Koszul-type complexes for commuting polynomial matrices

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Abstract

In this paper we show that it is possible to construct a Koszul-type complex for maps given by pairwise commuting matrices of polynomials. This result has applications to surjectivity theorems for constant coefficients differential operators of finite and infinite order.

1 Introduction

Let us consider a system of linear, constant coefficients partial differential equations of the form

$$(p_1(D), \ldots, p_r(D))f = P(D)f = 0$$

where $p_1, \ldots, p_r$ are polynomials in $R = \mathbb{C}[z_1, \ldots, z_n]$, $D = (-i\partial/\partial x_1, \ldots, -i\partial/\partial x_n)$ and $f : U \subseteq \mathbb{R}^n \to \mathbb{R}$ is an infinitely differentiable function. It is well known that some important information on the system is contained in the polynomial matrix P_1 whose rows generate the first syzygies of the $r-$ vector $P = [p_1, \ldots, p_r]$. When the sequence of polynomials $p_1, \ldots, p_r$ is regular (see [18]), the matrix P_1 can be constructed in a simple way because its rows consist of all the vectors of the type

$$[0, \ldots, \quad -\underbrace{p_j}_{i-\text{place}}, 0, \ldots, 0, \quad \underbrace{p_i}_{j-\text{place}}, \ldots, 0]$$

One can repeat this procedure to find the syzygies of P_1, and so on, until one obtains the so called Koszul complex that is nothing but the minimal free resolution ([18]) of the module $M = R/I$, where I is the ideal generated by $p_1, \ldots, p_r$. When considering more general systems, in which the unknown function f is a vector and the matrix $P(D)$ of differential operators acting on it is an $r_1 \times r_0$ matrix (in the scalar case $r_1 = 1$), the situation becomes more complicated and the matrix of the first syzygies cannot be easily computed. The theory of Gröbner bases offers some algorithms that can be useful to compute the first syzygy module, but the complexity of the problem is doubly exponential in the number of variables as shown in the paper [6]. Moreover, the procedure of computing syzygies with Gröbner bases does not take into account the structure of the matrix P: even though in some cases P can be seen as a block matrix, in which every block represents an operator (see e.g. the case of the Cauchy–Fueter system [1], [5], [3] and the case of the Dirac system [21]) a priori the matrix of the first syzygies is not necessarily written in terms of those blocks; this is dramatically demonstrated in [5] for the case of the Cauchy-Fueter system. On the other hand, given a matrix

$$P(D) = \begin{bmatrix} P_1(D) \\ \dots \\ P_r(D) \end{bmatrix}$$

where each $P_i(D)$ is a $n \times n$ matrix, it might be interesting to understand under which conditions it is possible to build a Koszul complex starting from the matrices P_i. Obviously, it is not always possible to construct such a complex mainly because of the non commutativity of the ring of matrices and one may wonder what are the conditions under which the procedure can be applied. There are several questions that can be answered if one can solve the problem of finding the syzygies of the operator $P(D)$, and ee devote the rest of this introduction to a quick discussion of a few applications.

Suppose we consider the system

$$P_1(D)f = g$$

where we want f and g to be in some space

$$\mathcal{Q} = \{f : \mathbb{R}^n \longrightarrow \mathbb{R}^m \mid f \text{ differentiable}, P_2(D)f = 0\},$$

for some $P_2(D)$. We ask when the operator $P_1(D) : \mathcal{Q} \to \mathcal{Q}$ is surjective. The equation above can be rewritten as the system for $\mathcal{C}^\infty$ functions

$$\begin{cases} P_1(D)f = g \\ P_2(D)f = 0 \end{cases}$$

and the surjectivity of $P_1(D)$ is equivalent to the request that the only compatibility condition on the datum g is $P_2(D)g = 0$. This is not true in general. When $\mathcal{Q}$ is the space of holomorphic functions, i.e., $P_2(D)$ is given by n Cauchy-Riemann operators and

$$\mathcal{Q} = \{f : \mathbb{C}^n \longrightarrow \mathbb{C} \mid \partial f/\partial \bar{z}_1 = \dots = \partial f/\partial \bar{z}_n = 0\},$$

the surjectivity of an arbitrary differential equation is a consequence of the Lindelöf theorem. If, on the other hand, $\mathcal{Q}$ is the space of regular functions of a quaternionic variable (i.e. $P_2(D)$ is given by n Cauchy-Fueter operators and $\mathcal{Q} = \{f : \mathbb{H}^n \longrightarrow \mathbb{H} \mid \partial f/\partial \bar{q}_1 = \dots = \partial f/\partial \bar{q}_n = 0\}$), then we have no immediate way to prove that any operator on $\mathcal{Q}$ is surjective. The results we prove in this paper will have this fact as an immediate corollary.

