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The Inverse Gröbner Basis Problem in Codimension Two*

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Generic linkage is used to compute a prime ideal such that the radical of the initial ideal of the prime ideal is equal to the radical of a given codimension two monomial ideal that has Cohen–Macaulay quotient ring.

Introduction

The *inverse Gröbner basis problem* is to find the ideals that have a given monomial ideal as its initial ideal. We consider the problem of finding when the given monomial ideal is the initial ideal of a *prime* ideal. Kalkbrenner and Sturmfels (1995), in Theorem 1, prove that the radical of the initial ideal of a prime ideal is equi-dimensional and connected in codimension one or equivalently, the initial complex of a prime ideal is pure and strongly connected. They ask if these necessary conditions are sufficient.

Dalbec (1998), in Theorem 2, proved that if I is an ideal generated by all the degree d square-free monomials in n variables, then there exists a prime ideal P such that the radical of the initial ideal of P is I. The ideals he considers are square-free, the generators have the same degree and their quotient is Cohen–Macaulay. We prove the following main theorem that establishes that the ring being Cohen–Macaulay is sufficient for ideals of codimension two.

Theorem 0.1. *Let $R = k[x_1, x_2, \dots, x_r]$ be a polynomial ring over a field k and I an ideal of R. Let $>$ be a monomial order that respects total degree. Assume I is a monomial ideal of codimension 2 and R/I is Cohen–Macaulay. Then there exists an extension field K of k and a prime ideal P contained in the polynomial ring $S = K[x_1, x_2, \dots, x_r]$ such that $\sqrt{in(P)} = \sqrt{I}S$.*

If I is a square-free monomial ideal and has minimal generators of the same degree, then the proof of Theorem 0.1 actually gives $in(P) = IS$. If I is only square-free then $\sqrt{in(P)} = IS$. We would like to have P in $k[x_1, x_2, \dots, x_r]$. For this we need to specialize indeterminates. The process of specializing does not necessarily preserve the property of the ideal being prime nor the structure of the initial ideal. A Bertini theorem (Flenner, 1977) can be used to preserve the prime property. Equations needed

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to preserve the Gröbner basis, in this setting, are given by Taylor (2000). However, the two are incompatible and obtaining P in $k[x_1, x_2, \ldots, x_r]$ remains open.

The proof of our main result uses two important ingredients. First, generic linkage is our tool for constructing a prime ideal. Second, we give a Gröbner basis for the ideal of maximal minors for a particular class of non-generic matrices. Gröbner bases for ideals of minors of generic matrices are known, however, if the matrix is not generic, finding a Gröbner basis for the ideal of maximal minors is, in general, difficult.

Since we use generic linkage to construct our prime ideal, in Section 1 we collect the relevant definitions and propositions needed from the theory of generic linkage. We prove a key theorem on Gröbner bases for the ideal of maximal minors for certain non-generic matrices in Section 2. In Section 3 we prove the main theorem. Finally, in Section 4 we give several examples, including examples that illustrate the construction, explore the necessity of the Cohen–Macaulay hypothesis and the necessity of the radicals.

Before we proceed to Section 1 we include the basic definitions and notation from Gröbner basis theory that we will use. A *monomial order* $\geq$ on a polynomial ring $R = k[x_1, x_2, \ldots, x_r]$ over a field k is a total order on the monomials in R such that $m \geq 1$ for each monomial m in R and if m_1, m_2, n are monomials in R with $m_1 \geq m_2$ then $nm_1 \geq nm_2$. A monomial order on a polynomial ring in several variables generalizes the notion of degree for a polynomial ring in one variable. The *initial term* of an element $f \in R$, denoted $in(f)$, is the largest term (including coefficient) of f with respect to a fixed monomial order. We use $lm(f)$ to denote the largest monomial of f when we do not want to include the coefficient. Given an ideal I of R, the *initial ideal* is defined to be $\langle \{in(f) : f \in I\} \rangle$, and is denoted $in(I)$. It should be noted that different monomial orders may yield different initial ideals, so whenever an initial ideal is referred to, it is assumed a monomial order has been fixed. A *Gröbner basis* is a subset $\{g_1, \ldots, g_n\}$ of I such that $in(I) = \langle in(g_1), \ldots, in(g_n) \rangle$.

1. Generic Linkage

Two varieties X and Y in $\mathbb{P}^n$ with no common components are *linked* if $X \cup Y$ is a complete intersection (Peskine and Szpiro, 1974). Algebraically, two ideals I and J in a local Cohen–Macaulay ring are linked if there exists a regular sequence $\alpha_1, \ldots, \alpha_s = \underline{\alpha}$ contained in the intersection $I \cap J$ such that $\langle \underline{\alpha} \rangle : I = J$ and $\langle \underline{\alpha} \rangle : J = I$ (Huneke and Ulrich, 1985, Definition 2.1). Huneke and Ulrich (1985) define a *generic link* of I and prove, under some hypotheses, that it is a prime ideal.

Definition 1.1. (Huneke and Ulrich, 1985, Definition 2.3) Let R be a Gorenstein ring and I an unmixed ideal of R of grade g. For $I = R$ we take $g > 0$ arbitrary and finite, although the convention in this case is grade$(I) = \infty$. Fix a generating set $f_1, \ldots, f_m$ of I. A generic link $L(\underline{f})$ of I is defined as follows: let $Y_{ij} (1 \leq i \leq g, 1 \leq j \leq m)$ be $g \cdot m$ variables and set $S = R[Y_{ij}]$ and $\alpha_i = \sum_{j=1}^{m} Y_{ij} f_j$, $1 \leq i \leq g$. We set $L(\underline{f}) = \langle \alpha_1, \ldots, \alpha_g \rangle : IS$, and call $(S, L(\underline{f}))$ a generic link to I.

Hochster (1973) proved that $\alpha_1, \ldots, \alpha_g$ is a maximal regular sequence in IS. Therefore if R is Gorenstein and I is unmixed, then $(S, L(\underline{f}))$ is linked to I (Huneke and Ulrich, 1985). Hochster (1973) also gives an equivalence relation on pairs (R, I), where I is an ideal of the ring R. The pairs (R_1, I_1) and (R_2, I_2) are equivalent if there exist integers r, s and indeterminates $Y_1, \ldots, Y_r$ over R_1 and $Z_1, \ldots, Z_s$ over R_2 such that

$(R_1/I_1)[Y_1,\ldots,Y_r]$ and $(R_2/I_2)[Z_1,\ldots,Z_s]$ are isomorphic. Huneke and Ulrich (1985) prove that for R a Gorenstein Noetherian ring and I an unmixed ideal of R, if $\underline{f}$ and $\underline{h}$ are two generating sets of I and $(Q_1, L(\underline{f}))$ and $(Q_2, L(\underline{h}))$ two generic links for I, then $(Q_1, L(\underline{f}))$ is equivalent to $(Q_2, L(\underline{h}))$. This alleviates the dependence of Definition 1.1 on the generating set of I and allows us to use the notation $L(I)$ for a generic link of I when the ring is understood. We use $L_n(I)$ to denote the nth generic link of I, defined inductively to be $L_1(L_{n-1}(I))$. The next proposition is the main property we need from the theory of generic linkage.

Proposition 1.2. (Huneke and Ulrich (1985), Proposition 2.6) *Let R be a Gorenstein local domain and let I be an unmixed ideal of R which is generically a complete intersection. Let $(S, L(I))$ be a generic link to I. Then $L(I)$ is a prime ideal.*

For any square-free monomial ideal $I \subseteq R = k[x_1, x_2, \ldots, x_r]$, R/I is reduced and the primary decomposition looks like $I = P_1 \cap \cdots \cap P_s$ where the P_i are generated by subsets of the variables $\{x_1, \ldots, x_r\}$. Hence any square-free monomial ideal $I \subseteq R$ is generically a complete intersection. If we further assume that R/I is Cohen–Macaulay then $L(I)$ is a prime ideal by Proposition 1.2. We use this in the proof of Theorem 0.1.

In the context of the main theorem, R is a polynomial ring and I is a homogeneous ideal so I has a graded minimal free resolution. Under these assumptions we can construct a free resolution of the generic link of I. Set S to be the ring for a generic link of I and $\mathbf{K} = \mathbf{K}(\underline{\alpha}; \mathrm{S})$ to be the Koszul resolution of $S/\langle\underline{\alpha}\rangle$. Let $\mathbf{F}$ be the minimal free resolution of S/I. A free resolution of $S/L(I)$ is the mapping cone of the dual of the map $u : \mathbf{K} \to \mathbf{F}$ induced by $S/\langle\underline{\alpha}\rangle \to S/I$ (Peskine and Szpiro, 1974, Proposition 2.6). This resolution has length $\mathrm{grade}(I) + 1$, but the last differential in the mapping cone splits. Taking the mapping cone of the dual of u modulo the subcomplex $S \to S$ gives a resolution of length equal to the grade of I.

