Vers

Preliminary draft of September 6, 1988 – not for publication

GRÖBNER BASES OF DETERMINANTAL IDEALS

Bernd Sturmfels
Research Institute for Symbolic Computation
Johannes-Kepler-Universität Linz
A–4040 Linz, AUSTRIA

Abstract. *Using methods from algebraic combinatorics, we prove that the set of $(r+1)\times(r+1)$-minors of a generic $m\times n$-matrix forms a reduced Gröbner basis (for certain term orders). This yields an efficient normal form algorithm and an explicit Stanley decomposition for the coordinate ring of matrices with rank $\le r$.*

1. Introduction

In this paper we study classical determinantal varieties from a computer algebra point of view. We obtain efficient algorithms for computing with polynomial functions on matrices of bounded rank by explicitly describing the reduced Gröbner bases of the corresponding vanishing ideals.

Recently A. Kovacec [Kov] has obtained a characterization for the reduced Gröbner bases of ideals generated by elementary symmetric functions. Both Kovacec's and our results fit into a larger program, suggested by B. Buchberger, which aims to "predict" Gröbner bases for important infinite families of ideals. Whenever such characterizations are possible, one can use many tools from Gröbner bases theory (e.g. normal form algorithms) without ever having to compute a Gröbner basis, which is usually the most time-consuming step. For introductions and references to Gröbner bases theory we refer to [Bu1],[Bu2],[SW1].

Let $K[\mathbf{X}]$ denote the polynomial algebra freely generated by the entries of a generic $m\times n$-matrix $\mathbf{X}=(x_{i,j})$ over a field K of characteristic 0. We view $K[\mathbf{X}]$ as the coordinate ring of the mn-dimensional vector space $K^{m\times n}$ of $m\times n$-matrices with entries from K. We write $\mathcal{I}_r\subset K[\mathbf{X}]$ for the vanishing ideal of the affine subvariety $K_r^{m\times n}$ of matrices of rank at most $r<m,n$. In this paper we prove the following result.

Theorem 1. *The set of $(r+1)\times(r+1)$-minors of $\mathbf{X}$ is the reduced Gröbner basis of $\mathcal{I}_r$ with respect to the lexicographic term order induced from the variable order $x_{1,n} >$*

$x_{1,n-1} > \ldots > x_{1,1} > x_{2,n} > x_{2,n-1} > \ldots > x_{2,1} > \ldots\ldots > x_{m,n} > x_{m,n-1} > \ldots > x_{m,1}.$

The proof of Theorem 1 will be given in Section 3. It is based on two well-known results from algebraic combinatorics, namely the Doubilet-Rota-Stein straightening algorithm and the Knuth-Robinson-Schensted correspondence for Young bitableaux. With an exposition of these techniques in Section 2 we express our belief that many interesting connections between combinatorics and computational algebraic geometry are yet to be explored.

In Section 4 we discuss some computational implications of Theorem 1. Section 5 contains a more combinatorial application. Using a theorem of A. Björner on the shellability of higher-order complexes [Bjö], we derive an explicit Stanley decomposition [SW2] of the quotient ring $K[\mathbf{X}]/\mathcal{I}_r$. This generalies a result of Billera, Cushman and Sanders [BCS] who settled the special case $r = 1$.

2. Tools from combinatorics and invariant theory

We use the abbreviation

$$[l_1 l_2 \ldots l_s | p_1 p_2 \ldots p_s] \quad := \quad \det \begin{pmatrix} x_{l_1,p_1} & x_{l_1,p_2} & \cdots & x_{l_1,p_s} \\ x_{l_2,p_1} & x_{l_2,p_2} & \cdots & x_{l_2,p_s} \\ \vdots & \vdots & \ddots & \vdots \\ x_{l_s,p_1} & x_{l_s,p_2} & \cdots & x_{l_s,p_s} \end{pmatrix} \tag{1}$$

for the minors of the generic $m \times n$-matrix $\mathbf{X}$. A product of minors $B :=$

$$[l_{11} l_{12} \ldots l_{1s_1} | p_{11} p_{12} \ldots p_{1s_1}] [l_{21} \ldots l_{2s_2} | p_{21} \ldots p_{2s_2}] \ldots [l_{\nu 1} \ldots l_{\nu s_\nu} | p_{\nu 1} \ldots p_{\nu s_\nu}] \tag{2}$$

will be called a *(Young) bitableaux* provided $s_1 \geq s_2 \geq \ldots \geq s_\nu$, and $l_{i,j} < l_{i,j+1}$ and $p_{i,j} < p_{i,j+1}$ for all i, j. (These expressions correspond to the bitableaux in the *letter place algebra* of [DKR]: "l" stands for "letters" and "p" stands for "places".) The integer s_1 will be called the *length* of the bitableau B. The bitableau B is said to be *standard* if $l_{i,j} \leq l_{i,j+1}$ and $p_{i,j} \leq p_{i+1,j}$ for all i, j.

