \cdot (\bar{P} - Q)^T|,$$

where  $\xi_{i,j,k} = \bar{P} + \theta(\bar{P} - Q)$  with  $\theta \in (0, 1)$ . Therefore, one concludes that

$$|f_{i,j,k}(Q)| \leq \varepsilon \|f\| - \lambda_{i,j,k} + 2\delta_1 M \leq \varepsilon \|f\| - \lambda + 2\delta_1 M < \varepsilon \|f\|,$$

and then,  $Q$  is an  $\varepsilon$ -singularity of multiplicity at least  $r$  of the surface  $V$ .

On the other side, since  $\bar{P}$  is an  $\varepsilon$ -singularity of multiplicity  $r$ , in particular one deduces that there exists a partial derivative of  $f(x, y, z)$  of order  $r$  not vanishing at  $\bar{P}$ . We consider that  $g(x, y, z) = \frac{\partial^r f}{\partial^{i_0} x \partial^{j_0} y \partial^{k_0} z}(x, y, z)$ , for any  $i_0, j_0, k_0 \in \mathbb{N}$  with  $i_0 + j_0 + k_0 = r$ , is this derivative. Note that since  $g(x, y, z)$  is continuous and  $g(\bar{P}) \neq 0$  one has that there exists  $\delta_2 > 0$  such that for any  $Q \in B(\bar{P}, \delta_2)$  it holds that  $g(Q) \neq 0$ . Then, let  $\delta = \min\{\delta_1, \delta_2\}$ . Thus, for each  $Q \in B(\bar{P}, \delta)$  one has that  $Q$  is an  $\varepsilon$ -singularity of multiplicity at least  $r$  of  $V$  and that there exists at least a partial derivative of  $f(x, y, z)$  of order  $r$  not vanishing in  $Q$ .  $\square$

By Lemma 4, it holds that  $V$  has infinitely many  $\varepsilon$ -singularities of multiplicity at least  $(d - 1)$  quite close to  $\bar{P}$ . For our purposes, we are interested in choosing the singularity appropriately. More precisely, one has the following definition.

**Definition 4.** Let  $\bar{P}$  be a real  $\varepsilon$ -singularity of multiplicity  $(d - 1)$  of  $V$ . Then, we say that the point  $Q = (a, b, c)$  in the ball of Lemma 4 is a *proper  $\varepsilon$ -singularity of multiplicity  $d - 1$*  of the surface  $V$ , if the polynomial

$$\sum_{j_1+j_2+j_3=d-1}^d \frac{\partial^{j_1+j_2+j_3} f}{\partial^{j_1} x \partial^{j_2} y \partial^{j_3} z}(Q)(x - a)^{j_1}(y - b)^{j_2}(z - c)^{j_3} \frac{1}{j_1! j_2! j_3!}$$

is irreducible over  $\mathbb{C}$ .

Note that by Lemma 4, one deduces that there exists a ball with center at the point  $\bar{P}$ , of  $\varepsilon$ -singularities of  $V$  which multiplicity is at least  $d - 1$ . If  $\bar{P}$  is not proper, then the polynomial used in Definition 4 is reducible over  $\mathbb{C}$ . However, since a small perturbation of the coefficients of a polynomial transforms it onto an irreducible polynomial, one always may take a new point on the ball, close to  $\bar{P}$ , and such that it is a proper  $\varepsilon$ -singularity. Therefore, in the following we assume w.l.o.g. that  $\bar{P}$  is proper and thus, the polynomial in Definition 4 is irreducible. In addition, some partial derivative of  $f(x, y, z)$  of order  $d - 1$  at the point  $\bar{P}$  is not zero.

In particular, this implies that for this proper  $\varepsilon$ -singularity  $\bar{P}$ , Lemmas 3 and 4 are satisfied. That is,  $\bar{P}(t, h) = (\bar{p}_1(t, h), \bar{p}_2(t, h), \bar{p}_3(t, h))$  obtained with the above construction but considering  $\bar{P}$  a proper  $\varepsilon$ -singularity, is always a proper rational parametrization.

The following theorem shows that the implicit equation of the rational surface defined by the parametrization  $\bar{P}(t, h)$  generated by the above process can also be obtained as in the quadric case, by Taylor expansions at the  $\varepsilon$ -singularity. In fact, the theorem includes as a particular case the result for quadrics (see Theorem 1). This result will avoid quotient computations and will be used to analyze the error in Section 4.

**Theorem 3.** Let  $f(x, y, z)$  be the implicit equation of the surface  $V \in S_\varepsilon^d$  and let  $\bar{P} = (\bar{a}, \bar{b}, \bar{c})$  be a proper  $\varepsilon$ -singularity of multiplicity  $d - 1$  of  $V$ . We consider a generic point  $Q(t, h) = (q_1(t, h), q_2(t, h), q_3(t, h))$  of a plane not containing the point  $\bar{P}$ , and we consider the pencil of lines defined by

$$\mathcal{H}_\lambda(t, h) = \bar{P} + \lambda(Q(t, h) - \bar{P}) = (\bar{a} + \lambda(q_1 - \bar{a}), \bar{b} + \lambda(q_2 - \bar{b}), \bar{c} + \lambda(q_3 - \bar{c})).$$



Let  $\mu(t, h)$  be the root in  $\mathbb{R}(t, h)$  of the quotient of  $f(\mathcal{H}_\lambda(t, h))$  and  $\lambda^{d-1}$  w.r.t.  $\lambda$ . Then, the implicit equation of the rational surface  $\bar{V}$  defined by the parametrization

$$\bar{\mathcal{P}}(t, h) = \bar{P} + \mu(t, h)(Q(t, h) - \bar{P}),$$

is

$$\bar{f}(x, y, z) = f(x, y, z) - T(x, y, z),$$

where  $T(x, y, z)$  is the Taylor expansion up to order  $d - 1$  of  $f(x, y, z)$  at  $\bar{P}$ .

**Proof.** Reasoning as we did in the proof of Lemma 4 we deduce that

$$f(\mathcal{H}_\lambda) = \lambda^{d-1}M(t, h, \lambda) + N(t, h, \lambda),$$

where

$$N(t, h, \lambda) = T(\mathcal{H}_\lambda(t, h)), \quad M(t, h, \lambda) = \frac{S(\mathcal{H}_\lambda(t, h))}{\lambda^{d-1}},$$

and  $S(x, y, z)$  is the Taylor expansion of  $f(x, y, z)$  from order  $d - 1$  up to order  $d$  at  $\bar{P}$ . Furthermore,  $M = U$  and  $V = N$  where  $U(t, h, \lambda)$  and  $V(t, h, \lambda)$  are the quotient and the remainder of  $f(\mathcal{H}_\lambda(t, h))$  and  $\lambda^{d-1}$  w.r.t.  $\lambda$ , respectively. In this situation,

$$\begin{aligned} \bar{f}(\bar{\mathcal{P}}(t, h)) &= f(\bar{\mathcal{P}}(t, h)) - T(\bar{\mathcal{P}}(t, h)) = f(\mathcal{H}_{\mu(t, h)}(t, h)) - T(\mathcal{H}_{\mu(t, h)}(t, h)) \\ &= \mu(t, h)^{d-1}U(t, h, \mu(t, h)) + N(t, h, \mu(t, h)) - T(\mathcal{H}_{\mu(t, h)}(t, h)) \\ &= N(t, h, \mu(t, h)) - T(\mathcal{H}_{\mu(t, h)}(t, h)) = T(\mathcal{H}_{\mu(t, h)}(t, h)) - T(\mathcal{H}_{\mu(t, h)}(t, h)) = 0. \end{aligned}$$

In addition, since  $\bar{P}$  is a proper  $\varepsilon$ -singularity of  $V$ , one has that  $\bar{f}$  is irreducible and then,  $\bar{\mathcal{P}}(t, h)$  parametrizes  $\bar{V}$ .  $\square$

Theorem 3 provides the following algorithm for parametrizing approximate algebraic surfaces  $V \in \mathcal{S}_\varepsilon^d$ .

**Algorithm 2.** Approximate parametrization by lines for surfaces.

- **Given** a tolerance  $\varepsilon > 0$  and an  $\varepsilon$ -irreducible polynomial  $f(x, y, z) \in \mathbb{R}[x, y, z]$ , defining a surface  $V \in \mathcal{S}_\varepsilon^d$ .
- **Compute** a rational parametrization  $\bar{\mathcal{P}}(t, h)$  of a rational surface  $\bar{V}$  close to  $V$ .