If R/I is Cohen–Macaulay, $I = \langle f_1, \ldots, f_m\rangle$ and $\mathrm{codim}(I) = 2$ then, by the Hilbert–Burch theorem (Bruns and Herzog, 1993, Theorem 1.4.17), the resolution $\mathbf{F}$ has the form

$$0 \longrightarrow R^{m-1} \xrightarrow{A} R^m \xrightarrow{B} R \longrightarrow 0$$

where $B = \begin{bmatrix} f_1 & f_2 & \ldots & f_m \end{bmatrix}$ and the $(m-1) \times (m-1)$-minors of A generate the ideal I, that is $I_{m-1}(A) = I$. Take $\{Y_{ij}\}_{1\le j\le m, 1\le i\le 2}$ and form the linear combinations $\alpha_1 = \sum_{j=1}^m Y_{1j} f_j$ and $\alpha_2 = \sum_{j=1}^m Y_{2j} f_j$.

In this particular case, after we mod out by $S \to S$, the resolution for $S/\langle\alpha_1, \alpha_2\rangle : IS$ is

$$0 \longrightarrow S^m \xrightarrow{A'} S^{m+1} \xrightarrow{B'} S \longrightarrow 0 \tag{1.1}$$

where

$$A' = \begin{bmatrix} & A^{\mathrm{T}} & \\ Y_{11} & \cdots & Y_{1m} \\ Y_{21} & \cdots & Y_{2m} \end{bmatrix} \quad \text{and } B' = \begin{bmatrix} (-1)\delta_1 & (-1)^2\delta_2 & \ldots & (-1)^{m+1}\delta_{m+1} \end{bmatrix} \text{ where } \delta_1, \ldots, \delta_{m+1} \text{ are the maximal minors of } A'. \tag{1.2}$$

Since α_1, α_2 is a regular sequence in IS we know that $\langle\alpha_1, \alpha_2\rangle : IS$ has grade at least 2. Hence its quotient is Cohen–Macaulay and the Hilbert–Burch theorem implies the maximal minors of A' generate $\langle\alpha_1, \alpha_2\rangle : IS$, the first generic link of I.

Repeating this process, the second generic link is generated by the maximal minors of the matrix

$$A'' = \begin{bmatrix} & & A'^{\mathrm{T}} & & \\ Z_{11} & \cdots & Z_{1m-1} & Y_{1m+1} & Y_{2m+1} \\ Z_{21} & \cdots & Z_{2m-1} & Y_{1m+2} & Y_{2m+2} \end{bmatrix} = \begin{bmatrix} & A & Y_{11} & Y_{21} \\ & & \vdots & \vdots \\ & & Y_{1m} & Y_{2m} \\ Z_{11} & \cdots & Y_{1m+1} & Y_{2m+1} \\ Z_{21} & \cdots & Y_{1m+2} & Y_{2m+2} \end{bmatrix}. \tag{1.3}$$

The indeterminates, $Z_{11}, \ldots, Z_{2m-1}, Y_{1m+1}, Y_{1m+2}, Y_{2m+1}, Y_{2m+2}$, used in forming the second generic link are labelled this way because it is useful in later sections.

We utilize the second generic link and its resolution in the proof of Theorem 0.1. There are two reasons to expect the second generic link to be better than the first generic link for our purposes. First, I is a link of $L_1(I)$ and $L_2(I)$ is a generic link of $L_1(I)$, so there is a specialization from the second generic link of I to I (Huneke and Ulrich, 1985, Proposition 2.13).

Second, the structure of the maximal minors of A'' (1.3) is better. We give an example. Let f_i denote the signed minor of A when the ith row is removed, for $1 \le i \le m$, and let δ_i denote the same signed minor for A''. Therefore,

$$\delta_i = f_i(Y_{1m+2}Y_{2m+1} - Y_{1m+1}Y_{2m+2}) + \beta_i \quad 1 \le i \le m. \tag{1.4}$$

where each term of β_i, $1 \le i \le m$ has higher degree in the new variables. For example, the following are the first three minors of a presentation matrix (1.3) for the second generic link of $I = \langle ac, ad, bd \rangle$:

$$\begin{aligned}
\delta_1 = {} & \mathbf{ac}(\mathbf{Y_{15}Y_{24}} - \mathbf{Y_{14}Y_{25}}) - aY_{15}Y_{22}Z_{12} + aY_{12}Y_{25}Z_{12} + aY_{14}Y_{22}Z_{22} \\
& - aY_{12}Y_{24}Z_{22} - bY_{15}Y_{23}Z_{12} + bY_{13}Y_{25}Z_{12} + bY_{14}Y_{23}Z_{22} - bY_{13}Y_{24}Z_{22} \\
& - cY_{15}Y_{23}Z_{11} + cY_{13}Y_{25}Z_{11} + cY_{14}Y_{23}Z_{21} - cY_{13}Y_{24}Z_{21} + Y_{13}Y_{22}Z_{12}Z_{21} \\
& - Y_{12}Y_{23}Z_{12}Z_{21} - Y_{13}Y_{22}Z_{11}Z_{22} + Y_{12}Y_{23}Z_{11}Z_{22} \\
\delta_2 = {} & \mathbf{ad}(\mathbf{Y_{15}Y_{24}} - \mathbf{Y_{14}Y_{25}}) + aY_{15}Y_{21}Z_{12} - aY_{11}Y_{25}Z_{12} - aY_{14}Y_{21}Z_{22} \\
& + aY_{11}Y_{24}Z_{22} - dY_{15}Y_{23}Z_{11} + dY_{13}Y_{25}Z_{11} + dY_{14}Y_{23}Z_{21} - dY_{13}Y_{24}Z_{21} \\
& - Y_{13}Y_{21}Z_{12}Z_{21} + Y_{11}Y_{23}Z_{12}Z_{21} + Y_{13}Y_{21}Z_{11}Z_{22} - Y_{11}Y_{23}Z_{11}Z_{22} \\
\delta_3 = {} & \mathbf{bd}(\mathbf{Y_{15}Y_{24}} - \mathbf{Y_{14}Y_{25}}) + bY_{15}Y_{21}Z_{12} - bY_{11}Y_{25}Z_{12} - bY_{14}Y_{21}Z_{22} \\
& + bY_{11}Y_{24}Z_{22} + cY_{15}Y_{21}Z_{11} - cY_{11}Y_{25}Z_{11} - cY_{14}Y_{21}Z_{21} + cY_{11}Y_{24}Z_{21} \\
& + dY_{15}Y_{22}Z_{11} - dY_{12}Y_{25}Z_{11} - dY_{14}Y_{22}Z_{21} + dY_{12}Y_{24}Z_{21} + Y_{12}Y_{21}Z_{12}Z_{21} \\
& - Y_{11}Y_{22}Z_{12}Z_{21} - Y_{12}Y_{21}Z_{11}Z_{22} + Y_{11}Y_{22}Z_{11}Z_{22}.
\end{aligned} \tag{1.5}$$

The non-boldface terms are the terms we call β_i in equation (1.4) and the boldface terms correspond to the remaining terms in equation (1.4).