There is a natural $\mathbf{N}^{n+m}$-grading on the ring $K[\mathbf{X}]$ which is defined as follows. The *degree* of a monomial $\mathbf{m} = \Pi_{i,j} x_{ij}^{a_{ij}}$ is the integer vector

$$deg(\mathbf{m}) \quad := \quad (\, \Sigma_j a_{1j},\, \Sigma_j a_{2j},\, \ldots,\, \Sigma_j a_{mj}\,;\, \Sigma_i a_{i1},\, \Sigma_i a_{i2},\, \ldots\,,\, \Sigma_i a_{in}\,).$$

In other words, $deg(\mathbf{m})$ is the vector of row and column sums of the $m \times n$-exponent matrix of the monomial $\mathbf{m}$. Every bitableau $B \in K[\mathbf{X}]$ is homogeneous of (multi-)degree $deg(B)$. In the combinatorics literature the integer vector $deg(B)$, viewed as a pair of multisets, is usually called the *content* of the bitableau B.

We now discuss a certain bijection between the monomials and the bitableaux of fixed degree. Such a one-to-one correspondence follows from a combinatorial algorithm of D.E. Knuth [Kn2] which generalizes earlier constructions due to G.B. Robinson and C. Schensted [Sch]. We refer to R. Stanley [Sta] for a discussion of the Knuth-Robinson-Schensted correspondence and its application in the theory of symmetric functions. Applications to sorting algorithms are given in [Kn1, Section 5.14].

The basic idea is the following. Every monomial $\mathbf{m} = \Pi_{i=1}^{d} x_{l_i,p_i}$ of total degree d corresponds uniquely to a "generalized permutation"

$$\begin{matrix} l_1 & l_2 & l_3 & \dots & l_d \\ p_1 & p_2 & p_3 & \dots & p_d \end{matrix} \tag{3}$$

where $l_1 \leq l_2 \leq l_3 \leq \ldots \leq l_d$, and if $l_i = l_{i+1}$ then $p_i \leq p_{i+1}$. The length μ of the longest strictly decreasing subsequence $p_{i_1} > p_{i_2} > \ldots > p_{i_\mu}$ in the sequence $p_1, p_2, p_3, \ldots, p_d$ is called the *width* of the monomial $\mathbf{m}$. Observe that $width(\mathbf{m}) = 1$ if and only if $\mathbf{m} = \Pi_j [l_j | p_j]$ is a standard bitableau.

Proposition 2. [Knuth-Robinson-Schensted] *There exists a bijection, denoted $\kappa : \mathbf{m} \mapsto B$, between all monomials and all standard bitableaux in $K[\mathbf{X}]$ such that*
(i) $deg(\mathbf{m}) = deg(B)$, and
(ii) $width(\mathbf{m}) = length(B)$.

As the second tool for our proof of Theorem 1 we will need the *straightening law* for bitableaux given by Doubilet, Rota and Stein [DRS]. For additional details we refer to Désarménien, Kung and Rota [DKR] and DeConcini, Eisenbud and Procesi [DEP]. In Section 3 we will discuss the straightening formula from a complexity point of view. Here we only need some of its theoretical properties.

Proposition 3. [Straightening law] *Every bitableau $B \in K[\mathbf{X}]$ can be written as a K-linear combination $B = \sum_{i=1}^{N} \lambda_i B_i$ of standard bitableaux B_i such that $deg(B) = deg(B_i)$ and $length(B) \leq length(B_i)$ for $i = 1, 2, \ldots, N$.*

The Knuth-Robinson-Schensted correspondence, together with an easy counting argument, implies that the representation in Proposition 3 is necessarily unique. This constitutes an alternative proof for the second part of the standard basis theorem which avoids both the use of the Capelli operator (as in [DKR]) and Hodge's straightening law for the Grassmann variety (as in [DEP]). Our counting argument will yield the following stronger version of the basis theorem.