- (1) Compute a proper  $\varepsilon$ -singularity  $\bar{P} = (\bar{a}, \bar{b}, \bar{c})$  of  $V$  of multiplicity  $d - 1$ .
- (2) Determine  $\bar{f}(x, y, z) = f(x, y, z) - T(x, y, z)$ , where  $T(x, y, z)$  is the Taylor expansion of  $f(x, y, z)$  up to order  $d - 1$  at  $\bar{P}$ .
- (3) Compute

$$A(x, y, z, t, h) = \frac{\sum_{r+s=0}^{d-1} \frac{\partial^{d-1} \bar{f}}{\partial^r x \partial^s y \partial^{d-1-r-s} z} t^r h^s}{(d-1-r-s)! r! s!},$$

$$\sum_{r+s=0}^d \frac{\partial^d \bar{f}}{\partial^r x \partial^s y \partial^{d-r-s} z} t^r h^s}{(d-r-s)! r! s!},$$

and return

$$\bar{\mathcal{P}}(t, h) = (-tA(\bar{P}, t, h) + \bar{a}, -hA(\bar{P}, t, h) + \bar{b}, -A(\bar{P}, t, h) + \bar{c}).$$

Taking into account Theorem 3 and the above algorithm, one deduces the following result.

**Theorem 4.** *The input surface and the output surface provided by the Algorithm 2 have the same degree. Furthermore, the input surface and the output surface have the same homogeneous form of maximum degree.*

### 3.1. Examples

In this subsection we illustrate Algorithm 2 by some examples. First, we give two examples (Examples 3 and 4) where we explain carefully how the algorithm is performed, and we remark the mentioned particular properties of the parametrization method. More precisely, we see that the polynomial  $\bar{f}$  is irreducible and the surface  $\bar{V}$  is rational. In addition, we check that the input polynomial  $f$  and the output polynomial  $\bar{f}$  have the same homogeneous form of maximum degree (see Theorem 4). Moreover, we can also check that the output parametrization  $\bar{\mathcal{P}}(t, h)$  is proper. Afterwards, we present three other examples (Examples 5, 6 and 7) where details are omitted, and we only give the input surface  $V$ , the  $\varepsilon$ -singularity  $\bar{P}$ , the output rational surface  $\bar{V}$ , and its rational parametrization  $\bar{\mathcal{P}}(t, h)$ .

In these examples one observes that the output surface  $\bar{V}$  is close to the input surface  $V$ . This fact will be theoretically studied in Section 4. Nevertheless, in order to illustrate this property, we estimate for this particular examples the distance of the input and the output surface. For this purpose, we proceed similarly as we did in Example 2. More precisely, we randomly generated 10000 points on the output surface. Since this surface is rational, and since we know a parametrization of it, the points on the surface are obtained by giving random values to the parameters. Once the points on the output surface are determined, we compute, by intersecting with a pencil of lines (see details in the Examples 3 and 4), points on the input surface, and we measure the corresponding distances. For the set of all the obtained distances we get the mean value as well as the statistical standard error, that tend to be very small.

**Example 3.** We consider  $\varepsilon = 0.001$  and the surface  $V$  defined by the polynomial

$$\begin{aligned} f(x, y, z) = & 34.03308880x + 10.11353500y - 178.7688488z + x^4 + z^4 - 15.99600000z^3 \\ & - 18.98589600x^2 + 85.94410400z^2 + y^4 + 12.00400000y^3 + 2.002000000y^2x \\ & + 8.017004000yx + 47.00299600zy - 4.004000000x^3 + 37.04200400y^2 \\ & - 6.002000000z^2y - 13.01099600zx + 2.002000000z^2x + 180.7991556 + xyz \\ & - 1.x^2z^2 - 1.y^2z^2 - 1.x^2y^2 - 6.002000000x^2y + 7.998000000y^2z \\ & + 7.998000000x^2z. \end{aligned}$$

First of all, by applying the algorithm developed in (Sasaki, 2001), we observe that the polynomial  $f(x, y, z)$  is  $\varepsilon$ -irreducible. Now, we apply step (1) of Algorithm 2, and we compute the  $\varepsilon$ -singularity. For this purpose, we determine the solutions of the system (see (Fortune, 2001; Krishnan and Manocha, 1996))

$$\mathcal{B} = \left\{ \frac{\partial^2 f}{\partial^2 x}(x, y, z) = \frac{\partial^2 f}{\partial^2 z}(x, y, z) = \frac{\partial^2 f}{\partial x \partial y}(x, y, z) = 0 \right\},$$

under precision  $\varepsilon \|f(x, y, z)\| = 0.1807991556$ . We get four solutions, namely

$$\begin{aligned} \bar{P}_1 &= (1.001000000, -3.011000000, 3.999000000), \\ \bar{P}_2 &= (1.001000000, -2.991000000, 3.999000000), \\ \bar{P}_3 &= (.8892860796, -2.751000000, 3.887286080), \\ \bar{P}_4 &= (.8892860796, -3.251000000, 4.110713920). \end{aligned}$$

The points  $\bar{P}_1$  and  $\bar{P}_2$  satisfy that

$$\left| \frac{\partial^{i+j} f}{\partial^i x \partial^j y}(\bar{P}_r) \right| < 0.1807991556, \quad i + j = 0, \dots, 2, \quad r = 1, 2,$$

and

$$\left| \frac{\partial^3 f}{\partial^2 x \partial z}(\bar{P}_r) \right| \geq 0.1807991556, \quad r = 1, 2.$$

Then, we take  $\bar{P} := \bar{P}_1$  as the  $\varepsilon$ -singularity of multiplicity 3 of the surface  $V$ . Thus,  $V \in \mathcal{S}_{0.001}^4$  (similarly if we take  $\bar{P} := \bar{P}_2$ ). In addition, since the corresponding polynomial in Definition 4

$$\sum_{j_1+j_2+j_3=3}^4 \frac{\partial^{j_1+j_2+j_3} f}{\partial^{j_1} x \partial^{j_2} y \partial^{j_3} z}(\bar{P})(x - 1.001000000)^{j_1} (y + 3.011000000)^{j_2} (z - 3.999000000)^{j_3}}{j_1! j_2! j_3!}$$

is irreducible over  $\mathbb{C}$  we have that  $\bar{P}$  is a proper  $\varepsilon$ -singularity. Now, applying step (2) of the algorithm we get the rational surface  $\bar{V}$  defined implicitly by

$$\begin{aligned} \bar{f}(x, y, z) &= 33.99279880x + 10.10942580y - 178.7788588z + x^4 + z^4 - 15.99600000z^3 \\ &\quad - 18.98589600x^2 + 85.94410400z^2 + y^4 + 12.00400000y^3 + 2.002000000y^2x \\ &\quad + 8.017004000yx + 47.00299600zy - 4.004000000x^3 + 37.04140400y^2 \\ &\quad + 180.8334578 - 6.002000000z^2y - 13.00099600zx + 2.002000000z^2x + xyz \\ &\quad - 1.x^2z^2 - 1.y^2z^2 - 1.x^2y^2 - 6.002000000x^2y + 7.998000000y^2z \\ &\quad + 7.998000000x^2z. \end{aligned}$$

Observe that polynomials  $f$  and  $\bar{f}$  have the same degree, and the same homogeneous form of maximum degree (see Theorem 4). In addition, note that

$$\bar{f}(\bar{P}) = 0, \quad \frac{\partial^{j_1+j_2+j_3} \bar{f}}{\partial^{j_1} x \partial^{j_2} y \partial^{j_3} z}(\bar{P}) = 0, \quad \text{for } j_1 + j_2 + j_3 \in \{1, 2\}.$$

Therefore,  $\bar{P}$  is a singularity of multiplicity 3 of the surface  $\bar{V}$ . Thus, since  $\bar{f}$  is irreducible one has that  $\bar{V}$  is rational. In order to compute a parametrization of  $\bar{V}$ , we apply step (3) of Algorithm 2 to obtain the rational parametrization

$$\bar{\mathcal{P}}(t, h) = (\bar{p}_1(t, h), \bar{p}_2(t, h), \bar{p}_3(t, h)),$$

where

$$\begin{aligned} \bar{p}_1(t, h) &= (.3456018242ht - .5005000000 \times 10^{-1}h^2t^2 - .1484481000ht^2 + .1182661000h^2t \\ &\quad + 3.653785283 + .6008002000h^3 + .5005000000 \times 10^{-1}h^4 + 2.558076611h \\ &\quad + 2.204172170h^2 + .7233870993t - .1424667228t^2 - .2034112000t^3 \end{aligned}$$

$$\begin{aligned}
& + .5005000000 \times 10^{-1}t^4 - .1000000000 \times 10^{-2}ht^3 + .2000000000 \times 10^{-2}h^3t) \\
& / (.6028022000ht - .5000000000 \times 10^{-1}h^2t^2 - .3011000000ht^2 + .1001000000h^2t \\
& + 3.222682417 + .6022000000h^3 + .5000000000 \times 10^{-1}h^4 + 2.449821864h \\
& + 2.220036200h^2 + 1.607217612t - .6024058000t^2 - .2002000000t^3 \\
& + .5000000000 \times 10^{-1}t^4),
\end{aligned}$$

$$\begin{aligned}
\bar{p}_2(t, h) = & (-.8999824802ht + .1495500000h^2t^2 + .9005901000ht^2 - .1494491000h^2t \\
& - 10.99055693 - 1.789136200h^3 - .1485500000h^4 - 8.122126733h \\
& - 6.735831498h^2 - 3.461717012t + 1.804777743t^2 + .6028022000t^3 \\
& - .1505500000t^4) / (.6028022000ht - .5000000000 \times 10^{-1}h^2t^2 - .3011000000ht^2 \\
& + .1001000000h^2t + 3.222682417 + .6022000000h^3 + .5000000000 \times 10^{-1}h^4 \\
& + 2.449821864h + 2.220036200h^2 + 1.607217612t - .6024058000t^2 \\
& - .2002000000t^3 + .5000000000 \times 10^{-1}t^4),
\end{aligned}$$

$$\begin{aligned}
\bar{p}_3(t, h) = & (1.954901950ht - .1999500000h^2t^2 - 1.201099900ht^2 + .4002999000h^2t \\
& + 14.16943772 + 2.402199800h^3 + .1999500000h^4 + 10.11382961h \\
& + 8.823744830h^2 + 5.055138342t - 2.399990805t^2 - .8005998000t^3 \\
& + .1999500000t^4) / (.6028022000ht - .5000000000 \times 10^{-1}h^2t^2 - .3011000000ht^2 \\
& + .1001000000h^2t + 3.222682417 + .6022000000h^3 + .5000000000 \times 10^{-1}h^4 \\
& + 2.449821864h + 2.220036200h^2 + 1.607217612t - .6024058000t^2 \\
& - .2002000000t^3 + .5000000000 \times 10^{-1}t^4).
\end{aligned}$$