More generally, given a matrix $P(D)$ we would like to compute not only the matrix of the first syzygies of the polynomial matrix P associated to $P(D)$ but also the free resolution of the module $M = \mathrm{coker} P^t$ where P^t is the transpose of P. In fact, by the Hilbert syzygy theorem, there is a finite free resolution

$$0 \longrightarrow R^{r_s} \xrightarrow{P^t_{r_s}} R^{r_{s-1}} \longrightarrow \ldots \xrightarrow{P^t_1} R^q \xrightarrow{P^t} R^{r_0} \longrightarrow M \longrightarrow 0$$

that together with its dual

$$0 \longrightarrow R^{r_0} \xrightarrow{P} R^{r_1} \xrightarrow{P_1} \ldots \longrightarrow R^{r_{s-1}} \xrightarrow{P_{r_s}} R^{r_s} \longrightarrow 0 \tag{1}$$

are key tools in the algebraic analysis of the system associated to $P(D)$. Our results will allow us to calculate the sequence (1) rather easily, at least in some cases. Finally one can show that, under suitable hypoteses, if we set $Q = P_{r_s}$ and we denote by $\mathcal{E}$ the sheaf of infinitely differentiable functions and by $\mathcal{D}'$ the sheaf of Schwartz distributions, then for any bounded open set $K \subset \mathbb{R}^n$

$$[H^0(K, \mathcal{E}^P)]' \cong H^s(\mathbb{R}^n, \mathbb{R}^n \setminus K, \mathcal{D}'^Q).$$

Our results can be used to construct such duality in many concrete cases (see [11] for a more elaborate discussion of this aspect of the theory).

2 Commutative matrices

In this section we study a system of differential equations of the form $P_1(D)f = \cdots = P_k(D)f = 0$ and we seek conditions on the polynomial square matrices $P_1 \ldots P_k$ such that the compatibility conditions on the system, i.e. the first syzygies of the rows of the matrix

$$P = \begin{bmatrix} P_1 \\ \vdots \\ P_k \end{bmatrix}$$

can be easily written in terms of the matrices P_i themselves. For example, given $k = 2$, if the matrices P_1 and P_2 commute, we may expect that the syzygies are the rows of the matrix $[-P_2 \ P_1]$. This is suggested by the fact that the following matrix product is zero:

$$\begin{bmatrix} -P_2 & P_1 \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = 0.$$

this commutativity, however, is not enough to guarantee that $[-P_2 \ P_1]$ contains all the syzygies we need as we can easily see for example taking $P_1 = P_2$. In general, the validity of this result depends on whether or not we have relations between rows of P_1 and rows of P_2, or among rows of the same matrix. According to well known results in algebra, the situation can be fully understood through suitable algebraic conditions on the matrices involving the notion of regular sequence. In the sequel $\mathcal{R} = \mathrm{Mat}_n(R)$ will denote the ring of $n \times n$ matrices with entries in the ring R; if R is an integral domain, $\mathrm{Frac}(R)$ will denote its field of fractions.

Definition 2.1. *Let R be a ring, and let $P_1, \dots, P_k$ be square matrices in $\mathcal{R}$. We say that the k-uple $(P_1, \dots, P_k)$ is a left regular sequence if*
1) P_1 is a left regular element of $\mathcal{R}$, i.e. the only $B \in \mathcal{R}$ such that $BP_1 = 0$ is $B = 0$;
2) P_i is not a zero divisor in $\mathcal{R}/(P_1, \dots, P_{i-1})\mathcal{R}$ for all $i = 2 \dots k$ where $(P_1, \dots, P_{i-1})\mathcal{R}$ is the left ideal in $\mathcal{R}$ generated by $P_1, \dots, P_{i-1}$.