Proposition 4. [Standard basis theorem] *Every monomial* $\mathbf{m} \in K[\mathbf{X}]$ *has a unique representation* $\mathbf{m} = \sum_{i=1}^{n} \lambda_i B_i$ *as a* K*-linear combination of standard bitableaux. In this representation we have* $length(B_i) \leq width(\mathbf{m})$ *for all* $i = 1, 2, \ldots, N$.

In order to prove Proposition 4, we need to introduce some notations. Fix $d \in \mathbf{N}^{n+m}$, and consider the graded component $K[\mathbf{X}]_d$ as the free K-vector space generated by the monomials of degree d. We can write $K[\mathbf{X}]_d = \bigoplus_{w=1}^{m} K[\mathbf{X}]_{d,w}$ where $K[\mathbf{X}]_{d,w}$ denotes the subspace generated by the monomials of degree d and width w.

Let S_d denote the free K-vector space generated by all standard bitableau of degree d, and write $S_d = \bigoplus_{w=1}^{m} S_{d,w}$ where $S_{d,w}$ denotes the subspace by all standard bitableau of degree d and length w. All these K-vector spaces are finite-dimensional, and Proposition 2 implies that

$$dim_K(K[\mathbf{X}]_{d,w}) \quad = \quad dim_K(S_{d,w}) \tag{4}$$

Expanding bitableaux as K-linear combinations of monomials of the same degree defines a linear map

$$expand \;:\; S_d \quad \longrightarrow \quad K[\mathbf{X}_d]. \tag{5}$$

Proposition 3 states that every monomial $\mathbf{m}$ of degree d is a linear combination of standard bitableaux with degree d. In other words, the expansion map (5) is surjective.

If follows from (4) (i.e. from the Knuth-Robinson-Schensted correspondence) that the expansion map (5) is a K-vector space isomorphism because every epimorphism between vector spaces of the same finite dimension is automatically injective. The (unique) inverse of (5) is the straightening map

$$straight \;:\; K[\mathbf{X}_d] \quad \longrightarrow \quad S_d. \tag{6}$$

Hence the standard tableau representation in Proposition 3 is automatically unique.

Proof of Proposition 4. Fix $d \in \mathbf{N}^{n+m}$ and $L \in \{1, 2, \ldots, m\}$. Restrict the expansion map (4) to the subspace $\bigoplus_{w=1}^{L} S_{d,w}$ generated by the standard bitableaux of degree d and length at most L. It is easy to see that every such bitableau expands into monomials of width at most L, and hence we obtain an induced map

$$expand \;:\; \bigoplus_{w=1}^{L} S_{d,w} \quad \longrightarrow \quad \bigoplus_{w=1}^{L} K[\mathbf{X}]_{d,w} \qquad (7).$$

Since the expansion map (5) is injective, also the induced map (7) is injective. Again by (4), the vector spaces on both sides of (7) have the same finite K-dimension, and therefore also the induced map (7) is both injective and surjective. Consequently the straightening map (6) satisfies $straight(K[\mathbf{X}]_{d,L}) \subset \bigoplus_{w=1}^{L} S_{d,w}$. This proves Proposition 4. $\triangle$

3. Proof of the main result

We are now prepared to prove Theorem 1. Let "$\prec$" denote the term order on $K[\mathbf{X}]$ defined in the statement of Theorem 1. Consider an $(r+1)\times(r+1)$-minor $D := [l_1 l_2 \ldots l_{r+1} | p_1 p_2 \ldots p_{r+1}]$ where $l_1 < l_2 < \ldots < l_{r+1}$ and $p_1 < p_2 < \ldots < p_{r+1}$.