In Fig. 5 one may compare the input and the output surfaces. Note that the surface  $\bar{V}$  is close to the input surface  $V$ . This behavior will be studied in Section 4. However, as we have mentioned above, we will statistically estimate the distance between  $\bar{V}$  and  $V$ .

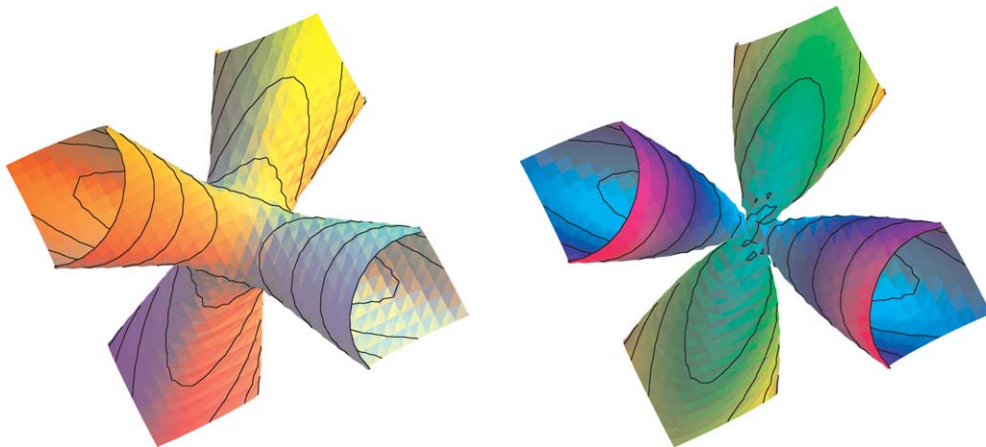


Fig. 5. Input surface  $V$  (left) and output surface  $\bar{V}$  (right).

For this purpose, we particularize the theoretical reasoning in Section 4 to this example. More precisely, giving random values to the parameters  $t, h$  in  $\overline{P}(t, h)$  we generate 10000 points on  $\overline{V}$ . Let  $\mathcal{A}$  denote the set of all these points. Now, for each point  $Q \in \mathcal{A}$  we consider the line  $\mathcal{L}$  of parametric equation  $Q + \lambda(Q - \overline{P})$ . Afterwards, we compute the intersection of the line  $\mathcal{L}$  and the surface  $V$ , i.e., we approximate the roots of  $f(Q + \lambda(Q - \overline{P}))$ . This computation yields to a finite set  $\mathcal{B}_Q$  of points on  $V$ , and we take the minimum of the Euclidean distances of  $Q$  to the points in  $\mathcal{B}_Q$ . After repeating this construction for all points in  $\mathcal{A}$ , we have a set of distances, and we compute their mean value  $\mu$  as well as the statistical standard error  $\rho$ . In this case, we have obtained

$$\mu = 0.03632991728, \quad \rho = 0.00005474193152,$$

from where one can statistically deduce that the distance is, in average, in the interval  $[\mu - 1.96\rho, \mu + 1.96\rho] = [0.03622262309, 0.03643721147]$ .

**Example 4.** We consider  $\varepsilon = 0.001$  and the surface  $V$  defined by the polynomial

$$\begin{aligned} f(x, y, z) = & 48.02701301x + 621.5981530y + y^5 - 9.005000000x^4 - 6.002000000z^4 \\ & - 6.003000000z^3 - 56.09505401x^2 + .1000000000 \times 10^{-2}z^2 - 17.005000000y^4 \\ & + 114.0620070y^3 - .2700900000 \times 10^{-1}y^2x + .8105400900 \times 10^{-1}yx \\ & + 32.03601000x^3 - 378.2880570y^2 + 2.z^4y + x^5 + 3.z^3x \\ & + .3000000000 \times 10^{-2}y^3x - 421.5251910. \end{aligned}$$

First of all, by applying the algorithm developed in (Sasaki, 2001), we observe that the polynomial  $f(x, y, z)$  is  $\varepsilon$ -irreducible. Now, we apply step (1) of Algorithm 2, and we compute the  $\varepsilon$ -singularity. For this purpose, we determine the solutions of the system (see (Fortune, 2001; Golub and Van Loan, 1989; Krishnan and Manocha, 1996))

$$\mathcal{B} = \left\{ \frac{\partial^3 f}{\partial^3 x}(x, y, z) = \frac{\partial^3 f}{\partial^3 y}(x, y, z) = \frac{\partial^3 f}{\partial^2 z \partial x}(x, y, z) = 0 \right\},$$

under precision  $\varepsilon \|f(x, y, z)\| = .6215981530$ . One gets the solutions

$$\begin{aligned} \overline{P}_1 &= (2.001000000, 3.001000000, 0), \\ \overline{P}_2 &= (2.001000000, 3.801000000, 0), \\ \overline{P}_3 &= (1.601000000, 3.000850028, 0). \end{aligned}$$

The point  $\overline{P}_1$  satisfies that

$$\left| \frac{\partial^{i+j} f}{\partial^i x \partial^j y}(\overline{P}_1) \right| < .6215981530, \quad i + j = 0, \dots, 3,$$

and

$$\left| \frac{\partial^4 f}{\partial^4 x}(\overline{P}_1) \right| \geq .6215981530.$$

Thus, we take  $\overline{P} := \overline{P}_1$ , and then  $V \in \mathcal{S}_{0.001}^5$ . Furthermore, since the corresponding polynomial in Definition 4 associate to  $\overline{P}$  is irreducible over  $\mathbb{C}$ , one has that  $\overline{P}$  is a proper  $\varepsilon$ -singularity. Now, we apply step (2) of Algorithm 2, and we get the rational surface  $\overline{V}$  defined by the polynomial

$$\begin{aligned}\bar{f}(x, y, z) = & 48.03101501x + 621.5941530y + y^5 - 9.005000000x^4 \\ & - 6.002000000z^4 - 6.003000000z^3 - 56.09605401x^2 - 17.00500000y^4 \\ & + 114.0620070y^3 - .2700900000 \times 10^{-1}y^2x + .8105400900 \times 10^{-1}yx \\ & + 32.03601000x^3 - 378.2880570y^2 + 2.z^4y + x^5 + 3.z^3x \\ & + .3000000000 \times 10^{-2}y^3x - 421.5071910.\end{aligned}$$

Observe that polynomials  $f$  and  $\bar{f}$  have the same degree, and the same homogeneous form of maximum degree (see Theorem 4). In addition, note that

$$\bar{f}(\bar{P}) = 0, \quad \frac{\partial^{j_1+j_2+j_3} \bar{f}}{\partial^{j_1}x \partial^{j_2}y \partial^{j_3}z}(\bar{P}) = 0, \quad \text{for } j_1 + j_2 + j_3 \in \{1, 2, 3\}.$$