When $k = 1$ we have the square system $P_1(D)f = 0$ and the condition that P_1 is a non-zerodivisor in $\mathcal{R}$ is fully equivalent to the fact that we do not have nonzero sygyzies for its rows, as stated by the following

Proposition 2.2. *Let R be an integral domain. The following are equivalent facts for a square matrix $P \in \mathcal{R}$:*
1) $\mathrm{Det}(P) \neq 0$*;*
2) P is a left regular element of $\mathcal{R}$;
3) $\mathrm{Syz}(P) = \langle 0 \rangle \subset R^n$ *where* $\mathrm{Syz}(P)$ *means the first module of the syzygies for the rows of the matrix P.*

Proof. 1)$\Rightarrow$ 2): let B be a nonzero square matrix such that $BP = 0$. Then, in particular, any row of B is a solution to the linear sistem $(x_1 \dots x_n)P = (0 \dots 0)$ which has only trivial solutions since $\mathrm{Det}(P) \neq 0$. Hence $B = 0$.
2)$\Rightarrow$ 3): if $x = (x_1 \dots x_n) \in \mathrm{Syz}(P)$ is a nonzero row, than the matrix X whose n rows are all equal to x is such that $XP = 0$ and that is a contradiction since P is left regular.
3)$\Rightarrow$ 1): suppose $\mathrm{Det}(P) = 0$. Then the linear system $xP = 0$ has a non trivial solution $(x_1 \dots x_n) \in \mathrm{Frac}(R)^n$, and multiplying this solution by the product of all nonzero denominators of its elements, we get a non trivial syzygy $(s_1 \dots s_n) \in R^n$. □

If we require the condition of regularity on the matrices of an overdetermined system, we are able to describe the module of the first syzygies. This fully corresponds to what we would obtain if we considered the syzygies of k polynomials $g_1, \dots, g_k$ forming a regular sequence. The free resolution for the ideal $I = (g_1, \dots, g_k)$ in this case is the Koszul complex, and in particular the first syzygies are of the form

$$\mathrm{Syz}(g_1, \dots, g_k) = \langle (0, \dots, -g_j, 0, \dots, 0, g_i, \dots, 0) |\, i < j \rangle$$

If we consider for example three commuting matrices P_1, P_2, P_3 forming a left regular sequence, the syzygies can be written as the rows of the following matrix:

$$\begin{bmatrix} -P_2 & P_1 & 0 \\ -P_3 & 0 & P_1 \\ 0 & -P_3 & P_2 \end{bmatrix}.$$

The commutativity of the matrices P_i implies that these are syzygies, but in general we could expect to find more relations among their rows. The following result assures that this is not the case.

Theorem 2.3. *Let $P_1, \dots, P_k \in \mathcal{R}$ be $k > 1$ square matrices such that*
1) $P_i P_j = P_j P_i$ for all $i, j = 1 \dots k$,
2) $(P_1, \dots, P_k)$ is a left regular sequence,

and let $\mathcal{M}$ be the R-submodule of R^n generated by the rows of $P_1, \dots, P_k$. Then the first syzygy module $\mathrm{Syz}(\mathcal{M})$ *is generated by the rows of the block matrix*

$$B = \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_{k-1} \end{bmatrix}, \qquad B_i = \begin{bmatrix} 0 & \dots & 0 & -P_{i+1} & P_i & 0 & \dots & 0 \\ 0 & \dots & 0 & -P_{i+2} & 0 & P_i & \dots & 0 \\ & \dots & & \dots & & & \dots & \\ 0 & \dots & 0 & -P_k & 0 & 0 & \dots & P_i \end{bmatrix}. \tag{2}$$

Proof. Let us define P to be the block matrix where we put the P_i's in a column. Then it is easy to see that we have $B \cdot P = 0$ since the P_i's commute, and that means that the rows of the matrix B are syzygies for the rows of the matrix P, i.e. they belong to the set of generators of $\mathcal{M}$. Conversely, let us consider a nonzero row vector

$$x = (x_{11}, \dots, x_{1n}, \dots, x_{k1}, \dots, x_{kn}) \in \mathrm{Syz}(\mathcal{M})$$