Lemma 5. *The leading term of D with respect to the term order "$\prec$" is given by*

$$init(D) \quad = \quad x_{l_1,p_{r+1}}\, x_{l_2,p_r}\, x_{l_3,p_{r-1}} \;\cdots\; x_{l_{r+1},p_1}.$$

Proof. Consider the Laplace expansion of the determinant D with respect to the first row. Clearly, $x_{l_1,p_{r+1}}$ is the most "expansive" variable occuring in D. By induction on r, we may assume that Lemma 5 holds the cofactor of x_{l_1,p_n}. Thus we have

$$init\big([l_2 \ldots l_{r+1} | p_1 \ldots p_r]\big) \quad = \quad x_{l_2,p_r}\, x_{l_3,p_{r-1}} \;\cdots\; x_{l_{r+1},p_1}.$$

This implies the claim. $\triangle$

Consider the set

$$\mathcal{G}_{r+1} \;:=\; \big\{ [l_1 l_2 \ldots l_{r+1} | p_1 p_2 \ldots p_{r+1}] \;:\; l_1 < l_2 < \ldots < l_{r+1} \text{ and } p_1 < p_2 < \ldots < p_{r+1} \big\}$$

of $(r+1)\times(r+1)$-minors of $\mathbf{X}$, and let $init(\mathcal{G}_{r+1})$ denote the mononial ideal generated by the initial terms of $\mathcal{G}_{r+1}$. From Lemma 5 we obtain

Lemma 6. *A monomial $\mathbf{m} \in K[\mathbf{X}]$ is in $init(\mathcal{G}_{r+1})$ if and only if $width(\mathbf{m}) \geq r+1$.*

Proof of Theorem 1. Since $\mathcal{G}_{r+1}$ generates the ideal $\mathcal{I}_r$, it suffices to show that every non-zero polynomial in $\mathcal{I}_r$ is reducible modulo the set $\mathcal{G}_{r+1}$ with respect to the term order "$\prec$".

Suppose that there exists an $f \in \mathcal{I}_r \setminus \{0\}$ which is is irreducible modulo $\mathcal{G}_{r+1}$, and let $f = \sum_i \lambda_i\, \mathbf{m}_i$ with $\lambda_i \neq 0$ be its monomial expansion. Since f is irreducible, Lemma 6 implies that $width(\mathbf{m}_i) \leq r$. Using the straightening algorithm (Proposition 4) we can write $\mathbf{m}_i = \sum_j \mu_{ij} B_{ij}$ where the B_{ij} are standard bitableaux with $length(B_{ij}) \leq width(\mathbf{m}_i)$. Hence the (unique) expansion

$$f \quad = \quad \sum_{i,j} \lambda_i\, \mu_{ij}\, B_{ij} \tag{8}$$

of f into standard bitableaux satisfies $length(B_{ij}) \leq r$.

On the other hand, f is contained in the ideal $\mathcal{I}_{r+1}$ which is generated by the bitableaux of length $\geq r+1$. Write

$$f \quad = \quad \sum_i \alpha_i C_i \tag{9}$$

where $\alpha_i \in K$ and the C_i are (possibly non-standard) bitableaux with $length(C_i) \geq r+1$. Applying the straightening algorithm (Proposition 3), we obtain $C_i = \sum_j \beta_{ij} D_{ij}$ where the D_{ij} are standard bitableaux with $length(D_{ij}) \geq length(C_i) \geq r+1$. From (9) we get

$$f \quad = \quad \sum_{i,j} \alpha_i \beta_{ij} D_{ij}. \tag{10}$$

In (8) we have written f as a K-linear combination of standard bitableaux of length $\leq r$, and in (10) we have written f as a K-linear combination of standard bitableaux of length $\geq r+1$. Since f was assumed to be non-zero, this is a contradiction to the standard basis theorem (Proposition 4). This proves Theorem 1. $\triangle$

Remark 7. The set $\mathcal{G}_{r+1}$ is the reduced Gröbner basis of $\mathcal{I}_r$ also with respect to the lexicographic extension of any variable order which respects the rows and columns of $\mathbf{X}$. This follows easily by symmetry arguments.

In particular, Theorem 1 remains valid if we replace the order "$\prec$" by the lexicographic term order induced from the (perhaps more natural) variable order $x_{1,1} > x_{1,2} > \ldots > x_{1,n} > x_{2,1} > x_{2,2} > \ldots > x_{2,n} > \ldots\ldots > x_{m,1} > x_{m,2} > \ldots > x_{m,n}$. For the purpose of this paper we prefer the term order "$\prec$" because it fits easier into the combinatorial scheme of Section 2.