The minors δ_{m+1} and δ_{m+2} do not have exactly the form given in equation (1.4), but as can be seen below each has terms of the form f_iM where M is a degree-two monomial in the new variables:

$$\begin{aligned}\delta_4 = &\ \mathbf{ac}(\mathbf{Y_{15}Y_{21}} - \mathbf{Y_{11}Y_{25}}) + \mathbf{ad}(\mathbf{Y_{15}Y_{22}} - \mathbf{Y_{12}Y_{25}}) + \mathbf{bd}(\mathbf{Y_{15}Y_{23}} - \mathbf{Y_{13}Y_{25}})\\ &-aY_{12}Y_{21}Z_{22} + aY_{11}Y_{22}Z_{22} - bY_{13}Y_{21}Z_{22} + bY_{11}Y_{23}Z_{22} - cY_{13}Y_{21}Z_{21}\\ &\quad + cY_{11}Y_{23}Z_{21} - dY_{13}Y_{22}Z_{21} + dY_{12}Y_{23}Z_{21}\\ \delta_5 = &\ \mathbf{ac}(\mathbf{Y_{14}Y_{21}} - \mathbf{Y_{11}Y_{24}}) + \mathbf{ad}(\mathbf{Y_{14}Y_{22}} - \mathbf{Y_{12}Y_{24}}) + \mathbf{bd}(\mathbf{Y_{14}Y_{23}} - \mathbf{Y_{13}Y_{24}})\\ &-aY_{12}Y_{21}Z_{12} + aY_{11}Y_{22}Z_{12} - bY_{13}Y_{21}Z_{12} + bY_{11}Y_{23}Z_{12} - cY_{13}Y_{21}Z_{11}\\ &+cY_{11}Y_{23}Z_{11} - dY_{13}Y_{22}Z_{11} + dY_{12}Y_{23}Z_{11}\end{aligned}$$

In this example, where the degrees of the generators of I have the same degree, the initial term of each δ_i, $1 \le i \le m+2$, using the inverse block order with respect to the added variables (see the beginning of Section 2 for a definition of this order), is of the form $f_i M$ where M is a degree two monomial in the new variables. We will use this structure in the proof of Theorem 0.1 as well as the proof of Theorem 2.2.

In contrast, the minors of a presentation matrix for the first generic link of I do not have this form. The following are the maximal minors of a presentation matrix (1.2) for the first generic link of $I = \langle ac, ad, bd \rangle$:

$$\begin{aligned}\delta_1 &= -cY_{12}Y_{21} + cY_{11}Y_{22} + dY_{13}Y_{22} - dY_{12}Y_{23}\\ \delta_2 &= aY_{13}Y_{22} - aY_{12}Y_{23} + bY_{13}Y_{21} - bY_{11}Y_{23}\\ \delta_3 &= acY_{21} + adY_{22} + bdY_{23}\\ \delta_4 &= acY_{11} + adY_{12} + bdY_{13}\end{aligned} \tag{1.6}$$

The first and second minors are linear in a, b, c, d. These linear terms will appear in any Gröbner basis computed from this generating set. The linear terms are not in I and all of this suggests that the second generic link, as opposed to the first, is the prime ideal we want to work with.

2. Gröbner Bases for Ideals of Maximal Minors

In this section we prove, under certain conditions, that the maximal minors of the presentation matrix of $L_2(I)$ given in equation (1.4) are a Gröbner basis for $L_2(I)$. First, we need to fix some notation. From now on assume that the monomial order on $R = k[x_1, x_2, \ldots, x_r]$ respects total degree. The *inverse block order* (Kredel and Weispfenning, 1988, Section 8) is a useful monomial order when adding additional variables to R. Let $<_R$ denote the monomial order on R. Let $<_T$ denote the monomial order on $T = k[y_1, \ldots, y_s]$. Set $S = k[x_1, \ldots, x_r, y_1, \ldots, y_s]$ and $Y = \{y_1, \ldots, y_s\}$. Let $s_1, s_2 \in R$ and $t_1, t_2 \in T$ be monomials. The *inverse block order on S with respect to Y* is defined as follows: if either $s_1 < s_2$, or $s_1 = s_2$ and $t_1 < t_2$, then $s_1 t_1 < s_2 t_2$. Given a monomial st in S, where $s \in R$ and $t \in T$, define $\deg_{\underline{x}}(st) = \deg s$.

For $I \subseteq R$ homogeneous use $\mathrm{H}(R/I, n) = \dim_k (R/I)_n$ to denote the Hilbert function for R/I and $\mathrm{H}_{R/I}(t) = \sum_{n=0}^{\infty} \mathrm{H}(R/I, n) t^n$ to denote the Hilbert series for R/I. We use the following standard fact. If $\mathbf{F}$ is a minimal graded free resolution of R/I, then the Hilbert series is the alternating sum of the Hilbert series of the modules in the resolution, so $\mathrm{H}_{R/I}(t) = \sum_{i=0}^{m} (-1)^i \mathrm{H}_{F_i}(t)$ where $m = pd_R(R/I)$.

We also use the following standard facts in the setup and proof of Theorem 2.2. Let $f_1, \ldots, f_m$ be a generating set for a homogeneous ideal I of R. If I is codimension two and R/I is Cohen–Macaulay then the Hilbert–Burch theorem along with Peskine and Szpiro (1974) implies that a minimal graded free resolution of I has the following form:

$$0 \longrightarrow \bigoplus_{j=1}^{m-1} R(-b_j) \xrightarrow{\phi} \bigoplus_{i=1}^{m} R(-a_i) \xrightarrow{A} R \longrightarrow R/I \longrightarrow 0. \tag{2.1}$$

Denote the ijth entry of ϕ by ϕ_{ij}. By Peskine and Szpiro (1974),

$$a_j = \deg(f_j), \qquad \text{for } 1 \leq j \leq m, \tag{2.2}$$

$$\deg(\phi_{ij}) = b_j - a_i, \text{ for the non-zero entries } \phi_{ij} \text{ of } \phi, \tag{2.3}$$

$$A = \begin{bmatrix} f_1 & f_2 & \cdots & f_m \end{bmatrix}, \tag{2.4}$$

$$I = I_{m-1}(\phi). \tag{2.5}$$

If $\deg(f_1) = \deg(f_2) = \cdots = \deg(f_m) = d$, then equation (2.3) implies $\deg(\phi_{ij}) = b_j - d$ for the non-zero entries of ϕ. Hence all of the non-zero entries in the jth column of ϕ have degree $b_j - d$, $1 \leq j \leq m-1$. And vice versa, if all of the non-zero entries in each column of ϕ have the same degree the maximal minors must be homogeneous and of the same degree.

Before proceeding to Theorem 2.2 we indicate a way to simplify equations. The need for this can be seen in the minors of the presentation matrix of the second generic link of $I = \langle ac, ad, bd \rangle$ (1.5). The first generic link may involve simpler computations, but yields polynomials that appear unhelpful (see equation 1.6) so we simplify in a different way. Let ϕ be a presentation matrix for I. The following matrix is the matrix that is useful for simplifying computations:

$$\Phi = \begin{pmatrix} & & & Y_1 \\ & \phi & & \vdots \\ & & & Y_m \\ Z_1 & \cdots & Z_{m-1} & Y_{m+1} \end{pmatrix}. \tag{2.6}$$

The minors of matrix (2.6) for the ideal $I = \langle ac, ad, bd \rangle$ are included to illustrate how much simpler they are while maintaining the structure of the minors of the second generic link. The boldface terms are the initial terms using the inverse block order and the remaining terms are what we call β_i in the proof of Theorem 2.2. Compare these equations to equation (1.5):

$$\begin{aligned} \delta_1 &= \mathbf{acY_{14}} - aY_{12}Z_{12} - bY_{13}Z_{12} - cY_{13}Z_{11} \\ \delta_2 &= \mathbf{adY_{14}} + aY_{11}Z_{12} - dY_{13}Z_{11} \\ \delta_4 &= \mathbf{bdY_{14}} + bY_{11}Z_{12} + cY_{11}Z_{11} + dY_{12}Z_{11} \\ \delta_5 &= \mathbf{acY_{11}} + adY_{12} + bdY_{13} \end{aligned} \tag{2.7}$$

Theorem 2.2 is about the minors of matrix (2.6). In Corollary 2.3 we add a second row and column of new variables to prove that under the given conditions, the maximal minors of the presentation matrix for the second generic link, form a Gröbner basis for the second generic link. The following definition will aid in stating the next theorem and corollary.

Definition 2.1. Let $R = k[x_1, x_2, \ldots, x_r]$ and $I = \langle f_1, \ldots, f_m \rangle \subseteq R$.
We say the pair $(R, \underline{f})$ has *Property A* if

(1) $\operatorname{codim}(I) = 2$.
(2) R/I Cohen–Macaulay.
(3) I homogeneous.
(4) $\deg(f_1) = \cdots = \deg(f_m)$.
(5) $\{f_1, \ldots, f_m\}$ is both a Gröbner basis and a minimal generating set for I.

The first two conditions in this definition are the assumptions for the main theorem. Conditions 3 and 5 are satisfied by any monomial ideal and help in the induction that follows. Condition 4 is seemingly strong, however, we reduce the main theorem to this case using Corollary 3.3. This condition allows us to say that the minors of Φ are homogeneous so we can use graded free resolutions. In Section 4 (see page 14) we give an example of what happens, besides no-longer having graded resolutions, if we do not make this assumption.