This means that the product $x \cdot P$ is the zero vector of R^{kn}. We will show that x belong to the R-module generated by the rows of B. Let us consider the matrix $X \in \mathrm{Mat}_{n,kn}(R)$ containing n rows, all equal to x. Obviously $X \cdot P$ is the zero n by n matrix. Then we can think of X as a block matrix of k square matrices:

$$X = \begin{bmatrix} X_1 & X_2 & \dots & X_k \end{bmatrix}$$

and so the product $X \cdot P = 0$ means

$$X_1 P_1 + \cdots + X_k P_k = 0. \tag{3}$$

We now proceed by induction on k.
Case k=2
Equation (3) for $k = 2$ is simply $X_1 P_1 + X_2 P_2 = 0$ that implies $X_2 P_2 = -X_1 P_1$, which is possible only if the matrix X_2 is a left multiple of P_1, given the fact that (P_1, P_2) is a regular sequence. So there exists a square matrix $F \in \mathcal{R}$ such that $X_2 = F P_1$ and hence we get

$$0 = X_1 P_1 + F P_1 P_2 = X_1 P_1 + F P_2 P_1 = (X_1 + F P_2) P_1 \tag{4}$$

which implies, P_1 being a left regular element, that $X_1 = -F P_2$, i.e. the matrix X is of the form

$$[X_1 \ \ X_2] = F \cdot [-P_2 \ \ P_1] = F \cdot \tilde{B}_1 \tag{5}$$

where $\tilde{B}_1$ is the only block in the matrix B that we have in this case. In particular (5) says that a row of X is a combination of the rows of $\tilde{B}_1$ whose coefficients are a row of F.
Case k>2
In this case we can rewrite equation (3) as

$$X_k P_k = -X_1 P_1 - \cdots - X_{k-1} P_{k-1}. \tag{6}$$

Using the hypotesis of regularity, in particular the fact that P_k is regular in the quotient $\mathcal{R}/(P_1, \dots, P_{k-1})\mathcal{R}$, we know that X_k has to be in the left ideal $(P_1, \dots, P_{k-1})\mathcal{R}$:

$$X_k = F_1 P_1 + \cdots + F_{k-1} P_{k-1}. \tag{7}$$

Plugging this equation into (6) we have

$$F_1P_1P_k + \cdots + F_{k-1}P_{k-1}P_k = -X_1P_1 - \cdots - X_{k-1}P_{k-1} \tag{8}$$

that can be rewritten using the commutativity of the matrices as

$$(F_1P_k + X_1)P_1 + \cdots + (F_{k-1}P_k + X_{k-1})P_{k-1} = 0 \tag{9}$$

which means that the rows of $[(F_1P_k + X_1) \ \ldots \ (F_{k-1}P_k + X_{k-1})]$ are syzygies for $[P_1 \ \ldots \ P_{k-1}]$. The first $k-1$ matrices of P obviously commute and are still a left regular sequence, so we can apply the inductive hypotesis and get their syzygies as combination of the rows of the matrix

$$B' = \begin{bmatrix} B'_1 \\ B'_2 \\ \vdots \\ B'_{k-2} \end{bmatrix}, \quad B'_i = \begin{bmatrix} 0 & \ldots & 0 & -P_{i+1} & P_i & 0 & \ldots & 0 \\ 0 & \ldots & 0 & -P_{i+2} & 0 & P_i & \ldots & 0 \\ & \ldots & & \ldots & & & \ldots & \\ 0 & \ldots & 0 & -P_{k-1} & 0 & 0 & \ldots & P_i \end{bmatrix}$$

where the blocks B'_i have the same rows and columns of the blocks B_i except for the last n rows and the last n columns. Every matrix of the type $[Y_1 \ \ldots \ Y_{k-1}]$ whose rows are syzygies for $P_1, \ldots, P_{k-1}$ is obtained by a suitable combination of the rows of B', in particular there exists a set of square matrices

$$\{F_{ji} \in \mathcal{R} | \ i = 1 \ldots k-1, \ j = i+1 \ldots k-1\}$$

such that we have $Y_1 = \sum_{j=2}^{k-1} F_{j1}P_j$, $Y_2 = -F_{21}P_1 + \sum_{j=3}^{k-1} F_{j2}P_j \ldots$ and in general

$$Y_l = -\sum_{i=1}^{l-1} F_{li}P_i + \sum_{j=l+1}^{k-1} F_{jl}P_j, \quad l = 1 \ldots k-1.$$