We close with the observation that $\mathcal{G}_{r+1}$ is, in general, not a Gröbner basis for lexicographic term orders which do not respect the rows and columns of the matrix $\mathbf{X}$. For example, if $n = m = 3$ and $r = 1$ then the reduced Gröbner basis of $\mathcal{I}_1$ for the lexicographic term order induced from $x_{11} > x_{22} > x_{33} > x_{23} > x_{12} > x_{21} > x_{13} > x_{32} > x_{31}$ equals

$$\begin{aligned}\{ &\underline{x_{11}x_{22}} - x_{12}x_{21},\ \underline{x_{11}x_{33}} - x_{13}x_{31},\ \underline{x_{11}x_{23}} - x_{13}x_{21},\ \underline{x_{11}x_{32}} - x_{12}x_{31},\\ &\underline{x_{22}x_{33}} - x_{23}x_{32},\ \underline{x_{13}x_{22}} - x_{12}x_{23},\ \underline{x_{22}x_{31}} - x_{21}x_{32},\ \underline{x_{12}x_{33}} - x_{13}x_{32},\\ &\underline{x_{21}x_{33}} - x_{23}x_{31},\ \underline{x_{12}x_{23}x_{31}} - x_{21}x_{13}x_{32}\}.\end{aligned}$$

The underlined leading terms show that the tenth polynomial is irreducible with respect to the nine others. Hence $\mathcal{G}_2$ is not a Gröbner basis for this term order.

4. Computational applications

In this section we discuss some computational applications of Theorem 1. We write $\mathcal{R}_r := K[\mathbf{X}]/\mathcal{I}_r$ for the ring of polynomial functions on $m \times n$-matrices of rank r or less. By standard abuse of notation, monomials in $K[\mathbf{X}]$ are identified with their images in the quotient ring $\mathcal{R}_r$.

Since $\mathcal{G}_{r+1}$ is a Gröbner basis of $\mathcal{R}_r$, the set of monomials not contained in $init(\mathcal{G}_{r+1})$ forms a K-vector space basis for the residue ring $\mathcal{R}_r$ [Bu1, Lemma 6.7]. From Lemma 6 we obtain

Corollary 8. *Every element f in $\mathcal{R}_r$ can be written uniquely as a K-linear combination $NF_r(f) = \sum_i \lambda_i \mathbf{m}_i$ of monomials $\mathbf{m}_i$ with $width(\mathbf{m}_i) \leq r$.*

This normal form $NF_r(f)$ can be computed easily with the normal form subroutines which are available in many Gröbner basis implementations (e.g. in MAPLE).

Example 9. We consider the case $m = n = 5$, that is, we are working in the polynomial ring $K[\mathbf{X}]$ with 25 variables x_{ij}. Let $r = 2$. Then the Gröbner basis $\mathcal{G}_3$ of $\mathcal{I}_2$ consists of all 100 3×3-subdeterminants of $\mathbf{X}$. Given a polynomial function $f \in \mathcal{R}_2$ on the subvariety of matrices with rank ≤ 2, then $NF(f)$ is its unique expansion in terms of monomials of width ≤ 2.

(a) For $f := x_{13}x_{22}x_{31} - x_{12}x_{23}x_{31}$ we have

$$NF(f) \quad = \quad x_{11}x_{22}x_{33} + x_{13}x_{21}x_{32} - x_{12}x_{21}x_{33} - x_{11}x_{23}x_{32}.$$

(b) For $g := x_{15}x_{24}x_{33}x_{42}x_{51}$ we have $NF(g) =$

$$\begin{aligned}
&3\,x_{11}x_{22}x_{33}x_{44}x_{55} - 2\,x_{11}x_{22}x_{33}x_{45}x_{54} - 3\,x_{11}x_{22}x_{34}x_{43}x_{55} + x_{11}x_{22}x_{34}x_{45}x_{53}\\
&+ x_{11}x_{22}x_{35}x_{43}x_{54} - 3\,x_{11}x_{23}x_{32}x_{44}x_{55} + 2\,x_{11}x_{23}x_{32}x_{45}x_{54} + 2\,x_{11}x_{23}x_{34}x_{42}x_{55}\\
&- x_{11}x_{23}x_{34}x_{45}x_{52} + 2\,x_{11}x_{24}x_{32}x_{43}x_{55} - x_{11}x_{24}x_{35}x_{42}x_{53} - x_{11}x_{25}x_{32}x_{43}x_{54}\\
&- 2\,x_{12}x_{21}x_{33}x_{44}x_{55} + 2\,x_{12}x_{21}x_{33}x_{45}x_{54} + 2\,x_{12}x_{21}x_{34}x_{43}x_{55} - x_{12}x_{21}x_{34}x_{45}x_{53}\\
&- x_{12}x_{21}x_{35}x_{43}x_{54} + x_{12}x_{23}x_{31}x_{44}x_{55} - x_{12}x_{23}x_{31}x_{45}x_{54} - x_{12}x_{23}x_{34}x_{41}x_{55}\\
&+ x_{12}x_{23}x_{34}x_{45}x_{51} + x_{13}x_{21}x_{32}x_{44}x_{55} - x_{13}x_{21}x_{32}x_{45}x_{54} - x_{13}x_{24}x_{31}x_{42}x_{55}\\
&+ x_{13}x_{24}x_{35}x_{41}x_{52} - x_{14}x_{21}x_{32}x_{43}x_{55} + x_{14}x_{25}x_{31}x_{42}x_{53} + x_{15}x_{21}x_{32}x_{43}x_{54}. \qquad \triangle
\end{aligned}$$