Theorem 2.2. *Let $R = k[x_1, x_2, \ldots, x_r]$ and $I = \langle f_1, \ldots, f_m\rangle \subseteq R$. Assume $(R, \underline{f})$ has Property A. Let ϕ be a graded free presentation matrix for I and e_j the degree of the non-zero entries in the* jth *column of ϕ. Let $Y = \{Y_1, \ldots, Y_{m+1}\}$, $Z = \{Z_1, \ldots, Z_{m-1}\}$ and $S = R[Y, Z]$. Give S the inverse block order with respect to $\{Y, Z\}$. Set* $\deg(Y_i) = 1$ *for $1 \le i \le m+1$ and* $\deg(Z_j) = e_j$ *for $1 \le j \le m-1$. Set*

$$\Phi = \begin{pmatrix} & & & Y_1 \\ & \phi & & \vdots \\ & & & Y_m \\ Z_1 & \cdots & Z_{m-1} & Y_{m+1} \end{pmatrix}.$$

Then the maximal minors of Φ form a Gröbner basis for the ideal $I_m(\Phi)$.

Proof. For $1 \le i \le m+1$, let δ_i denote the signed minor of Φ when the ith row is removed. The generators, $f_1, \ldots, f_m$ of I are a Gröbner basis by assumption, so $in(f_1), \ldots, in(f_m)$ generate $in(I)$. We prove that the Hilbert series for $S/I_m(\Phi)$ and $S/\langle in(\delta_1), \ldots, in(\delta_{m+1})\rangle$ are equal. Therefore $in(I_m(\Phi)) = \langle in(\delta_1), \ldots, in(\delta_{m+1})\rangle$ and hence the maximal minors of Φ form a Gröbner basis for $I_m(\Phi)$.

We can order $f_1, \ldots, f_m$ such that $in(f_1) \ge in(f_2) \ge \cdots \ge in(f_m)$ and assume f_i is the signed $(m-1) \times (m-1)$ minor of ϕ when the ith row is removed. Let $d = \deg(f_i)$ for $1 \le i \le m$.

The non-zero entries in each column of Φ have the same degree, so by the remarks before the theorem, δ_i, $1 \le i \le m+1$, is homogeneous and has the form

$$\begin{aligned} \delta_1 &= f_1 Y_{m+1} + \beta_1 \\ &\vdots \\ \delta_m &= f_m Y_{m+1} + \beta_n \\ \delta_{m+1} &= f_1 Y_1 + f_2 Y_2 + \cdots + f_m Y_m. \end{aligned} \tag{2.8}$$

Each β_i is at least degree two in the new variables so $\deg_{\underline{x}}(\beta_i) < d$, $1 \le i \le m$. Also $\operatorname{codim}(I_m(\Phi)) \le 2$ (Eagon and Northcott, 1962, Theorem 3). Suppose $\operatorname{codim}(I_m(\Phi)) \le 1$. Localize at a codimension one prime ideal P containing $I_m(\Phi)$. Since $I_{m-1}(\phi)$ is codimension two, at least one $(m-1) \times (m-1)$ minor of ϕ is invertible in S_P. Using row and column operations, rewrite Φ as

$$\Phi' = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & Y'_m \\ 0 & \cdots & 0 & Y'_{m+1} \end{pmatrix}.$$

The terms $Y'_m = Y_m + \beta$ and $Y'_{m+1} = Y_{m+1} + \gamma$ are such that β and γ are polynomials not involving Y_m and Y_{m+1} respectively. Then Y'_m and Y'_{m+1} are a regular sequence and $I_m(\Phi)S_P = I_m(\Phi')S_P = \langle Y'_m, Y'_{m+1}\rangle S_P$ is a codimension two ideal in S_P. This is a contradiction and hence $\operatorname{codim}(I_m(\Phi)) = 2$. Since S is a regular ring, $\operatorname{grade}(I_m(\Phi)) = \operatorname{codim}(I_m(\Phi)) = 2$. Therefore, the Hilbert–Burch Theorem implies

$$0 \longrightarrow S^m \xrightarrow{\Phi} S^{m+1} \xrightarrow{A} S \longrightarrow S/I_m(\Phi) \longrightarrow 0$$

is a minimal free resolution for $S/I_m(\Phi)$ and $A = \begin{bmatrix}(-1)\delta_1 & \cdots & (-1)^{m+1}\delta_{m+1}\end{bmatrix}$.

Each δ_i, $1 \le i \le m+1$, is homogeneous of degree $d+1$, by construction, and if $\Phi_{ij} \ne 0$ then b_j, as defined in equation (3.1), is

$$b_j = \begin{cases} \deg(\Phi_{ij}) + (d+1) = e_j + (d+1) & 1 \le j \le m-1, \\ d+2 & j = m. \end{cases}$$

The following is a graded free resolution of $S/I_m(\Phi)$ with the twists:

$$0 \longrightarrow \bigoplus_{j=1}^{m-1} S(-d-e_j-1) \oplus S(-d-2) \xrightarrow{\Phi} S(-d-1)^{m+1} \xrightarrow{A} S \longrightarrow S/I_m(\Phi) \longrightarrow 0.$$

Therefore

$$H_{S/I_m(\Phi)}(t) = \frac{1-(m+1)t^{d+1} + t^{d+2} + \sum_{j=1}^{m-1} t^{d+e_j+1}}{(1-t)^N(1-t^{e_1})(1-t^{e_2})\cdots(1-t^{e_{m-1}})},$$

where $N = r+m+1$.

Each β_i, $1 \le i \le m+1$, in equation (2.8) has the property that $\deg_{\underline{x}}(\beta_i) < d$. Also, R has an order that respects total degree and S has the inverse block order, therefore $\langle in(\delta_1), \ldots, in(\delta_{n+1})\rangle = \langle in(f_1)Y_{m+1}, \ldots, in(f_m)Y_{m+1}, in(f_1)Y_1\rangle$. Let K denote this ideal. The following is a standard exact sequence:

$$0 \longrightarrow \frac{S}{(K : Y_{m+1})}(-1) \longrightarrow \frac{S}{K} \longrightarrow \frac{S}{(K, Y_{m+1})} \longrightarrow 0. \tag{2.9}$$

Since $\langle K, Y_{m+1}\rangle = \langle Y_{m+1}, in(f_1)Y_1\rangle$ and $(K : Y_{m+1}) = in(I)$, equation (2.9) gives

$$H_{S/K}(t) = H_{S/(Y_{m+1}, in(f_1)Y_1)}(t) + H_{S/in(I)(-1)}(t). \tag{2.10}$$

The monomials Y_{m+1} and $in(f_1)Y_1$ consist of distinct variables and therefore form a regular sequence. Hence

$$H_{S/(Y_{m+1}, in(f_1)Y_1)}(t) = \frac{1-t-t^{d+1}+t^{d+2}}{(1-t)^N(1-t^{e_1})(1-t^{e_2})\cdots(1-t^{e_{m-1}})}. \tag{2.11}$$

Viewing I as an ideal in S, the Hilbert–Burch Theorem and equations (2.1)–(2.5) provide the following graded free resolution of S/I:

$$0 \longrightarrow \bigoplus_{j=1}^{m-1} S(-d-e_j) \xrightarrow{\phi} S(-d)^m \longrightarrow S/I \longrightarrow 0.$$

Therefore,

$$\frac{1 - mt^d + \sum_{j=1}^{m-1} t^{d+e_j}}{(1-t)^N(1-t^{e_1})(1-t^{e_2})\cdots(1-t^{e_{m-1}})} = H_{S/I}(t) = H_{S/in(I)}(t), \tag{2.12}$$

where the last equality is standard (Eisenbud, 1996, Theorem 15.26). To shift this series by 1, multiply by t. Combining equations (2.10), (2.11) and (2.12)

$$\begin{aligned} \mathrm{H}_{S/K}(t) &= \frac{1 - t - t^{d+1} + t^{d+2} + t - mt^{d+1} + \sum_{j=1}^{m-1} t^{d+e_j+1}}{(1-t)^N(1-t^{e_1})(1-t^{e_2})\cdots(1-t^{e_{m-1}})} \\ &= \frac{1-(m+1)t^{d+1} + t^{d+2} + \sum_{j=1}^{m-1} t^{d+e_j+1}}{(1-t)^N(1-t^{e_1})(1-t^{e_2})\cdots(1-t^{e_{m-1}})} = H_{S/I_m(\Phi)}(t). \quad \square \end{aligned}$$

Corollary 2.3. *Assume R, I, S, ϕ, Φ and $\{\delta_i\}_{i=1}^{m+1}$ are as in Theorem* 2.2. *Then*

(*1*) *$(S, \underline{\delta})$ has Property A.*
(*2*) *The generators for $L_2(I)$ from the presentation matrix* (1.3) *form a Gröbner basis for $L_2(I)$.*

Proof. (1): In the proof of Theorem 2.2 we established that $I_m(\Phi)$ is codimension two and $S/I_m(\Phi)$ is Cohen–Macaulay. The construction of Φ implies δ_i is homogeneous of degree $d+1$, for $1 \leq i \leq m+1$. No entry in ϕ is a unit by assumption, so the same is true of Φ by construction. Hence $\{\delta_i\}_{i=1}^{m+1}$ form a minimal generating set for $I_m(\Phi)$. Theorem 2.2 implies $\{\delta_i\}_{i=1}^{m+1}$ is a Gröbner basis for $I_m(\Phi)$.