Hence, using the coefficients $F_lP_k + X_l$ from (9) as Y_l's and defining $F_{kl} := -F_l$ for $l = 1 \ldots k-1$ such that (7) becomes $X_k = -F_{k1}P_1 - \cdots - F_{kk-1}P_{k-1}$, we have the following equalities:

$$X_l = -\sum_{i=1}^{l-1} F_{li}P_i + \sum_{j=l+1}^{k} F_{jl}P_j, \quad l = 1 \ldots k,$$

which express the matrix X as a $\mathcal{R}$-combination of elements of B, namely

$$[(F_{21} \ \ldots \ F_{k1}) \ (F_{32} \ \ldots \ F_{k2}) \ \ldots \ (F_{k-1k-2} \ \ F_{kk-2}) \ (F_{kk-1})] \begin{bmatrix} B_1 \\ B_2 \\ \ldots \\ B_{k-2} \\ B_{k-1} \end{bmatrix}$$

To conclude the proof, it suffices to note that x is a row of X and so it is itself a combination of rows of B. $\square$

Remark 2.4. Since the matrices P_i commute, and since they are a regular sequence in $\mathcal{R}$, it is obvious (and follows from general theory) that the syzygies of the ideal generated by $P_1, \ldots, P_k$ in $\mathcal{R}$ are given by the matrix B. What our result proves is the much stronger fact that B actually gives the module of syzygies for the module generated in R^n by the rows of $P_1, \ldots, P_k$.

Remark 2.5. Note that our construction works also in the case in which R is any other commutative ring, for example the ring $\mathrm{Exp}_0(\mathbb{C}^n)$ of entire functions of infraexponential type, which is the ring of the symbols of infinite order differential operators (see [9]).

As anticipated before, we can now construct the whole free resolution for the module $\mathcal{M}$ using the hypotesis of regularity.

Theorem 2.6. *Let $P_1, \ldots, P_k \in \mathcal{R}$ be $k > 1$ square matrices such that*
1) $P_iP_j = P_jP_i$ for all $i, j = 1 \ldots k$,
2) $(P_1, \ldots, P_k)$ is a left regular sequence,
and let $\mathcal{M}$ be the R-submodule of R^n generated by the rows of $P_1, \ldots, P_k$. The resolution of $\mathcal{M}$ is formally constructed as the classical Koszul complex:

$$0 \to R^{n \cdot \binom{k}{k}} \to R^{n \cdot \binom{k}{k-1}} \to \ldots \to R^{n \cdot \binom{k}{2}} \to R^{n \cdot \binom{k}{1}} \to 0.$$

If we consider the case of just two matrices P and Q, which is the situation we are generally interested in, it is possible to rewrite conditions *1)* and *2)* of theorem 2.3 in an equivalent more operational way as follows:

Proposition 2.7. *Let R be an integral domain and let us denote with $\mathcal{R} = \mathrm{Mat}_n(R)$ as before. Let $P, Q \in \mathcal{R}$ be square matrices such that $PQ = QP$. Then the following conditions are equivalent:*
a) P and Q form a left regular sequence in $\mathcal{R}$
b) Q is invertible in $\mathrm{Mat}_n(\mathrm{Frac}(R))$ and for every $A \in \mathcal{R}$ such that $AQ^{-1}P \in \mathcal{R}$, we have $AQ^{-1} \in \mathcal{R}$.