This normal form reduction can be used to decide whether a given polynomial $f \in K[\mathbf{X}]$ is zero in the ring $\mathcal{R}_r$ because $NF(f) = 0$ if and only if $f \in \mathcal{I}_r$. It follows from Proposition 3 that also straightening algorithm for bitableaux could answer this question, at least in principle. Indeed, a polynomial f is contained in $\mathcal{I}_r$ if and only if $length(B_i) \geq r+1$ for all standard bitableaux B_i in the expansion $f = \sum_i \lambda_i B_i$.

Comparing these two methods for verifying identities in $\mathcal{R}_r$, we find that from a practical point of view there are many good arguments against the straightening algorithm.

(a) The straightening algorithm for an $n \times m$-matrix $\mathbf{X}$ requires to work in a polynomial algebra with as many as $\binom{n+m}{n} - 1$ variables.

(b) The relations or ***straightening syzygies*** for computing with these "compound" variables are very difficult to describe (see [DKR, Section 3] and Eisenbud [Eis, Introduction]).

(c) Implementating tableaux algorithms requires data structures and subroutines which are usually not supported by computer algebra systems.

(d) Practical experiences show that straightening has a very high space complexity and time complexity (see N. White [Whi]).

In contrast to the straightening algorithm, the above normal form computation is very practical and conceptually simple.

(a') It requires only the mn original variables. No additional variables are necessary.

(b') The syzygies for computing in $\mathcal{R}_r$ are trivial: they are just the (expanded) minors themselves.

(c') The Gröbner basis normal form algorithm can be implemented using requires only polynomial arithmetic and determinant expansions. In the enviroment of a computer algebra system (such as MAPLE) no extra programming is necessary.

(d') The normal form reduction is much faster than straightening. Example 1 (b) has been computed using MAPLE on a MICROVAX II in 5 CPU seconds. If we wanted to do the same reduction with the (projective version of the) straightening algorithm, we need to transform Example 1 (b) into a rank 5 bracket polynomial in 10 letters of degree 5. This is a problem in $\binom{10}{5} = 252$ variables and hence almost too big for the currently fastest straightening software, namely N. White's implementations in FORTRAN [Wh1, Table 1].

Let us point out that there is a certain disadvantage to using a (general purpose) normal form subroutine in a computer algebra system because one does not take advantage of the special features of our determinantal ideals. Moreover, as the parameters m and n increase, the cardinality $\binom{n}{r}\binom{m}{r}$ of the Gröbner basis $\mathcal{G}_{r+1}$ grows very fast. Administrating the large set $\mathcal{G}_{r+1}$ then costs a lot of time and slows down the normal form reduction.

With relatively little extra programming effort we can substantially improve this situation: The combinatorial tools developed above allow an easy implicit representation of the set $\mathcal{G}_{r+1}$. We summarize the resulting reduction procedure.