(2): By (1), we can apply Theorem 2.2 to $(S, \underline{\delta})$. The matrix that arises in this process is the same as the presentation matrix for $L_2(I)$ given in Section 1 (see 1.3) and hence the maximal minors form a Gröbner basis for $L_2(I)$. $\square$

3. Sufficient Conditions in Codimension 2

For a monomial ideal $I = \langle f_1, \ldots, f_m\rangle$ in $R = k[x_1, x_2, \ldots, x_r]$ inductively define a *polarization* of I as follows. Let α_j denote the exponent of x_1 in f_j for $1 \leq j \leq m$. Write each f_j, $1 \leq j \leq m$, as $x_1^{\alpha_j} m_j$ where x_1 does not divide m_j. Set $\alpha = \max_{1\leq j\leq m} \{\alpha_j\}$ and $Y_1, \ldots, Y_{\alpha-1}$ to be $\alpha - 1$ new indeterminates. Set

$$P(f_j) = \begin{cases} x_1 Y_1 Y_2 \cdots Y_{\alpha_j-1} m_j & \text{if } \alpha_j \geq 2, \\ f_j & \text{if } \alpha_j = 0, 1. \end{cases} \tag{3.1}$$

For each f_j, $1 \leq j \leq m$, repeat this process for each x_i, $1 \leq i \leq r$, and call the resulting monomial the *polarization of f_j*. A *polarization of I* is the ideal generated by the polarizations of $f_1, \ldots, f_m$. Let $P(\underline{f})$ denote the polarization of I, formed from the generating set $\underline{f} = f_1, \ldots, f_m$.

The polarization is a square free monomial ideal by construction. Let $\mathbf{Y}$ denote the indeterminates needed to form the polarization of I and set

$$\mathbf{Y}-\mathbf{x}=\{Y_i-x_j|Y_i \text{ replaces } x_j \text{ in the polarization}\}.$$

Then $R[\mathbf{Y}]/\langle P(\underline{f}), \mathbf{Y}-\mathbf{x}\rangle \simeq R[\mathbf{Y}]/\langle I, \mathbf{Y}-\mathbf{x}\rangle$. The following proposition, which is folklore, uses the polarization of I in its proof. We use this proposition in our proof of Theorem 0.1 to reduce to the case where the monomial ideal I is square-free.

Proposition 3.1. *Let $R=k[x_1,x_2,\ldots,x_r]$ and I be a monomial ideal of R. If R/I is Cohen–Macaulay then $R/\sqrt{I}$ is also Cohen–Macaulay.*

Proof. Set a minimal generating set for $I=\langle f_1,\ldots,f_m\rangle$. First, using induction, we prove that $R[\mathbf{Y}]/P(\underline{f})$ is Cohen–Macaulay. Denote the degree of x_1 in f_j by α_j for $1\le j\le m$. Reorder the generators of I so that $\alpha_1,\ldots,\alpha_s>1$, $\alpha_{s+1}=\cdots=\alpha_{s+r}=1$ and $\alpha_{s+r+1}=\cdots=\alpha_m=0$. Write $f_i=x_1^{\alpha_i}h_i$, $1\le i\le s+r$, such that x_1 does not divide h_i. Use Y_1 to denote the first variable used to replace x_1. We claim Y_1-x_1 is a non-zero divisor on the ideal

$$J=\langle x_1^{\alpha_1-1}Y_1h_1,\ldots,x_1^{\alpha_s-1}Y_1h_s,x_1h_{s+1},\ldots,x_1h_{s+r},f_{s+r+1},\ldots,f_m\rangle.$$

Suppose Y_1-x_1 is a zero divisor on J. Thus Y_1-x_1 is in some associated prime ideal P of J. There exists $g\notin J$ such that $P=(J:g)$. Since J is monomial, P is monomial and g can be taken to be monomial. So there exists a monomial $g\notin J$ such that $g(Y_1-x_1)\in J$.

Since J is a monomial ideal $Y_1g\in J$ and $x_1g\in J$. Since Y_1 is a non-zero divisor on $\langle x_1h_{s+1},\ldots,x_1h_{s+r},f_{s+r+1},\ldots,f_m\rangle$, if Y_1g is in that ideal, then $g\in J$ which is a contradiction. Therefore we can assume $x_1^{\alpha_i-1}Y_1h_i$ divides Y_1g for some $1\le i\le s$. Write $g=x_1^nY_1^lg'$ where $l,n\ge 0$ and Y_1 and x_1 do not divide g'. Since $x_1^{\alpha_i-1}Y_1h_i$ divides Y_1g for some $1\le i\le s$, $x_1^{\alpha_i-1}h_i$ divides $g=x_1^nY_1^lg'$. Thus, x_1 divides g so $n>0$. Also, $x_1^{\alpha_i-1}h_i$ divides x_1^ng'. Therefore, $x_1^{\alpha_i-1}Y_1h_i$ divides $x_1^nY_1g'$. If $l>0$ this divides g and then g is in J which is a contradiction. Hence we may assume $n>0$ and $l=0$. Now we use that $x_1g\in J$. The monomials $x_1^{\alpha_i-1}Y_1h_i$ cannot divide x_1g since Y_1 does not divide x_1g. This implies $x_1^ng'\in\langle h_{s+1},\ldots,h_{s+r},f_{s+r+1},\ldots,f_m\rangle$ and x_1 is a non-zero divisor on the ideal, so $g'\in\langle h_{s+1},\ldots,h_{s+r},f_{s+r+1},\ldots,f_m\rangle$. Since $n>0$, x_1g' divides g and therefore $g\in J$ a contradiction. Hence Y_1-x_1 is a non-zero divisor on J.

Assume S is a graded ring and m is the irrelevant ideal, then S is Cohen–Macaulay if and only if S_m is Cohen–Macaulay (Matijevic and Roberts, 1974). Let m_1 denote the irrelevant ideal for $R[Y_1]$. Since I is homogeneous and R/I is Cohen–Macaulay, the previous two statements combine to imply $R[Y_1]_{m_1}/IR[Y_1]_{m_1}$ is Cohen–Macaulay. The ring $R[Y_1]_{m_1}/IR[Y_1]_{m_1}$ is local and $Y_1-x_1\in m_1$ is a non-zero divisor, so

$$R[Y_1]_{m_1}/\langle I,Y_1-x_1\rangle R[Y_1]_{m_1}\simeq R[Y_1]_{m_1}/\langle J,Y_1-x_1\rangle R[Y_1]_{m_1}$$

is Cohen–Macaulay. The ring $R[Y_1]_{m_1}/JR[Y_1]_{m_1}$ is Cohen–Macaulay since the element Y_1-x_1 is a non-zero divisor on J. Moreover, this implies $R[Y_1]/JR[Y_1]$ is Cohen–Macaulay. By induction on the variables used to form a polarization of I, both $R[\mathbf{Y}]/P(\underline{f})$ and $R[\mathbf{Y}]/\langle P(\underline{f}),\mathbf{Y}-\mathbf{x}\rangle$ are Cohen–Macaulay.