Therefore,  $\bar{P}$  is a singularity of multiplicity 4 of the surface  $\bar{V}$ . Thus, since  $\bar{f}$  is irreducible one has that  $\bar{V}$  is rational. In order to compute a parametrization of  $\bar{V}$ , we apply step (3) of Algorithm 2 to obtain the rational parametrization

$$\bar{P}(t, h) = (\bar{p}_1(t, h), \bar{p}_2(t, h), \bar{p}_3(t, h)),$$

where

$$\begin{aligned}\bar{p}_1(t, h) = & (.2541379019t + 1.247811900h - .8105400900 \times 10^{-4}ht^2 + .2700900000 \times 10^{-4}h^2t^2 \\ & + .1079639220h^2t - .2399599400 \times 10^{-1}h^3t + .2001000000 \times 10^{-2}h^5 \\ & - .8311909904 \times 10^{-1}t^2 + .1001000000 \times 10^{-2}t^5 - .1001500500 \times 10^{-1}t^4 \\ & + .4008005001 \times 10^{-1}t^3 - .3402700500 \times 10^{-1}h^4 + .2282380760h^3 \\ & - .7569544021h^2 - .2158916939ht - .3000000000 \times 10^{-5}h^3t^2 \\ & + .2000000000 \times 10^{-2}th^4 - .8674578942)/(.1000000000 \times 10^{-2}h^5 \\ & + .4075402701h - .2814873501 - .8012006001 \times 10^{-1}t^2 + .1000000000 \times 10^{-2}t^5 \\ & - .1000500000 \times 10^{-1}t^4 + .4004001000 \times 10^{-1}t^3 + .8016012004 \times 10^{-1}t \\ & - .2702700900h^2 - .1500500000 \times 10^{-1}h^4 + .9006001000 \times 10^{-1}h^3),\end{aligned}$$

$$\begin{aligned}\bar{p}_2(t, h) = & (.1531440760t + 2.023430894h - .2402400600 \times 10^{-1}ht^2 - .1000000000 \times 10^{-2}ht^4 \\ & + .8004000000 \times 10^{-2}ht^3 - .1621080180 \times 10^{-3}h^2t + .3601200000 \times 10^{-4}h^3t \\ & + .5001000000 \times 10^{-2}h^5 - .1683442581t^2 + .3001000000 \times 10^{-2}t^5 \\ & - .2702400500 \times 10^{-1}t^4 + .9614006601 \times 10^{-1}t^3 - .7503400200 \times 10^{-1}h^4 \\ & + .4503180500h^3 - 1.351296342h^2 + .2937234811 \times 10^{-1}ht - .300000000010^{-5}th^4 \\ & - 1.300970085)/(.1000000000 \times 10^{-2}h^5 + .4075402701h - .2814873501 \\ & - .8012006001 \times 10^{-1}t^2 + .1000000000 \times 10^{-2}t^5 - .1000500000 \times 10^{-1}t^4 \\ & + .4004001000 \times 10^{-1}t^3 + .8016012004 \times 10^{-1}t - .2702700900h^2 \\ & - .1500500000 \times 10^{-1}h^4 + .9006001000 \times 10^{-1}h^3),\end{aligned}$$

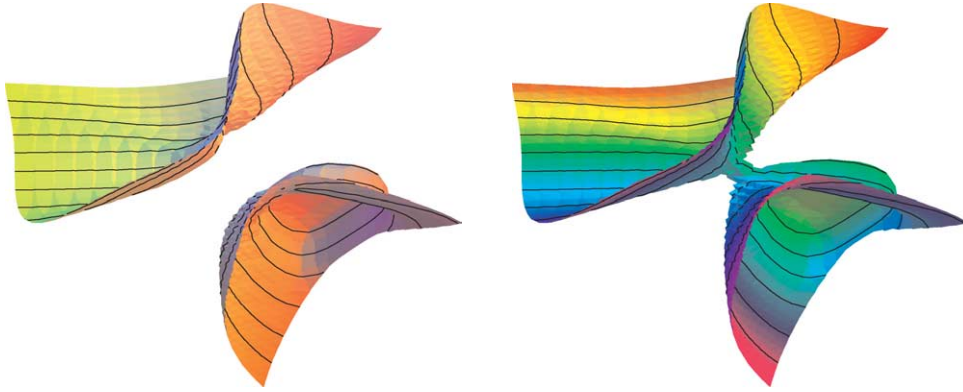


Fig. 6. Input surface  $V$  (left) and output surface  $\bar{V}$  (right).

$$\begin{aligned} \bar{p}_3(t, h) = & (.2700900000h^2t - .8105400900ht + 291.2910503t + 20.00000000h^4 \\ & - 2160.538829h + 1080.179670h^2 + 80.04000000t^3 - 240.0199700h^3 \\ & - 10.00000000t^4 - .3000000000 \times 10^{-1}h^3t - 240.2400600t^2 + 1520.248409) \\ & / (10.00000000h^5 + 4075.402701h - 2814.873501 - 801.2006001t^2 \\ & + 10.00000000t^5 - 100.0500000t^4 + 400.4001000t^3 + 801.6012004t \\ & - 2702.700900h^2 - 150.0500000h^4 + 900.6001000h^3). \end{aligned}$$

In Fig. 6 one may check that the input and the output surfaces are close. In fact, reasoning as we did in Example 3, one gets that in this case

$$\mu = 0.01420600280, \quad \rho = 0.0003023364344,$$

from where one can statistically deduce that the distance is, in average, in the interval  $[\mu - 1.96\rho, \mu + 1.96\rho] = [0.01361342339, 0.01479858221]$ .

**Example 5.** Let  $\varepsilon = 0.1$  and the surface  $V$  defined by the polynomial

$$\begin{aligned} f(x, y, z) = & x^4 + 2x^2y^2 + y^4 + 9xz^2y^2 - 3x^3z^2 + .100000x + .100000y \\ & + .100000x^2 + .100000z^2 + .300000. \end{aligned}$$

This surface has an  $\varepsilon$ -singularity of multiplicity 4 at  $\bar{P} = (0, 0, 0)$ ; hence,  $V \in \mathcal{S}_{0,1}^5$ . Furthermore, this  $\varepsilon$ -singularity is proper because the polynomial in Definition 4

$$\sum_{j_1+j_2+j_3=3}^5 \frac{\partial^{j_1+j_2+j_3} f}{\partial^{j_1} x \partial^{j_2} y \partial^{j_3} z}(\bar{P}) x^{j_1} y^{j_2} z^{j_3} \frac{1}{j_1! j_2! j_3!}$$

is irreducible over  $\mathbb{C}$ . Thus, applying step (2) of Algorithm 2, we obtain the surface  $\bar{V}$  defined by the irreducible polynomial

$$\bar{f}(x, y, z) = x^4 + 2x^2y^2 + y^4 + 9xz^2y^2 - 3x^3z^2.$$

Finally, we apply step (3) to get the proper parametrization of the surface  $\bar{V}$

$$\bar{P}(t, h) = \left( \frac{t^4 + 2t^2h^2 + h^4}{3(-3h^2 + t^2)}, \frac{(t^4 + 2t^2h^2 + h^4)h}{3t(-3h^2 + t^2)}, \frac{t^4 + 2t^2h^2 + h^4}{3t(-3h^2 + t^2)} \right).$$

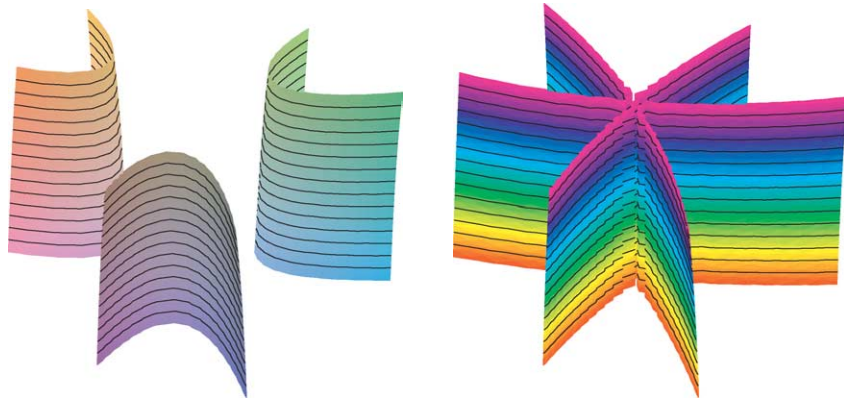


Fig. 7. Input surface  $V$  (left) and output surface  $\bar{V}$  (right).

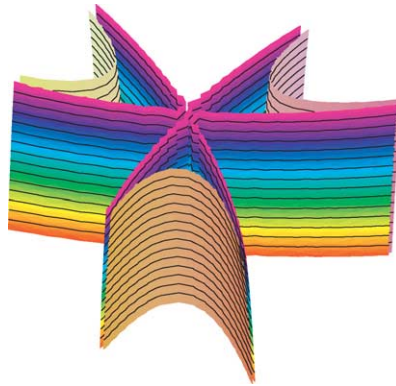


Fig. 8. Input surface  $V$  and output surface  $\bar{V}$ .

See Figs. 7 and 8 to compare the input and the output surfaces. Reasoning as in the previous examples, one gets that in this case

$$\mu = 0.1204696300 \times 10^{-8}, \quad \rho = 0.4229716566 \times 10^{-9},$$

from where one can statistically deduce that the distance is, in average, in the interval  $[\mu - 1.96\rho, \mu + 1.96\rho] = [0.3756718531 \times 10^{-9}, 0.2033720747 \times 10^{-8}]$ .

**Example 6.** Let  $\varepsilon = 0.001$  and the surface  $V$  defined by

$$f(x, y, z) = y^5 + x^5 + x^4 - 2y^4 + .000100z^4 - .001000x^3 - .001000z^3 - .001000 - .001000xy^2 - .001000y - .000010z^2 - .000100z.$$

This surface has an  $\varepsilon$ -singularity of multiplicity 4 at  $\bar{P} = (0.001, 0, 0.01)$ ; hence,  $V \in \mathcal{S}_{0.001}^5$ . In addition, this  $\varepsilon$ -singularity is proper because the polynomial in Definition 4

$$\sum_{j_1+j_2+j_3=4}^5 \frac{\partial^{j_1+j_2+j_3} f}{\partial^{j_1} x \partial^{j_2} y \partial^{j_3} z}(\bar{P})(x - 0.001)^{j_1} y^{j_2} (z - 0.01)^{j_3} \frac{1}{j_1! j_2! j_3!}$$

is irreducible over  $\mathbb{C}$ .