Proof. The fact that Q is invertible in $\mathrm{Mat}_n(\mathrm{Frac}(R))$ is equivalent to the request that $\mathrm{Det}(Q) \neq 0$ which is the same as Q being a regular element by Proposition 2.2. The second part of condition b) can be written as

$$AQ^{-1}P = B \implies AQ^{-1} = C$$

for matrices $B, C \in \mathcal{R}$ and so since P and Q^{-1} clearly commute, we have that

$$AP = BQ \implies A = CQ$$

i.e. the regularity of the sequence (P, Q). Conversely, if (P, Q) is left regular, it follows that for every A such that $AP = BQ$ we have $A = CQ$, and so, inverting the matrix Q, the second part of statement b) is true in $\mathrm{Mat}_n(\mathrm{Frac}(R))$. □

Remark 2.8. Note that the ring R of the entries of the matrices P and Q could be any integral domain. If P and Q, as we will consider in several examples of the next section, are the matrices of symbols of a finite order linear differential operator, we can take R as the ring of polynomials over $\mathbb{C}$, but it is also possible to consider infinite order differential operators, so that in this case we have holomorphic functions (in fact infraexponential entire functions) as entries. In that case condition *b)* of the previous proposition 2.7 becomes

$$AQ^{-1}P \text{ with infraexponential entries } \Longrightarrow AQ^{-1}\text{with infraexponential entries.}$$

Remark 2.9. It is worth to mention what can be done in the non commutative situation. What follows was developed by Kawai and Takei [13] to deal with the case of several non commuting matrices. Let us start with two matrices Q_1, Q_2 and let us introduce the commutators

$$R_0 = [Q_1, Q_2], \qquad R_1 = [Q_1, R_0], \qquad R_2 = [Q_2, R_0].$$

It can be directly proved that the system associated to $(Q_1, Q_2, R_0, R_1, R_2)$ is equivalent to the system associated to (Q_1, Q_2). Let us introduce the notation

$$L_j = Q_j, \quad \text{for } j = 1, 2 \qquad L_j = R_{j-3} \quad \text{for } j > 2.$$

We will say that the pair (Q_1, Q_2) is weakly commutative if there exist polynomials c_{jkl} such that for all j, k

$$[L_j, L_k] = \sum_{1 \leq l \leq 5} c_{jkl} L_l. \tag{10}$$

This notion of weak commutativity allows to construct the complex: for example, the first step of the resolution is a 10×5 matrix whose entries are matrices computed using the relation (10). Note that this approach is extremely useful, see [13], though it does not allow to treat systems like the Cauchy–Fueter system or the Dirac system in several variables, in fact in those cases, the procedure does not recover not even the first syzygies of the system. Additional ideas concerning this approach can also be found in [16] and [17].

3 Some applications

In the case in which P_1 and P_2 satisfy the hypothesis of theorem 2.3 or the equivalent reformulation in proposition 2.7 then one immediately has that the syzygies of $[P_1\ P_2]^t$ are given by $[-P_2\ P_1]$. We have the following corollaries of theorem 2.3

Corollary 3.1. *Let P_1, P_2 be two commuting matrices forming a regular sequence in $\mathcal{R} = \mathrm{Mat}_n(R)$. Then the range of the operator $P(D) = [P_1(D)\ P_2(D)]^t$ is given by*

$$\{(g_1, g_2) \in \mathcal{C}^\infty \mid P_2(D)g_1 = P_1(D)g_2\}.$$

Corollary 3.2. *Let P_1, P_2 be two commuting matrices forming a regular sequence in $\mathcal{R} = \mathrm{Mat}_n(R)$. Let $\mathcal{Q} = \ker(P_2)$ and let U be an open convex (or compact convex) set; then the operator*

$$P_1(D)\ : \mathcal{Q}(U) \to \mathcal{Q}(U)$$

is surjective.

More in general:

Corollary 3.3. *Let $P_1, \ldots, P_r$ be commuting matrices forming a regular sequence in $\mathcal{R}$ and let $\mathcal{Q} = \{f \mid P_2(D)f = \ldots = P_r(D)f = 0\}$. Let U be an open convex (or compact convex) set, then the operator*

$$P_1(D): \ \mathcal{Q}(U) \to \mathcal{Q}(U)$$

is surjective.

Remark 3.4. As we pointed out in the introduction, an immediate application of Corollary 3.2 is the surjectivity of a differential operator on the space of holomorphic functions on a convex open set. Our result shows, therefore, an independent proof of this well known theorem.