Let W be the multiplicatively closed set $k[\mathbf{Y}]\setminus\{0\}$ in $S=R[\mathbf{Y}]$ and let $K=k(\mathbf{Y})$. Then the localization of $S/P(\underline{f})$ at W is isomorphic to $K[x_1,\ldots,x_r]/\sqrt{I}$. So, $S/P(\underline{f})$

Cohen–Macaulay implies $K[x_1, \dots, x_r]/\sqrt{I}$ is Cohen–Macaulay. The ring

$$K[x_1, \dots, x_r]/\sqrt{I} \simeq k[x_1, x_2, \dots, x_r]/\sqrt{I} \otimes_k K$$

is Cohen–Macaulay if and only if $k[x_1, x_2, \dots, x_r]/\sqrt{I}$ is Cohen–Macaulay (Bruns and Herzog, 1993, Theorem 2.1.10). Hence $k[x_1, x_2, \dots, x_r]/\sqrt{I}$ is Cohen–Macaulay. □

Recall that Proposition 1.2 states that $L_2(I)$ is a prime ideal assuming I is generically a complete intersection. In general, monomial ideals with Cohen–Macaulay quotient ring are not generically complete intersections, however, if the ideal is a square-free monomial ideal then it is generically a complete intersection. Proposition 3.1 allows us to reduce to the square-free case in the proof of Theorem 0.1.

In the statement of Theorem 0.1 we do not assume the generators of I have the same degree, but this is required by Theorem 2.2 and we use Theorem 2.2 in the proof of Theorem 0.1. Proposition 3.2 and Corollary 3.3 allow us to reduce to the case where the degrees of the generators have the same degree.

For $R = k[x_1, x_2, \dots, x_r]$ and $I = \langle f_1, \dots, f_m \rangle$ monomial the relations on the generators of I are generated, due to the natural multi-grading, by relations of the form $mf_i - nf_j$ where $m, n \in R$ are monomials. Hence, a presentation matrix of I can be given with exactly two non-zero monomial entries in each column. We call a determinant of an $n \times n$ matrix $A = (a_{ij})$ a *simple determinant* if at most one term of $\det(A) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) a_{1\sigma(1)} \cdots a_{n\sigma(n)}$ is non-zero.

Proposition 3.2. *Let $R = k[x_1, x_2, \dots, x_r]$. Let ϕ be a $m \times (m-1)$ matrix with exactly two non-zero monomial entries in each column and assume the maximal minors of ϕ are all non-zero. Then each minor of ϕ is a simple determinant or zero.*

Proof. We use induction on the size of ϕ. If $m = 2$, the minors of ϕ are the two monomial entries of ϕ and hence are simple determinants. Assume the statement for m. Let ϕ be a $(m+1) \times m$ matrix satisfying the hypotheses of the proposition. Each column has exactly two non-zero entries so there are exactly $2m$ non-zero entries in ϕ. Since the maximal minors are all non-zero, every row has at least one non-zero element in it and therefore there must be at least one row with exactly one non-zero entry. Choose a row with exactly one non-zero entry and denote that entry M. Set N to be the other non-zero entry in that column. Reorder the rows and columns so that ϕ looks like

$$\phi = \begin{bmatrix} M & 0 & 0 & \cdots & 0 \\ N & & & & \\ 0 & & \psi & & \\ \vdots & & & & \\ 0 & & & & \end{bmatrix},$$

where ψ is the $m \times (m-1)$ matrix obtained by removing the first row and column from ϕ. Every column of ψ must have exactly two non-zero entries since otherwise we contradict this fact for ϕ. The entries of ψ are monomial since ψ is a submatrix of ϕ. Suppose a maximal minor of ψ is zero. Let i denote the row that was removed to form the maximal minor that is zero, and let δ_i denote this minor. The maximal minor of ϕ when $i+1^{st}$ row is removed is M times δ_i. Therefore the $i+1^{\text{st}}$ maximal minor of ϕ is $M\delta_i = M \cdot 0 = 0$, which is a contradiction. Hence ψ satisfies the induction hypotheses and therefore we can assume all of the minors of ψ are simple determinants or zero. Consider the minors of ϕ. If a square submatrix of ϕ is also a submatrix of ψ then the determinant is simple or zero. Suppose the submatrix is not contained in ψ. If the submatrix does not contain M or N then, since it is not contained in ψ, it must have a row or column of zeros and hence the determinant is zero. Suppose the submatrix includes M (and may or may not include N), if the determinant is expanded along the top row of the submatrix the determinant is M times the determinant of a submatrix contained in ψ and is hence zero or simple. Last, assume N is in the submatrix, but M is not. In this case the submatrix of ϕ is a submatrix of

$$\begin{bmatrix} N & & \\ 0 & \psi & \\ \vdots & & \\ 0 & & \end{bmatrix}.$$

Expanding the determinant along the first column, the determinant is N times the determinant of a submatrix of ψ and hence is zero or simple. □

This proposition is particularly interesting because it yields the following corollary that is very useful for the proof of the main theorem and maybe useful for other problems.

Corollary 3.3. *Let $R = k[x_1, x_2, \ldots, x_r]$ and $I \subseteq R$ be a codimension two monomial ideal such that R/I is Cohen–Macaulay. Then there exists a monomial ideal J such that $\sqrt{J} = \sqrt{I}$, R/J is Cohen–Macaulay, J is generically a complete intersection and the generators of J are all of the same degree.*

Proof. The ring $R/\sqrt{I}$ is Cohen–Macaulay by Proposition 3.1 and if such a J exists for $\sqrt{I}$ then it satisfies the properties for I as well. Therefore replacing I with $\sqrt{I}$, we can assume I is square-free and generically a complete intersection.

Let $f_1, \ldots, f_m$ be a minimal generating set for I. Let ϕ denote a presentation matrix for I from the Hilbert–Burch theorem such that there are exactly two non-zero entries in each column. Proposition 3.2 allows us to "homogenize" ϕ in the following way. Fix a column of ϕ and compare the two non-zero entries in that column. Raise the exponents in the monomial of smaller degree until the two monomials have the same degree. Do this for all of the columns of ϕ. Call this new matrix $h(\phi)$ (see example 4.3). The non-zero entries in each column of $h(\phi)$ are the same degree, by construction, so the maximal minors of $h(\phi)$ all have the same degree. By Proposition 3.2, each minor of both ϕ and $h(\phi)$ is a simple determinant or zero. This implies the maximal minors of $h(\phi)$ are monomial. By construction, a minor of ϕ is non-zero if and only if the corresponding minor of $h(\phi)$ is non-zero. The construction of $h(\phi)$ and the fact that the non-zero minors of each matrix are simple, implies that if α is a non-zero minor of ϕ and β is the corresponding minor of $h(\phi)$ then α divides β and for $N \gg 0$, β divides α^N. Hence

$$I_n(\phi) = \sqrt{I_n(h(\phi))} \qquad \text{for } 1 \leq n \leq m-1. \tag{3.2}$$

This implies $\operatorname{codim}(I_{m-1}(h(\phi))) = 2$ and hence $R/I_{m-1}(h(\phi))$ is Cohen–Macaulay by the Hilbert–Burch Theorem, since R is a regular ring.

In the context of this corollary $\operatorname{codim}(I_{m-2}(\phi)) \geq 3$ if and only if I is generically a complete intersection. Since I is square-free and monomial, I is generically a complete intersection and hence $\operatorname{codim}(I_{m-2}(\phi)) \geq 3$. By equation (3.2) $I_{m-2}(\phi) = \sqrt{I_{m-2}(h(\phi))}$.

Therefore $\operatorname{codim}(I_{m-2}(h(\phi))) \geq 3$ and $I_{m-1}(h(\phi))$ is generically a complete intersection. □

We now give the proof of Theorem 0.1 and restate the theorem for the reader's convenience.

THEOREM 0.1. *Let $R = k[x_1, x_2, \ldots, x_r]$ be a polynomial ring over a field k and I an ideal of R. Let $>$ be a monomial order that respects total degree. Assume I is a monomial ideal of codimension 2 and R/I is Cohen–Macaulay. Then there exists an extension field K of k and a prime ideal P contained in the polynomial ring $S = K[x_1, x_2, \ldots, x_r]$ such that $\sqrt{in(P)} = \sqrt{I}S$.*

PROOF. By Corollary 3.3 there exists a monomial ideal J that is generically a complete intersection, the generators are all the same degree, $\sqrt{J} = \sqrt{I}$ and R/J is Cohen–Macaulay. If we prove the theorem for J we get a prime ideal P such that $\sqrt{in(P)} = \sqrt{J}S = \sqrt{I}S$ as desired.

Replace I with J and let ϕ denote the presentation matrix for J constructed from a presentation matrix of I as in Corollary 3.3. Let m denote the number of rows of ϕ and f_i, $1 \leq i \leq m$ be the signed minors of ϕ. By the Hilbert–Burch theorem, $f_1, \ldots, f_m$ generate J and by Corollary 3.3 they are the same degree. Denote that degree d. Also, order the generators so that $f_1 > f_2 > \cdots > f_m$.