Now, applying step (2) of Algorithm 2 we get the rational surface  $\bar{V}$  defined by the irreducible polynomial

$$\begin{aligned}\bar{f}(x, y, z) = &-.4015000000 \times 10^{-8}x - .4000000000 \times 10^{-9}z + .2000000000 \times 10^{-11} \\ &+ y^5 + x^5 + x^4 - 2y^4 - .4010000000 \times 10^{-2}x^3 - .4000000000 \times 10^{-5}z^3 \\ &+ .1000000000 \times 10^{-3}z^4 + .6000000000 \times 10^{-7}z^2 + .6020000000 \times 10^{-5}x^2.\end{aligned}$$

Finally, applying step (3) we get the parametrization

$$\bar{P}(t, h) = (\bar{p}_1(t, h), \bar{p}_2(t, h), \bar{p}_3(t, h)),$$

$$\begin{aligned}\bar{p}_1(t, h) = &(-.100400t^3 + 50.2000t^4 + 10.0000h^5 - 10040.00t^5 + .100400 \times 10^{-3}t^2 \\ &- .9605960604t + .96059602 \times 10^{-3} + 20000.00th^4 - 20.0000h^4) \\ &/(.1000000000t^3 - 50.0000t^4 + 10000.00h^5 + 10000.00t^5 - .100000 \times 10^{-3}t^2 \\ &+ .5000000 \times 10^{-7}t - .100000 \times 10^{-10}),\end{aligned}$$

$$\begin{aligned}\bar{p}_2(t, h) = &(-.500000 \times 10^{-10}(-804000t - .8040000000 \times 10^{12}t^3 + 12060000t^2 \\ &+ .2010000 \times 10^{15}t^4 + .192119204 \times 10^{11} - .400000 \times 10^{15}h^4)h) \\ &/(.1000000t^3 - 50.0000t^4 + 10000.00h^5 + 10000.00t^5 - .100000 \times 10^{-3}t^2 \\ &+ .5000000 \times 10^{-7}t - .100000 \times 10^{-10}),\end{aligned}$$

$$\begin{aligned}\bar{p}_3(t, h) = &(39.799000t^3 - 9950.0000t^4 + 100.0000h^5 + 100.0000t^5 - .059698000t^2 \\ &+ .3979850000 \times 10^{-4}t - .95099006 + 19800.000h^4) \\ &/(.1000000t^3 - 50.0000t^4 + 10000.00h^5 + 10000.00t^5 \\ &- .100000 \times 10^{-3}t^2 + .5000000 \times 10^{-7}t - .100000 \times 10^{-10}).\end{aligned}$$

See Fig. 9 to compare the input and the output surfaces. Reasoning as in the previous examples, one gets that in this case

$$\mu = 0.05044250369, \quad \rho = 0.001564587377,$$

from where one can statistically deduce that the distance is, in average, in the interval  $[\mu - 1.96\rho, \mu + 1.96\rho] = [0.04737591243, 0.05350909495]$ .

**Example 7.** Let  $\varepsilon = 0.001$  and the surface  $V$  defined by

$$\begin{aligned}f(x, y, z) = &y^7 + x^7z + 2yx^5z^2 - 4y^5xz - .001000xz + .001000 - .001000y - .001000x^4z \\ &- .001000y^4.\end{aligned}$$

This surface has an  $\varepsilon$ -singularity of order 6 at  $\bar{P} = (0, 0.001, 0.001)$ ; hence,  $V \in \mathcal{S}_{0.001}^7$ . In addition, this  $\varepsilon$ -singularity is proper because the polynomial in Definition 4 is irreducible over  $\mathbb{C}$ .

Thus, applying step (2) of Algorithm 2, one obtains the rational surface  $\bar{V}$  defined by the irreducible polynomial

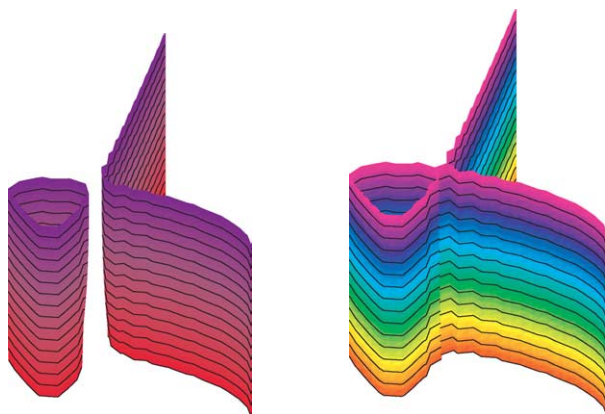


Fig. 9. Input surface  $V$  (left) and output surface  $\bar{V}$  (right).

$$\begin{aligned} \bar{f}(x, y, z) = &-.400000 \times 10^{-11}x + .700000 \times 10^{-11}y + .200000y^4zx + x^7z + .400000 \times 10^{-9}xz \\ &+ y^7 - .350000 \times 10^{-4}y^4 - .200000 \times 10^{-6}yxz + .400000 \times 10^{-4}y^2xz \\ &- .400000 \times 10^{-2}y^3xz - .210000 \times 10^{-8}y^2 - .100000 \times 10^{-13} \\ &+ .350000 \times 10^{-6}y^3 - .700000 \times 10^{-1}y^6 + .210000 \times 10^{-2}y^5 + .200000 \times 10^{-8}yx \\ &- .400000 \times 10^{-6}y^2x + .400000 \times 10^{-4}y^3x - .200000 \times 10^{-2}y^4x \\ &- .200000 \times 10^{-3}yx^5 - .400000 \times 10^{-3}zx^5 + .400000 \times 10^{-1}y^5x + 2yx^5z^2 \\ &- 4y^5xz + .400000 \times 10^{-5}x^5. \end{aligned}$$

Finally, we apply step (3) of the algorithm to get the parametrization

$$\bar{\mathcal{P}}(t, h) = (\bar{p}_1(t, h), \bar{p}_2(t, h), \bar{p}_3(t, h)),$$

where

$$\begin{aligned} \bar{p}_1(t, h) = &(-200.0000t^5h - 50.50505051t^7 - .200000 \times 10^{-5}t - 97.00000001t^5 \\ &+ 20000.00000th^5 - .20000000th^2 + .100000000 \times 10^{-2}th - 1000.000th^4 \\ &+ 20.00000th^3 + .5050505051 \times 10^{-10} - .3535353536 \times 10^{-7}h \\ &+ 353.5353536h^6 - .1767676768 \times 10^{-2}h^3 + .1060606061 \times 10^{-4}h^2 \\ &- 10.60606061h^5 + .1767676768h^4 - 5050.505051h^7)/(t^4(9900h - 99 + 5000t^2)), \end{aligned}$$

$$\begin{aligned} \bar{p}_2(t, h) = &(.200000 \times 10^{-7}t + .4040404041 \times 10^{-9}h + 4.000000t^5h - 1200.0000th^5 \\ &+ .300000 \times 10^{-2}th^2 - .1200000 \times 10^{-4}th + 30.0000th^4 - .40000th^3 \\ &+ 50.50505051t^7 - .020000000r^5 - 14.14141414h^6 + .2828282829 \times 10^{-4}h^3 \\ &- .1414141414 \times 10^{-6}h^2 + .2828282829h^5 - .003535353536h^4 + 404.04040410h^7 \\ &- .5050505051 \times 10^{-12} - 50.50505051ht^7 - 200.0000000t^5h^2 + 20000.0000th^6 \\ &- 5050.505051h^8)/(t^5(9900h - 99 + 5000t^2)), \end{aligned}$$

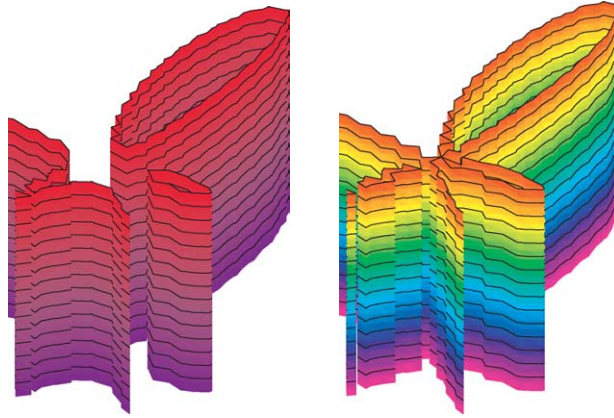


Fig. 10. Input surface  $V$  (left) and output surface  $\bar{V}$  (right).

$$\begin{aligned} \bar{p}_3(t, h) = & (-99.000000t^5h - 97.0200t^5 - .198000000 \times 10^{-5}t + 19800.00th^5 - .1980000th^2 \\ & + .99000000 \times 10^{-3}th - 990.00000th^4 + 19.80000th^3 + .5000000 \times 10^{-10} \\ & - .3500000 \times 10^{-7}h + 350.0000h^6 - .0017500000h^3 + .1050000 \times 10^{-4}h^2 \\ & - 10.50000h^5 + .1750000h^4 - 5000.000h^7)/(t^5(9900h - 99 + 5000t^2)). \end{aligned}$$

See Fig. 10 to compare the input and the output surfaces. Reasoning as in the previous examples, one gets that in this case

$$\mu = 0.01115123452, \quad \rho = 0.001626349788,$$

from where one can statistically deduce that the distance is, in average, in the interval  $[\mu - 1.96\rho, \mu + 1.96\rho] = [0.007963588936, 0.01433888010]$ .