Algorithm 10. *The following algorithm rewrites any element f in $\mathcal{R}_r$ as a unique K-linear combination of monomials with $width(\mathbf{m}_i) \leq r$.*

---

1. Given: An element $f = \sum_i \lambda_i \mathbf{m}_i$ of $\mathcal{R}_r$, expanded into monomials.
2. Pick $\mathbf{m}_i$ such that $width(\mathbf{m}_i) \geq r+1$. If there is no such $\mathbf{m}_i$, then f is already in normal form and we can STOP.
3. Otherwise, write

$$\mathbf{m}_i \quad = \quad \widetilde{\mathbf{m}_i} \cdot x_{l_1,p_{r+1}} x_{l_2,p_r} \; \cdots \; x_{l_{r+1},p_1}$$

   where $\widetilde{\mathbf{m}_i}$ is a monomial, and $l_1 < l_2 < \ldots < l_{r+1}$, and $p_1 < p_2 < \ldots < p_{r+1}$.
4. Replace f by

$$f \; - \; \sum_{\sigma} sign(\sigma) \; \widetilde{\mathbf{m}_i} \; x_{l_1,p_{\sigma(r+1)}} x_{l_2,p_{\sigma(r)}} \; \cdots \; x_{l_{r+1},p_{\sigma(1)}}$$

   where the sum ranges over all permutations σ of $\{1,\ldots,r+1\}$. GO TO 1.

The most difficult parts of Algorithm 10 are, of course, the computation of $width(\mathbf{m}_i)$ in step 2 and the subsequent factorization of $\mathbf{m}_i$ in step 3. This computation amounts to finding and isolating the longest decreasing subsequence in the monomial representation (3). This is done precisely with the well-known *tableau insertion algorithm* which realizes Knuth's bijection κ in Proposition 2. See Schensted [Sch] and D.E. Knuth [Kn1, Section 5.14] for details.

5. Stanley decompositions from shellings

TO BE INSERTED

Acknowledments. I am indebted to A. Björner and G. Ziegler for introducing me to the Knuth-Robinson-Schensted correspondence. Special thanks go to T. Havel and A. Kovacec for many inspiring discussions related to this paper.

References

[BCS] L.J. BILLERA, R. CUSHMAN, and J.A. SANDERS: The Stanley decomposition of the harmonic oscillator, Manuscript, March 1988.

[Bjö] A. BJÖRNER: Shellable and Cohen-Macaulay partially ordered sets, *Transactions of the Amer. Math. Soc.* **260** (1980) 159–183.

[Bu1] B. BUCHBERGER: Gröbner bases - an algorithmic method in polynomial ideal theory, Chapter 6 in N.K. Bose (ed.): "Multidimensional Systems Theory", D. Reidel, Doordrecht, 1985.

[Bu2] B. BUCHBERGER: Applications of Gröbner bases in non-linear computational geometry, in J.R. Rice (ed.): "Scientific Software", I.M.A. Volumes in Mathematics and its Applications, # 14, Springer, New York, 1988

[DEP] C. DE CONCINI, D. EISENBUD, and C. PROCESI: Young diagrams and determinantal varieties, Inventiones math. **56** (1980) 129–165.

[DEK] J. DÉSARMÉNIEN, J.P.S. KUNG, and G.C. ROTA: Invariant theory, Young bitableaux, and combinatorics, *Advances in Mathematics* **27** (1978) 63–92.

[DRS] P. DOUBILET, G.C. ROTA, and J. STEIN: On the foundation of combinatorial theory: IX. Combinatorial methods in invariant theory, *Studied in Applied Math.* **53** (1974) 185–216.

[Eis] D. EISENBUD: Introduction to algebras with straightening laws, in B.R. McDonald (ed.): "Ring Theory and Algebra", Proc. of the 3rd Oklahoma conf., New York, 1980.

[Kn1] D.E. KNUTH: "The Art of Computer Programming", Vol. 3: "Sorting and Searching", Addison-Wesley, Reading, Massachusetts, 1973.

[Kn2] D.E. KNUTH: Permutations, matrices, and generalized Young tableaux, *Pacific J. Math.* **34** (1970) 709–727.

[Kov] A. KOVACEC: Gröbner bases of ideals generated by elementary symmetric functions, presented at *"Computing and Algebraic Geometry"*, Marseille, June 1988.

[Sch] C. SCHENSTED: Longest increasing and decreasing subsequences, *Canadian J. Math.* **13** (1961) 179–191.

[Sta] R.P. STANLEY: Theory and application of plane partitions: Part 1, *Studies in Applied Math.* **50** (1971) 167–188.

[SW1] B. STURMFELS and N. WHITE: Gröbner bases and invariant theory, *Advances in Mathematics*, to appear.

[SW2] B. STURMFELS and N. WHITE: Computing combinatorial decompositions of rings, Manuscript, 1988.

[Whi] N. WHITE: Implementations of the straightening algorithm, Manuscript, 1988.