By Proposition 1.2 $L_2(I_{m-1}(\phi))$ is a prime ideal. Set $Y = \{Y_{11}, \ldots, Y_{2m+2}\}$, $Z = \{Z_{11}, \ldots, Z_{2m-1}\}$ and $Q = R[Y, Z]$ with $\deg(Y_{ij}) = 1$ and $\deg(Z_{ij}) = e_j$. Give Q the inverse block order with respect to $\{Y, Z\}$ and any order on the new variables. Set

$$\Phi = \begin{bmatrix} & & & Y_{11} & Y_{21} \\ & \phi & & \vdots & \vdots \\ & & & Y_{1m} & Y_{2m} \\ Z_{11} & \cdots & Z_{1m-1} & Y_{1m+1} & Y_{2m+1} \\ Z_{21} & \cdots & Z_{2m-1} & Y_{1m+2} & Y_{2m+2} \end{bmatrix}. \tag{3.3}$$

The matrix Φ is a presentation matrix for $L_2(I_{m-1}(\phi))$. Let δ_i, $1 \leq i \leq m+2$ denote the signed minor of Φ formed when the ith row is removed. Then,

$$\begin{aligned} \delta_1 &= f_1(Y_{1m+2}Y_{2m+1} - Y_{1m+1}Y_{2m+2}) + \beta_1 \\ &\ \vdots \\ \delta_m &= f_m(Y_{1m+2}Y_{2m+1} - Y_{1m+1}Y_{2m+2}) + \beta_m \\ \delta_{m+1} &= f_1(Y_{11}Y_{2m+2} - Y_{12}Y_{1m+2}) + f_2(Y_{12}Y_{2m+2} - Y_{22}Y_{1m+2}) + \cdots + \\ &\quad + f_m(Y_{1m}Y_{2m+2} - Y_{2m}Y_{1m+2}) + \beta_{m+1} \\ \delta_{m+2} &= f_1(Y_{11}Y_{2m+1} - Y_{21}Y_{1m+1}) + f_2(Y_{12}Y_{2m+1} - Y_{22}Y_{1m+1}) + \cdots + \\ &\quad + f_m(Y_{1m}Y_{2m+1} - Y_{2m}Y_{1m+1}) + \beta_{m+2} \end{aligned}$$

where $\deg_{\underline{x}}(\beta_i) < d$, $1 \leq i \leq m+2$. These form a Gröbner basis for $L_2(I_{m-1}(\phi))$, by Theorem 2.2.

Let g be a polynomial in $L_2(I_{m-1}(\phi)) \cap k[Y, Z]$ and assume $g \neq 0$. Then the initial term of g is in $in(L_2(I_{m-1}(\phi))) \cap k[Y, Z]$. Since $in(g)$ is in $in(L_2(I_{m-1}(\phi)))$ and $\delta_1, \ldots, \delta_{m+2}$ form a Gröbner basis for $L_2(I_{m-1}(\phi))$ it must be true that $in(\delta_i)$ divides $in(g)$ for some $1 \leq i \leq m+2$. However, the initial terms of $\delta_1, \ldots, \delta_{m+2}$ are of the form $f_i M$ for some $1 \leq i \leq m$, where M is a degree two monomial in $k[Y]$, since we are using the inverse block order with respect to $\{Y, Z\}$ and $\deg_{\underline{x}}(\beta_i) < d$ for each $1 \leq i \leq m+2$. Thus the initial terms of $\delta_1, \ldots, \delta_{m+2}$ are not in $k[Y, Z]$ and therefore cannot divide $in(g)$. Hence $L_2(I_{m-1}(\phi)) \cap k[Y, Z] = 0$. Let W be the multiplicatively closed set $k[Y, Z] \setminus \{0\}$. Hence $L_2(I_{m-1}(\phi))$ is disjoint from W and the image of $L_2(I_{m-1}(\phi))$ in $Q_W = k(Y, Z)[x_1, \ldots, x_r]$ is a prime ideal.

Denote the images of $\delta_1, \ldots, \delta_{m+2}$ in $S = k(Y, Z)[x_1, \ldots, x_r]$ by $\widetilde{\delta_1}, \ldots, \widetilde{\delta_{m+2}}$ and let $lm(\widetilde{\delta_i})$ denote the leading monomial of $\widetilde{\delta_i}$, that is the leading term without the coefficient. The images $\widetilde{\delta_1}, \ldots, \widetilde{\delta_{m+2}}$ are a Gröbner basis for the image of $L_2(I_{m-1}(\phi))$ in S because Q has the inverse block order with respect to $\{Y, Z\}$ (Becker and Weispfenning, 1993, Lemma 8.93). Moreover, $\widetilde{\delta_i}$, $1 \leq i \leq m+2$ in S is δ_i with the part of each term that is a monomial in $k[Y, Z]$ considered to be part of the coefficient. We are using the inverse block order and $\deg_{\underline{x}}(\beta_i) < \deg(f_i)$ and therefore, $lm(\widetilde{\delta_i}) = f_i$, for $1 \leq i \leq m$ and $lm(\widetilde{\delta_j}) = f_1$ for $j = m+1, m+2$. Hence $in(L_2(I_{m-1}(\phi))S) = \langle f_1, f_2, \ldots, f_n \rangle S = IS$. $\square$

4. Examples

The ideal $I = \langle bc, bd, acd \rangle$ is a nice test case for many of the theorems and assumptions. This ideal is one of the simplest examples that illustrates the necessity of each of the steps we have taken. We will use this ideal for the first and third examples. The first example illustrates the need for the assumption from Theorem 2.2 that the generating set have elements of the same degree, as well as the necessity of one of the radicals. The second example illustrates the necessity of the other radical. For the third example we use I to illustrate the entire algorithm for constructing the desired prime ideal. The fourth example looks at the necessity of the assumption that the quotient ring be Cohen–Macaulay.

Example 4.1. This example illustrates what happens if we drop the assumption in Theorem 2.2 that the generators of I have the same degree. This example also illustrates why we need one of the radicals. Let $I = \langle bc, bd, acd \rangle$. Using Macaulay2, written by Grayson and Stillman (2001), we computed the minors of the matrix given in Theorem 2.2 for I:

$$\begin{aligned} \delta_1 &= acY_{13}Z_{11} - \mathbf{bcY_{14}} + bY_{12}Z_{11} + cY_{13}Z_{12} \\ \delta_2 &= bdY_{14} + bY_{11}Z_{11} - dY_{13}Z_{12} \\ \delta_3 &= acdY_{14} + acY_{11}Z_{11} + cY_{11}Z_{12} + dY_{12}Z_{12} \\ \delta_4 &= acdY_{13} + bcY_{11} + bdY_{12}. \end{aligned} \tag{4.1}$$

The elements δ_1, δ_3 and δ_4 are not homogeneous and the bold face term is the one we would like to have as the initial term of δ_1. Set $\deg(Y_{13}) = \deg(Z_{11}) = 1$ and $\deg(Y_{11}) = \deg(Y_{12}) = \deg(Y_{21}) = \deg(Y_{22}) = \deg(Y_{14}) = \deg(Z_{12}) = 2$. The polynomials in (4.1) are now quasi-homogeneous, meaning they are homogeneous with respect to the weights. However, the initial term is still not the desired one. If we now use the reverse lexicographic order with Y_{14} as the largest of the new variables, δ_1 now has initial term $-bcY_{14}$. For any ideal where the generators are not all of the same degree we can weight the new variables so that the second generic link is quasi-homogeneous and use a particular reverse lexicographic order to get a statement like Theorem 2.2 and Corollary 2.3 for ideals where the generators are not the same degree. However, in the proof of Theorem 0.1, after constructing the second generic link of I, we invert all of the new variables to get an ideal in $K[x_1, \dots, x_r]$ where K is an extension field of k. After inverting the new variables the leading monomial of the image of δ_1 is ac not bc as needed. This happens regardless of the monomial order and the weights on the new variables. Also, one might suggest sending the new variables to elements in k so that the problematic terms go to zero, however, this process will not necessarily preserve the property that the ideal is prime. Hence $in(P) \neq IS$ if we do not reduce to the case where the generators all have the same degree. Therefore we raise the degree of b in the presentation matrix using Proposition 3.2 and get $\sqrt{in(P)} = IS$.