#### 4. Error analysis

Examples in Sections 2 and 3 show that, in practice, the output surface of our algorithm is quite close to the input one. In this section we theoretically analyze how far these two affine surfaces are. For this purpose, we distinguish two subsections: the first one devoted to explain the general strategy, and the second one dealing with the theoretical results.

##### 4.1. General strategy

Let  $V \in \mathcal{S}_\varepsilon^d$  be the input surface of degree  $d$ , and let  $f(x, y, z)$  be its defining polynomial. In addition, let  $\bar{V}$  be the output surface generated by either Algorithm 1 or Algorithm 2, and let  $\bar{P}(t, h)$  be the output rational parametrization of  $\bar{V}$ . We first observe that, since we will measure distances, we may assume that the  $\varepsilon$ -singularity  $\bar{P}$  of the surface  $V$  is the origin; otherwise, one can apply a translation such that it is moved to the origin and distances are preserved. Also we assume that  $\|f(x, y, z)\| = 1$ , otherwise we consider  $\frac{f(x, y, z)}{\|f(x, y, z)\|}$ . If one does not normalize the input polynomial  $f(x, y, z)$ , a similar treatment with relative errors can be done.

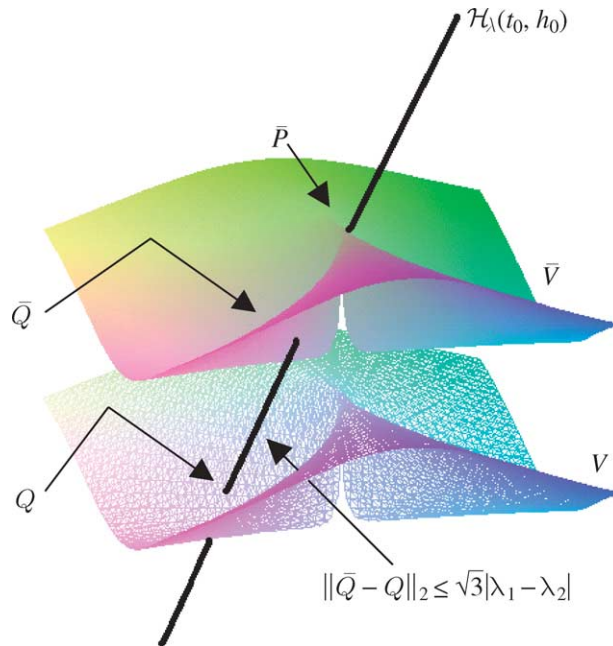


Fig. 11. General strategy.

In this situation, our general strategy consists in showing that almost all affine real points on the surface  $\bar{V}$  are at small distance of an affine point on  $V$ . For this purpose, we observe that  $\bar{P}(t, h)$  is an exact parametrization of  $\bar{V}$  obtained by lines, and therefore almost all affine real points on  $\bar{V}$  are obtained as the intersection of the surface  $\bar{V}$  with a pencil of lines. In fact, this pencil of lines is defined as

$$\mathcal{H}_\lambda(t, h) = \bar{P} + \lambda(Q(t, h) - \bar{P}) = \lambda Q(t, h), \quad \lambda \in \mathbb{R},$$

where  $Q(t, h)$  represents a generic point of a plane not containing the  $\varepsilon$ -singularity  $\bar{P}$  of  $V$ . In this error analysis, we consider w.l.o.g. that the plane is  $z = 1$ . Then,

$$Q(t, h) = (t, h, 1),$$

and

$$\mathcal{H}_\lambda(t, h) = (\lambda t, \lambda h, \lambda).$$

Therefore, almost all the real affine points on the surface  $\bar{V}$  are obtained as the intersection of the line  $y - hz = x - tz = 0$ , for  $t, h \in \mathbb{R}$ , with the surface  $\bar{V}$ . Then, if one intersects the surface  $V$  with the same line one gets, in almost all cases, finitely many points on  $V$ , and we show that at least one of these intersection points on  $V$  is close to the initial point on  $\bar{V}$ . Also, we observe that it is enough to reason with slope parameter values of  $t$  and  $h$  in the interval  $[-1, 1]$ , because if  $|t| > 1$  (similarly if  $|h| > 1$ ) one may apply a similar strategy considering the plane  $z = tx$  instead of  $x = tz$  in the definition of the line (similarly,  $z = hy$  instead of  $y = hz$ ).

Therefore, let  $t_0 \in \mathbb{R}$  and  $h_0 \in \mathbb{R}$  be such that  $|t_0| \leq 1$ ,  $|h_0| \leq 1$  and such that  $\bar{\mathcal{P}}(t, h)$  is defined at  $(t_0, h_0)$ . Then, the line  $\mathcal{H}_\lambda(t_0, h_0)$  intersects  $\bar{V}$  at  $\bar{P}$  and at an additional point  $\bar{Q}$ ; indeed  $\bar{Q} = \bar{P}(t_0, h_0)$ . Now, because of the construction, one has that  $\bar{Q}$  can be expressed as

$$\bar{Q} = (\lambda_0 t_0, \lambda_0 h_0, \lambda_0) \quad \text{where } \lambda_0 = \frac{m_0}{n_0} \text{ with some } m_0, n_0 \in \mathbb{R}.$$

If we write the affine point  $\bar{Q}$  projectively one has the point  $(m_0 t_0 : m_0 h_0 : m_0 : n_0)$ . Now, observe that if  $|m_0|$  and  $|n_0|$  are simultaneously very small, i.e., very close to  $\varepsilon$ , this point is not well defined as an element in  $\mathbb{P}^3(\mathbb{R})$ . For this reason, we will assume that either  $|m_0|$  or  $|n_0|$  is bigger than a certain bound that depends on the tolerance. In fact, for our error analysis, we fix that

$$|m_0| > \varepsilon^{1/d} \quad \text{or} \quad |n_0| > \varepsilon^{1/d}.$$

Furthermore, we observe that the defining polynomials of  $\bar{V}$  and  $V$  have the same homogeneous form of maximum degree (see Theorem 4), and hence both surfaces have the same points at infinity.

Now, let  $Q$  be any affine point in  $V \cap \mathcal{H}_\lambda(t_0, h_0)$ . Note that here it also holds that  $Q$  can be expressed as

$$Q = (\lambda_1 t_0, \lambda_1 h_0, \lambda_1) \quad \text{for some } \lambda_1 \in \mathbb{C}.$$

We want to compute the Euclidean distance between  $\bar{Q}$  and  $Q$ . In order to do that, we observe that

$$\begin{aligned} \|\bar{Q} - Q\|_2 &= \sqrt{(\lambda_1 t_0 - \lambda_0 t_0)^2 + (\lambda_1 h_0 - \lambda_0 h_0)^2 + (\lambda_1 - \lambda_0)^2} = |\lambda_1 - \lambda_0| \sqrt{t_0^2 + h_0^2 + 1} \\ &\leq \sqrt{3} |\lambda_1 - \lambda_0|. \end{aligned}$$

Therefore, we focus on the problem of computing a good bound for  $|\lambda_1 - \lambda_0|$ .

#### 4.2. Theoretical reasoning

Once we have described the general strategy, we proceed to bound the distance between  $V$  and  $\bar{V}$ . The bound that we present is given in terms of the degree  $d$  of the  $V$ , the tolerance  $\varepsilon$ , and the number  $e = \exp(1)$ . For this purpose we first prove two different lemmas.