A large class of examples to which this theory applies can be found in the quaternionic setting. Let us consider a pair of operators $(\partial/\partial\bar{q}, p(\partial/\partial q))$ where $\partial/\partial\bar{q}$ is the Cauchy-Fueter operator while $p(\partial/\partial q)$ is a polynomial in $\partial/\partial q$ with complex coefficients. Considering their Fourier transforms we obtain the pair $(\bar{q}, p(q))$ in the ring $R = \mathbb{C}[q, \bar{q}]$ forming a regular sequence since $\bar{q}$ is not a factor of $p(q)$. We wish to give an alternative proof of this fact using the theory we have developed. Let us consider the 4×4 matrices Q and P representing the Fourier transforms of the operators $\partial/\partial\bar{q}$ and $p(\partial/\partial q)$ respectively. We have the following result:

Proposition 3.5. *The pair (Q, P) form a regular sequence in the ring $\mathcal{R} = \mathrm{Mat}_n(R)$, $R = \mathbb{C}[x_0, \ldots, x_3]$.*

Proof. To prove the statement we will use Proposition 2.7. First of all note that P and Q commute. Let A be a matrix such that $AQ^{-1}P \in \mathcal{R}$; the elements in the matrix Q^{-1} all have denominator equal to $\sum_{j=0}^{3} x_j^2 = q\bar{q}$. The matrix P represents the polynomial $p(q)$ that does not contain the factor $\bar{q}$. So the only possibility to have $AQ^{-1}P$ with polynomial entries is that A represents a quaternionic expression containing the factor $\bar{q}$. It follows that $AQ^{-1} \in \mathcal{R}$ and this concludes the proof. □

Combining the previous discussion with Proposition 3.5, we obtain that

Corollary 3.6. *For any polynomial $p(q)$ with complex coefficients, the operator $p(\partial/\partial q)$ is surjective on the space of regular functions.*

Remark 3.7. Note that an analogous result can be obtained for the pair of operators $(\partial_{\underline{x}}, p(\partial_{\underline{x}}))$ where $\partial_{\underline{x}} = \sum_{j=1}^{m} e_j \partial/\partial x_j$ is the Dirac operator acting on functions f defined in $\mathbb{R}^m$ with values in the Clifford algebra over the m units $e_1, \ldots, e_m$), and $p(\partial_{\underline{x}}) = \sum_{l=0}^{N} a_l \partial_{\underline{x}}^l$ is a polynomial in $\partial_{\underline{x}}$ with complex coefficients. Note in fact that the conjugate of the Dirac operator is $\bar{\partial}_{\underline{x}} = -\partial_{\underline{x}}$ and moreover $\bar{\partial}_{\underline{x}}\partial_{\underline{x}} = \partial_{\underline{x}}\bar{\partial}_{\underline{x}} = \Delta_m$.

The previous approach can be also used to treat another interesting case. Let us introduce the following notations: let $\nu = \{\lambda_1, \ldots, \lambda_m\}$ be a set of integers with $\lambda_1, \ldots, \lambda_m \in \{1, 2, 3\}$ and let n_i be the number of λ_i's. We will denote by σ_m the set of triples $[n_1, n_2, n_3]$ such that $n_1 + n_2 + n_3 = m$. Let us consider the pair $(\partial/\partial\bar{q}, F(D))$ where

$$F(D) = \sum_{m=0}^{N} \sum_{\nu \in \sigma_m} p_\nu(D) a_\nu, \quad a_\nu \in \mathbb{H}$$

and

$$p_\nu(D) = \frac{1}{m!} \sum_{1\le\lambda_1,\dots,\lambda_m\le 3} \left(\frac{\partial}{\partial x_{\lambda_1}} - \frac{\partial}{\partial x_0} i_{\lambda_1}\right) \cdots \left(\frac{\partial}{\partial x_{\lambda_m}} - \frac{\partial}{\partial x_0} i_{\lambda_m}\right),$$

where the sum is taken over the $\frac{n!}{n_1!n_2!n_3!}$ different alignments of n_i elements equal to i, with $i = 1, 2, 3$. As above, let us denote by Q, F the matrices representing the Fourier transform of $\partial/\partial\bar{q}$ and $F(D)$. We have the following result:

Proposition 3.8. *The pair (F, Q) forms a regular sequence in the ring $\mathcal{R} = \mathrm{Mat}_n(R)$, $R = \mathbb{C}[x_0, \dots, x_3]$.*

Proof. The two matrices Q and F commute since the two operators of which they are the Fourier transform commute. Let A be a matrix such that $AQ^{-1}F \in \mathcal{R}$; the elements in the matrix Q^{-1} all have denomintor equal to $\sum_{j=0}^{3} x_j^2 = q\bar{q}$. Note that the matrix F represents a quaternionic polynomial $F(q)$ that is regular (see for example [9]) so it cannot contain $\bar{q}$ as a (left) factor. It follows that the only possibility is that A contains $\bar{q}$ as a factor, so that AQ^{-1} has polynomial entries. □

We know that there is another natural class of differential operators acting on regular functions: those are operators of the form

$$P = \frac{\partial^m}{\partial_{x_1}^{m_1}\partial_{x_2}^{m_2}\partial_{x_3}^{m_3}} I_4, \quad m = \sum_{i=1}^{3} m_i,$$

where I_4 is the 4×4 identity matrix. We have the following:

Corollary 3.9. *Let us consider the system*

$$\begin{cases} Pf = g \\ Qf = 0 \end{cases}$$

where Q is the Cauchy-Fueter system and P is as above. Then the system has a solution on open convex sets if and only if g is regular.

Proof. First, we show that P, Q satysfy the condition *b)* of proposistion 2.7. Let A be a matrix such that $M = AQ^{-1}P$ has polynomial entries. Note that if m_{ij} is the entry of M at the place ij, we have

$$m_{ij} = [AQ^{-1}P]_{ij} = \sum_{k,l} a_{ik} \frac{Q^*_{kl}}{DetQ} p_{lj} = \sum_{k} a_{ik} \frac{Q^*_{kj}}{DetQ} p_{jj} = p \sum_{k} a_{ik} \frac{Q^*_{kj}}{DetQ},$$

where $p = p_{11} = \cdots = p_{44}$. Since p involves only x_i with $i = 1, 2, 3$, it cannot have factors in common with $DetQ = (x_0^2 + x_1^2 + x_2^2 + x_3^2)^2$. Then, in order for m_{ij} to be polynomial it is necessary that $\sum_k a_{ik} \frac{Q^*_{kj}}{DetQ}$ is a polynomial for every i, j, and that is exactly $[AQ^{-1}]_{ij}$. □

We conclude with a problem involving the eigenvalue equation for the Dirac operator, but the same can be shown for both the Cauchy-Fueter and the Moisil-Theodorescu operators. Let us denote by $\mathcal{T}$ any of these three operators. Note that $\mathcal{T} - \lambda$ factorizes the

Helmholtz operators $\Delta - \lambda^2$. We consider the problem: determine a function f satisfying the system

$$\begin{cases} \mathcal{T} f = \lambda f \\ p(\overline{\mathcal{T}}) f = g \end{cases} \tag{11}$$

where g is a given function in $\mathcal{C}^\infty$, $\lambda \in \mathbb{C}$ is an eigenvalue, p is a polynomial with complex coefficients and $\overline{\mathcal{T}}$ denotes the conjugate of $\mathcal{T}$.

Proposition 3.10. *Let U be an open convex set in $\mathbb{R}^m$. Then the problem (11) admits a solution if and only if g is an eigenfunction of $\mathcal{T}$ related to the same eigenvalue.*

Proof. Once again the proof is based on proposition 2.7. We consider the Fourier transform of the two equations in (11) and their representative matrices $Q = T - \lambda I$, (I the identity matrix) and P, which obviously commute. For any matrix A such that $AQ^{-1}P$ has polynomial entries, the matrix AQ^{-1} already has polynomial entries in fact the elements in P cannot have any factor containing the term λ. Then the syzygies of $[Q, P]^t$ are $[-P, Q]$ and the only compatibility condition on the datum g is $Qg = 0$ which implies $\mathcal{T} g = \lambda g$. □

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