Example 4.2. Let $I = \langle x^2, xy, y^2 \rangle$, then I is codimension two and the quotient ring $k[x, y]/I$ is Cohen–Macaulay, but I is not generically a complete intersection. Therefore in order for the second generic link to be a prime ideal we must pass to the radical of I, $\langle x, y \rangle$. Since this ideal is square-free and the generators are of the same degree, the algorithm will yield a prime ideal P with initial ideal $\langle x, y \rangle$ and hence $in(P) = \sqrt{I}$.

Example 4.3. Using the ideal $I = \langle bc, bd, acd \rangle$ we work through the entire algorithm for constructing the desired prime ideal. In this example we are able to take the process one step further, as we are able to specialize the new variables to elements in k and verify that the image is indeed a prime ideal. In every example we have computed, specializing is possible, however, as we already mentioned, the fact that we can always specialize and preserve both the Gröbner basis and the property of being prime is open.

The following proposition is one way to verify that the ideal we construct is a prime ideal.

Proposition 4.4. (Vasconcelos, 1998, Proposition 3.5.6) *Suppose $A = k[\mathbf{z}, \mathbf{x}]/I$ is a Cohen–Macaulay, equidimensional ring. Let $B = k[\mathbf{z}]$ be a Noether normalization of A. The degree of A over B is the dimension of the vector space*

$$l = \dim_k(k[\mathbf{z}, \mathbf{x}]/(I, \mathbf{z})).$$

If there exists a subring

$$B \hookrightarrow S = k[\mathbf{z}, U]/\langle f(\mathbf{z}, U) \rangle \hookrightarrow A,$$

where $f(\mathbf{z}, U)$ is an irreducible polynomial of degree l, then A is an integral domain.

First, compute a presentation matrix for $I = \langle bc, bd, acd \rangle \subseteq R = k[a, b, c, d]$. This ideal is small enough we can find a presentation by hand, or using Macaulay2. A matrix is

$$\phi = \begin{bmatrix} -d & 0 \\ c & -ac \\ 0 & b \end{bmatrix}. \tag{4.2}$$

Since the generators of I are not all of the same degree we use Corollary 3.3 and form the following matrix:

$$h(\phi) = \begin{bmatrix} -d & 0 \\ c & -ac \\ 0 & b^2 \end{bmatrix} \tag{4.3}$$

where b is squared in the second column so that the non-zero entries in each column of ϕ now have the same degree. The maximal minors of $h(\phi)$ are b^2c, b^2d, acd, thus verifying that $\sqrt{I_2(h(\phi))} = I_2(\phi)$. The ideal generated by the maximal minors of $h(\phi)$ is still generically a complete intersection by Corollary 3.3 and hence the second generic link of $h(\phi)$ is a prime ideal. Generators for the second generic link are the maximal minors of the following matrix:

$$\Phi = \begin{bmatrix} -d & 0 & Y_{11} & Y_{21} \\ c & -ac & Y_{12} & Y_{22} \\ 0 & b^2 & Y_{13} & Y_{23} \\ Z_{11} & Z_{12} & Y_{14} & Y_{24} \\ Z_{21} & Z_{22} & Y_{15} & Y_{25} \end{bmatrix}.$$

Corollary 2.3 implies the maximal minors of this matrix form a Gröbner basis for $L_2(I)$. We can use Macaulay2 to verify this. Using Macaulay2 generate a random sequence of numbers from the base field. For this example we generated a random sequence of rational numbers. Specialize the new variables $Y_{11}, \ldots, Y_{25}, Z_{11}, \ldots, Z_{22}$ to the following numbers, respectively:

$$\frac{-9}{5}, \quad \frac{3}{10}, \quad \frac{-1}{2}, \quad \frac{7}{5}, \quad -1, \quad -7, \quad 6, \quad 7, \quad 1, \quad \frac{-1}{7}, \quad \frac{-1}{4}, \quad \frac{2}{5}, \quad 1, \quad \frac{3}{2}.$$

The image of three of the elements in the Gröbner basis have acd as their leading monomial. Therefore, the Gröbner basis reduces to include only one of these generators and the Gröbner basis simplifies to three elements. The following three elements form a reduced Gröbner basis for the image of $L_2(I)$ after specializing the variables added when the second generic link was formed.

$$\begin{aligned} g_1 &= \boldsymbol{acd} + \frac{807}{-440}ac + \frac{3777}{-1375}c + \frac{12123}{-5500}d - \frac{51429}{220000} \\ g_2 &= \boldsymbol{b^2d} - \frac{807}{440}b^2 - \frac{139}{22}d + \frac{27993}{-22000} \\ g_3 &= \boldsymbol{b^2c} + \frac{137}{88}b^2 + \frac{103}{-22}ac + \frac{139}{-22}c - \frac{10633}{-22000}. \end{aligned}$$

Homogenize with respect to t. Set $S = k[a,b,c,d,t]$ and $G_i = g_i(\frac{a}{t}, \frac{b}{t}, \frac{c}{t})t^3$ for $1 \leq i \leq 3$. The ideal $\langle g_1, g_2, g_3\rangle$ is a prime ideal if and only if the ideal $\langle G_1, G_2, G_3\rangle$ is a prime ideal. Let $J = \langle G_1, G_2, G_3\rangle$. The degree of S/J is 7. This can be found either by using Macaulay2 or by computing a Noether normalization of S/J, say $k[\underline{z}]$ and computing $\dim_k S/\langle J, \underline{z}\rangle$, utilizing Proposition 4.4. Reorder the variables in S so that $t > a > b > c > d$ and recompute the Gröbner basis using an elimination order for t. There is one polynomial in the Gröbner basis which is contained in $k[a,b,c,d]$. Denote this polynomial $f(a,b,c,d)$. A Noether normalization of S/J is $k[a,c,b+d]$. Hence

$$k[a,c,b+d] \hookrightarrow k[a,b,c,d]/\langle f(a,b,c,d)\rangle = k[a,b,c,d,t]/J \cap k[a,b,c,d] \hookrightarrow S/J.$$

The polynomial we found has degree 7. Using Maple we checked that it is irreducible and hence J is a prime ideal.

Example 4.5. We consider the necessity of the quotient ring being Cohen–Macaulay. Since the two ideals in this example are not perfect we use the definition of generic link to compute the ideals.

Let $R = k[a,b,c,d,e]$ and $I = (ad, ace, bcd, bce) = (a,b) \cap (a,c) \cap (c,d) \cap (d,e)$. This ideal is pure and strongly connected which are the necessary conditions given by Kalkbrenner and Sturmfels (1995). Localizing at $P = (a,b,d,e)$, a codimension 2 prime ideal in R/I, R_P/I_P is not depth 2 so R/I is not S_2. The leading terms of a minimal generating set for $L_2(I)$ are given below.

$$\begin{array}{rl}
b^2cY_{1,3}Z_{1,4}Z_{2,3} & \boldsymbol{-a^2eY_{1,2}Z_{1,4}Z_{2,3}} \\
abcY_{1,3}Z_{1,4}Z_{2,3} & -a^2Y_{1,2}Z_{1,4}Z_{2,3} \\
a^2eY_{1,2}^2Z_{1,4}Z_{2,3} & -abcY_{1,2}Z_{1,4}Z_{2,3} \\
\boldsymbol{-a^2bY_{1,2}Z_{1,4}Z_{2,3}} & a^2Y_{1,2}Z_{1,4}^2Z_{2,3}^2 \\
abcY_{1,3}Y_{2,2}Z_{1,3} & abcY_{1,3}Y_{2,2}Z_{2,3}.
\end{array}$$

The portion of each of these monomials that is in $k[a,b,c,d,e]$ is not in I. Moreover, if we carefully check each generator we see that the two elements in the Gröbner basis with the boldface monomials as leading terms contain no term whose a,b,c,d,e part is in I. Any Gröbner basis will preserve this bad structure.

The case when I is S_2 but not Cohen–Macaulay is both more and less encouraging. The second generic link in this case does not give a counter example, but we cannot compute it. Every example we have tried is too computationally complex for the computer we use. Let $R = k[a,b,c,d,e]$ and $I = (abd, bde, ace, acd, bce) = (a,b) \cap (a,c) \cap (c,d) \cap (d,e)$. The ideal I is one such example. The leading terms for a minimal generating set for the first generic link are not promising, but the first generic link was not promising for the Cohen–Macaulay case either. There is other evidence in a paper by Hochster and Huneke (1994) that suggests S_2 may be the desired necessary condition.

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