**Lemma 5.** *It holds that*

$$|\lambda_1 - \lambda_0| \leq \varepsilon \cdot C,$$

where

$$C = \frac{\sum_{j_1+j_2+j_3=0}^{d-2} |\lambda_0|^{j_1+j_2+j_3} |t_0|^{j_1} |h_0|^{j_2} \frac{1}{j_1! j_2! j_3!}}{|\lambda_0|^{d-1} |n_0|}.$$

**Proof.** First of all, we note that  $\lambda_0 = \frac{m_0}{n_0}$  is a root of the univariate polynomial  $\bar{f}(\lambda t_0, \lambda h_0, \lambda) = \lambda^{d-1}(\lambda n_0 - m_0)$ , and that  $\lambda_1$  is a root of the univariate polynomial

$$f(\lambda t_0, \lambda h_0, \lambda) = \lambda^{d-1}(\lambda n_0 - m_0) + \sum_{j_1+j_2+j_3=0}^{d-2} \frac{\partial^{j_1+j_2+j_3} f}{\partial^{j_1} x \partial^{j_2} y \partial^{j_3} z}(0, 0, 0) \lambda^{j_1+j_2+j_3} t_0^{j_1} h_0^{j_2} \frac{1}{j_1! j_2! j_3!}.$$

Since  $(0, 0, 0)$  is the  $(d - 1)$ -fold  $\varepsilon$ -singularity of  $V$  it holds that

$$\begin{aligned} & \|f(\lambda t_0, \lambda h_0, \lambda) - \bar{f}(\lambda t_0, \lambda h_0, \lambda)\| \\ &= \max_{j_1+j_2+j_3=0, \dots, d-2} \left\{ \left| \frac{\partial^{j_1+j_2+j_3} f}{\partial^{j_1} x \partial^{j_2} y \partial^{j_3} z}(0, 0, 0) \right| |t_0|^{j_1} |h_0|^{j_2} \frac{1}{j_1! j_2! j_3!} \right\} \\ &\leq \max_{j_1+j_2+j_3=0, \dots, d-2} \left\{ \left| \frac{\partial^{j_1+j_2+j_3} f}{\partial^{j_1} x \partial^{j_2} y \partial^{j_3} z}(0, 0, 0) \right| \right\} < \varepsilon \|f(x, y, z)\| = \varepsilon, \end{aligned}$$

and thus,  $\bar{f}(\lambda t_0, \lambda h_0, \lambda)$  can be written as

$$\bar{f}(\lambda t_0, \lambda h_0, \lambda) = f(\lambda t_0, \lambda h_0, \lambda) + R(\lambda) \quad \text{where } R \in \mathbb{R}[\lambda] \text{ y } \|R(\lambda)\| < \varepsilon.$$

Therefore, by applying standard numerical techniques to measure  $|\lambda_1 - \lambda_0|$  by means of the condition number (see for instance (Bulirsch and Stoer, 1993, p. 303)), one deduces that

$$|\lambda_1 - \lambda_0| \leq \varepsilon \cdot C,$$

where

$$\begin{aligned} C &= \frac{\sum_{j_1+j_2+j_3=0}^{d-2} |\lambda_0|^{j_1+j_2+j_3} |t_0|^{j_1} |h_0|^{j_2} \frac{1}{j_1! j_2! j_3!}}{\left| \frac{\partial \bar{f}(\lambda t_0, \lambda h_0, \lambda)}{\partial \lambda}(\lambda_0) \right|} \\ &= \frac{\sum_{j_1+j_2+j_3=0}^{d-2} |\lambda_0|^{j_1+j_2+j_3} |t_0|^{j_1} |h_0|^{j_2} \frac{1}{j_1! j_2! j_3!}}{|\lambda_0|^{d-1} |n_0|}. \quad \square \end{aligned}$$

**Lemma 6.** *Let*

$$h(x) = c \prod_{i=1}^n (x - c_i) \in \mathbb{C}[x] \quad \text{with } \deg(h) = n,$$

and let  $\lambda \in \mathbb{C}$  be such that  $|h(\lambda)| \leq \delta$ . Then, there exists a root  $c_{i_0}$  of  $h(x)$  such that

$$|\lambda - c_{i_0}| \leq (\delta/|c|)^{1/n}.$$

**Proof.** Let us assume that for  $i = 1, \dots, n$ ,  $|\lambda - c_i| > (\delta/|c|)^{1/n}$ . Then,

$$|h(\lambda)| = |c| \prod_{i=1}^n |\lambda - c_i| > \delta,$$

which contradicts that  $|h(\lambda)| \leq \delta$ .  $\square$

Now, we proceed to analyze  $|\lambda_0 - \lambda_1|$  by using the previous lemmas. For this purpose, we distinguish different cases depending on the values of  $|m_0|$  and  $|n_0|$ .

**Lemma 7.** *Let  $|n_0| \geq 1$ . Then, it holds that:*

1. *If  $|\lambda_0| > 1$ , then  $|\lambda_0 - \lambda_1| \leq \varepsilon \cdot e^3$ .*
2. *If  $|\lambda_0| \leq 1$ , then  $|\lambda_0 - \lambda_1| \leq (\varepsilon \cdot e^3)^{1/d}$ .*

**Proof.** 1. If  $|\lambda_0| > 1$ , taking into account the equality

$$\sum_{i+j+k=0}^r \frac{a^i b^j c^k}{i! j! k!} = \sum_{s=0}^r \frac{(a + b + c)^s}{s!},$$

one has that the constant  $C$  of Lemma 5 can be bounded as

$$\begin{aligned} C &= \frac{\sum_{j_1+j_2+j_3=0}^{d-2} |\lambda_0|^{j_1+j_2+j_3} |t_0|^{j_1} |h_0|^{j_2} \frac{1}{j_1!j_2!j_3!}}{|\lambda_0|^{d-1} |n_0|} \\ &\leq \frac{\sum_{j_1+j_2+j_3=0}^{d-2} |\lambda_0|^{j_1+j_2+j_3} \frac{1}{j_1!j_2!j_3!}}{|\lambda_0|^{d-1}} \\ &= \frac{\sum_{k=0}^{d-2} \frac{(3|\lambda_0|)^k}{k!}}{|\lambda_0|^{d-1}} = \sum_{k=0}^{d-2} \frac{3^k}{k! |\lambda_0|^{d-1-k}} \leq \sum_{k=0}^{d-2} \frac{3^k}{k!} \leq e^3. \end{aligned}$$

Thus, by Lemma 5 we deduce that

$$|\lambda_0 - \lambda_1| \leq \varepsilon \cdot e^3.$$

2. If  $|\lambda_0| \leq 1$ , one has that

$$\begin{aligned} &|f(\lambda_0 t_0, \lambda_0 h_0, \lambda_0)| \\ &= \left| \bar{f}(\lambda_0 t_0, \lambda_0 h_0, \lambda_0) + \sum_{j_1+j_2+j_3=0}^{d-2} \frac{\partial^{j_1+j_2+j_3} f}{\partial^{j_1} x \partial^{j_2} y \partial^{j_3} z} (0, 0, 0) \frac{\lambda_0^{j_1+j_2+j_3} |t_0|^{j_1} |h_0|^{j_2}}{j_1!j_2!j_3!} \right| \\ &\leq \sum_{j_1+j_2+j_3=0}^{d-2} \left| \frac{\partial^{j_1+j_2+j_3} f}{\partial^{j_1} x \partial^{j_2} y \partial^{j_3} z} (0, 0, 0) \right| |\lambda_0|^{j_1+j_2+j_3} \frac{1}{j_1!j_2!j_3!} \\ &\leq \varepsilon \cdot \sum_{j_1+j_2+j_3=0}^{d-2} \frac{|\lambda_0|^{j_1+j_2+j_3}}{j_1!j_2!j_3!}. \end{aligned}$$

Thus, taking into account the equality

$$\sum_{i+j+k=0}^r \frac{a^i b^j c^k}{i!j!k!} = \sum_{s=0}^r \frac{(a+b+c)^s}{s!},$$

one gets that

$$|f(\lambda_0 t_0, \lambda_0 h_0, \lambda_0)| \leq \varepsilon \cdot \sum_{k=0}^{d-2} \frac{(3|\lambda_0|)^k}{k!} \leq \varepsilon \cdot e^{3|\lambda_0|} \leq \varepsilon \cdot e^3.$$

In this situation, applying Lemma 6 we deduce that there exists a root of the univariate polynomial  $f(\lambda t_0, \lambda h_0, \lambda)$ , that we can assume w.l.o.g. that is  $\lambda_1$ , such that

$$|\lambda_1 - \lambda_0| \leq \left( \frac{\varepsilon \cdot e^3}{|n_0|} \right)^{1/d} \leq (\varepsilon \cdot e^3)^{1/d}. \quad \square$$

**Lemma 8.** Let  $|n_0| < 1$  and  $|m_0| \geq 1$ . Then, it holds that  $|\lambda_1 - \lambda_0| \leq \varepsilon \cdot e^3$ .

**Proof.** Since  $|n_0| < 1$  and  $|m_0| \geq 1$ , one has that the constant  $C$  in Lemma 5 can be bounded as

$$C = \frac{\sum_{j_1+j_2+j_3=0}^{d-2} |\lambda_0|^{j_1+j_2+j_3} |t_0|^{j_1} |h_0|^{j_2} \frac{1}{j_1!j_2!j_3!}}{|\lambda_0|^{d-1} |n_0|} \leq \frac{\sum_{j_1+j_2+j_3=0}^{d-2} |\lambda_0|^{j_1+j_2+j_3} \frac{1}{j_1!j_2!j_3!}}{|\lambda_0|^{d-1} |n_0|}.$$

Now, applying the equality

$$\sum_{i+j+k=0}^r \frac{a^i b^j c^k}{i!j!k!} = \sum_{s=0}^r \frac{(a+b+c)^s}{s!},$$

one deduces that

$$\begin{aligned} C &\leq \frac{\sum_{k=0}^{d-2} \frac{(3|\lambda_0|)^k}{k!}}{|\lambda_0|^{d-1} |n_0|} = \frac{\sum_{k=0}^{d-2} \frac{(3|m_0|)^k |n_0|^{(d-2-k)}}{k!}}{|m_0|^{d-1}} \\ &\leq \sum_{k=0}^{d-2} \frac{3^k}{k! |m_0|^{d-1-k}} \leq \sum_{k=0}^{d-2} \frac{3^k}{k!} \leq e^3. \end{aligned}$$

Therefore, by Lemma 6 we conclude that

$$|\lambda_1 - \lambda_0| \leq \varepsilon \cdot e^3. \quad \square$$

Finally, it only remains to analyze the case where  $|n_0| < 1$  and  $|m_0| < 1$ . In order to do that, we recall that we have assumed that either  $|m_0|$  or  $|n_0|$  is bigger than  $\varepsilon^{1/d}$ . In the next lemma, we study these cases.

**Lemma 9.** *Let  $|m_0| < 1$  and  $|n_0| < 1$ . Then, it holds that*

1. *If  $|n_0| < 1$  and  $\varepsilon^{1/d} < |m_0| < 1$ , then  $|\lambda_0 - \lambda_1| \leq \varepsilon^{1/d} \cdot e^3$ .*
2. *If  $|m_0| < 1$  and  $\varepsilon^{1/d} < |n_0| < 1$ , then  $|\lambda_0 - \lambda_1| \leq (\varepsilon^{1/2} \cdot e^3)^{1/d}$ .*

**Proof.** 1. If  $|n_0| < 1$  and  $\varepsilon^{1/d} < |m_0| < 1$ , one has that the constant  $C$  in Lemma 5 can be bounded as

$$\begin{aligned} C &= \frac{\sum_{j_1+j_2+j_3=0}^{d-2} |\lambda_0|^{j_1+j_2+j_3} |t_0|^{j_1} |h_0|^{j_2} \frac{1}{j_1!j_2!j_3!}}{|\lambda_0|^{d-1} |n_0|} \leq \frac{\sum_{j_1+j_2+j_3=0}^{d-2} |\lambda_0|^{j_1+j_2+j_3} \frac{1}{j_1!j_2!j_3!}}{|\lambda_0|^{d-1} |n_0|} \\ &= \frac{\sum_{j_1+j_2+j_3=0}^{d-2} |n_0|^{d-j_1-j_2-j_3-2} \frac{1}{j_1!j_2!j_3!}}{|m_0|^{d-j_1-j_2-1-j_3}} \leq \frac{\sum_{j_1+j_2+j_3=0}^{d-2} \frac{1}{j_1!j_2!j_3!}}{|m_0|^{d-1}}. \end{aligned}$$

In these conditions, taking into account the equality

$$\sum_{i+j+k=0}^r \frac{a^i b^j c^k}{i!j!k!} = \sum_{s=0}^r \frac{(a+b+c)^s}{s!},$$

one deduces that

$$C \leq \frac{\sum_{k=0}^{d-2} \frac{3^k}{k!}}{|m_0|^{d-1}} \leq \frac{e^3}{|m_0|^{d-1}} \leq e^3 \cdot \varepsilon^{-1+1/d}.$$

Thus, Lemma 5 implies that

$$|\lambda_0 - \lambda_1| \leq \varepsilon^{1/d} \cdot e^3.$$

2. Let  $\varepsilon^{1/d} < |n_0| < 1$  and  $|m_0| < 1$ . First, we assume w.l.o.g. that  $|m_0| \leq \varepsilon^{1/d}$ , because otherwise we would reason as in case (1). In these conditions one has that  $|m_0| \leq \varepsilon^{1/d} < |n_0| < 1$ , and we deduce that



$$\begin{aligned}
 |f(\lambda_0 t_0, \lambda_0 h_0, \lambda_0)| &= \left| \bar{f}(\lambda_0 t_0, \lambda_0 h_0, \lambda_0) + \sum_{j_1+j_2+j_3=0}^{d-2} \frac{\partial^{j_1+j_2+j_3} f}{\partial^{j_1} x \partial^{j_2} y \partial^{j_3} z}(0, 0, 0) \lambda_0^{j_1+j_2+j_3} t_0^{j_1} h_0^{j_2} \frac{1}{j_1! j_2! j_3!} \right| \\
 &= \left| \sum_{j_1+j_2+j_3=0}^{d-2} \frac{\partial^{j_1+j_2+j_3} f}{\partial^{j_1} x \partial^{j_2} y \partial^{j_3} z}(0, 0, 0) \lambda_0^{j_1+j_2+j_3} t_0^{j_1} h_0^{j_2} \frac{1}{j_1! j_2! j_3!} \right| \\
 &\leq \sum_{j_1+j_2+j_3=0}^{d-2} \left| \frac{\partial^{j_1+j_2+j_3} f}{\partial^{j_1} x \partial^{j_2} y \partial^{j_3} z}(0, 0, 0) \right| |\lambda_0|^{j_1+j_2+j_3} \frac{1}{j_1! j_2! j_3!} \\
 &\leq \varepsilon \cdot \sum_{j_1+j_2+j_3=0}^{d-2} \frac{|\lambda_0|^{j_1+j_2+j_3}}{j_1! j_2! j_3!}.
 \end{aligned}$$

Now, taking into account the equality

$$\sum_{i+j+k=0}^r \frac{a^i b^j c^k}{i! j! k!} = \sum_{s=0}^r \frac{(a+b+c)^s}{s!},$$

one deduces that

$$|f(\lambda_0 t_0, \lambda_0 h_0, \lambda_0)| \leq \varepsilon \cdot \sum_{k=0}^{d-2} |\lambda_0|^k \frac{1}{k!} \leq \varepsilon \cdot e^{3|\lambda_0|} \leq \varepsilon \cdot e^3.$$

By Lemma 6 we conclude that there exists a root of the univariate polynomial  $f(\lambda t_0, \lambda h_0, \lambda)$ , that we may assume w.l.o.g. it is  $\lambda_1$ , such that

$$|\lambda_0 - \lambda_1| \leq \left( \frac{\varepsilon \cdot e^3}{|n_0|} \right)^{1/d} \leq (\varepsilon \cdot e^3)^{1/d} \frac{1}{\varepsilon^{1/d^2}} \leq (\varepsilon \cdot e^3)^{1/d} \frac{1}{\varepsilon^{1/(2d)}} = (\varepsilon^{1/2} \cdot e^3)^{1/d}. \quad \square$$

From the previous lemmas, one deduces the following theorem.

**Theorem 5.** For almost all affine real point  $\bar{Q} \in \bar{V}$  there exists an affine point  $Q \in V$  such that

$$\|\bar{Q} - Q\|_2 \leq \sqrt{3} \varepsilon^{1/(2d)} e^3.$$

**Proof.** Applying Lemmas 7, 8, and 9 one deduces that

$$\begin{aligned}
 \|\bar{Q} - Q\|_2 &= \sqrt{(\lambda_1 t_0 - \lambda_0 t_0)^2 + (\lambda_1 h_0 - \lambda_0 h_0)^2 + (\lambda_1 - \lambda_0)^2} \\
 &= |\lambda_1 - \lambda_0| \sqrt{t_0^2 + h_0^2 + 1} \leq \sqrt{3} |\lambda_1 - \lambda_0| \leq \sqrt{3} \varepsilon^{1/(2d)} e^3. \quad \square
 \end{aligned}$$

Now, let  $\bar{Q} = (\lambda_0 t_0, \lambda_0 h_0, \lambda_0)$  be a regular point on  $\bar{V}$  such that there exists  $Q = (\lambda_1 t_0, \lambda_1 h_0, \lambda_1) \in V$  with  $\|\bar{Q} - Q\|_2 \leq \sqrt{3} \varepsilon^{1/(2d)} e^3$  (see Theorem 5). In this situation, we consider the tangent plane to  $\bar{V}$  at  $\bar{Q}$ ; i.e.,

$$T(x, y, z) = n_x(x - \lambda_0 t_0) + n_y(y - \lambda_0 h_0) + n_z(z - \lambda_0),$$

where  $(n_x, n_y, n_z)$  is the unitary normal vector to  $\bar{V}$  at  $\bar{Q}$ . Then, taking into account Theorem 5 we bound the value  $|T(Q)|$  as follows:

$$\begin{aligned} |T(Q)| &\leq |n_x| \cdot |\lambda_1 t_0 - \lambda_0 t_0| + |n_y| \cdot |\lambda_1 h_0 - \lambda_0 h_0| + |n_z| \cdot |\lambda_1 - \lambda_0| \\ &\leq \|\bar{Q} - Q\|_2 (|n_x| + |n_y| + |n_z|) \leq 3\sqrt{3} \varepsilon^{1/(2d)} e^3. \end{aligned}$$

Therefore, reasoning as in Section 2.4 of (Farouki and Rajan, 1988) one deduces the following theorem.

**Theorem 6.** *The surface  $\bar{V}$  is contained in the offset region of the surface  $V$  at distance  $3\sqrt{3} \varepsilon^{1/(2d)} e^3$ .*

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