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Properties of Entire Functions Over Polynomial Rings via Gröbner Bases

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Abstract. In this paper it is shown that the extension ideals of polynomial prime and primary ideals in the corresponding ring of entire functions remain prime or primary, respectively. Moreover, we will prove that a primary decomposition of a polynomial ideal can be extended componentwise to a primary decomposition of the extended ideal. In order to show this we first prove the flatness of the ring of entire functions over the corresponding polynomial ring by use of Gröbner basis techniques. As an application we give an elementary proof of a generalization of Hilbert's Nullstellensatz for entire functions (cf. [10]).

Keywords: Flat module, Gröbner basis, Entire function, Hilbert's Nullstellensatz.

1 Introduction

In this paper we will investigate some properties of ring extensions from polynomial rings to rings of entire functions. Our considerations are based mainly on elementary algebraic arguments, in particular on the theory of Gröbner bases in rings of entire functions.

In the same way as the decomposition of polynomial ideals into primary components has applications in commutative algebra and algebraic geometry decompositions of ideals in rings of entire functions are of great interest in complex analysis and analytic geometry. We will give a partial solution of the latter problem by providing effective methods for the computation of primary decompositions of ideals generated by polynomials in a ring of entire functions.

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In fact, it will turn out that the problem reduces to the decomposition of the ideal restricted to the underlying polynomial ring. Moreover, we will show that the restriction ideal is generated by the same polynomials in this situation.

We introduce some notions and notations which will be used throughout this paper. By "ring" we always mean "commutative ring with unit". For an arbitrary integral domain A we denote by $Q(A)$ its quotient field. The basic algebraic structures involved in this paper are the polynomial ring $R := \mathbb{K}[X_1, \ldots, X_n]$, $X_1, \ldots, X_n$ indeterminates, the ring $S := \mathbb{K}[[X_1, \ldots, X_n]]$ of formal power series, and the ring $E := \{f \in S | f \text{ is convergent in } \mathbb{K}^n\}$. Since we are interested in convergence, we restrict ourself to the fields of complex ($\mathbb{K} = \mathbb{C}$) or real ($\mathbb{K} = \mathbb{R}$) numbers. Clearly, there are the inclusions $R \subset E \subset S$. In this paper convergence of power series always means convergence at the entire space $\mathbb{K}^n$. The set $T := T(X_1, \ldots, X_n)$ of all terms $X^\alpha := X_1^{\alpha_1} \cdots X_n^{\alpha_n}$ corresponding to $\alpha \in \mathbb{N}^n$ forms a $\mathbb{K}$-vector space basis of R.

For $f = \sum_{\alpha \in \mathbb{N}^n} c_\alpha X^\alpha \in S$ we define the *support* of f by $\operatorname{supp} f := \{X^\alpha \mid c_\alpha \neq 0\}$. The elements of R are just these of finite support. For the use of Gröbner techniques it is necessary to order the monomials X^α in such a way that the multiplication is (strongly) monotone with respect to the order. Such orders are called *admissible*. In what follows the symbol $\prec$ will always denote an admissible term order on T. An admissible term order $\prec$ is called an *elimination order* for the nonempty proper subset $Z \subset X$ if for any two terms $t \in T(Y)$ and $s \in T(X) \setminus T(Y)$, where $Y := X \setminus Z$, it holds $t \prec s$.

We can define a function $lt : R \to R$ assigning to each non-zero polynomial $f \in R$ the largest term with respect to $\prec$ in $\operatorname{supp} f$. The term $lt(f)$ will be called the *leading term* of f (with respect to $\prec$). Each polynomial ideal $I \subseteq R$ can be associated the so-called *leading term ideal* $lt(I) \subseteq R$ which is defined as the ideal generated by all leading terms of non-zero elements of I. A subset $F \subset I$ of non-zero polynomials belonging to I is called a *Gröbner basis* of I (with respect to $\prec$) if the leading terms of the elements of F generate $lt(I)$. An elimination order $\prec$ for a nonempty proper subset $Z \subset X$ has the following nice property. If G is a Gröbner basis with respect to $\prec$ of the ideal $I \subseteq \mathbb{K}[X]$ then the set $G \cap \mathbb{K}[Y]$ is a Gröbner basis of the elimination ideal $I \cap \mathbb{K}[Y]$ with respect to the restriction of $\prec$ to the set $T(Y)$ of terms in Y, where as above Y denotes the set difference $X \setminus Z$. Finally, let $\mathfrak{D}_I := T \setminus lt(I)$ denote the set of *standard terms* modulo I. The residue classes of the elements of $\mathfrak{D}_I$ modulo I form a $\mathbb{K}$-vector space basis of the quotient ring R/I. For a comprehensive introduction to the theory of Gröbner basis we refer to the survey article [4] or one of the text books [3], [5] or [6].

The theory of Gröbner bases can be generalized to ideals generated by polynomials in the ring E of entire functions. Each entire function $g \in E$ can be uniquely represented as an infinite sum $g = \sum_{u \in T} c_u u$, $c_u \in \mathbb{K}$. In analogy to the polynomial case g is called *irreducible* modulo IE (with respect to $\prec$) if $c_u = 0$ for all $u \in lt(I)$. The central result is a division theorem [1, Theorem 3.7] which can be written in the following form:

Theorem 1 *Let $I \subseteq R$ be an ideal, F a Gröbner basis of I with respect to an arbitrary term order $\prec$ and $g \in E$ an entire function. Then there exists a uniquely determined entire function $g_{IE,red} \in E$ which is irreducible modulo IE with respect to $\prec$ and satisfies $g - g_{IE,red} \in IE$. Moreover, there exist entire functions $h_f \in E$ such that*

$$g = \sum_{f \in F} h_f f + g_{IE,red}$$

and all terms $u \cdot lt(f)$, where $f \in F$ and $u \in \mathrm{supp} h_f$, are pairwise distinct.

Note, if g is a polynomial then also the cofactors h_f and the remainder $g_{IE,red}$ are polynomials and we have a classical Gröbner division formula satisfying $h_f = 0$ or $lt(h_f f) \preceq lt(g)$ for all $f \in F$. The operator assigning to each element $g \in E$ its remainder $g_{IE,red}$ modulo IE (with respect to $\prec$) is linear and continuous and we have $g \in IE$ if and only if $g_{IE,red} = 0$. Hence, any ideal of E generated by polynomials is closed, for details see [1].

Corollary 1 *For any ideal I of R we have*

$$(IE) \cap R = I$$

Proof. The inclusion $I \subseteq IE \cap R$ is obvious. Now, consider $g \in IE \cap R$ and choose a Gröbner basis B of I (with respect to an arbitrary term order $\prec$). Then $g_{IE,red} = 0$ (considered as an entire function) and therefore g reduces to 0 modulo B (considered as a polynomial), i.e. $g \in I$. □

Remark 1. We note that

1. R is integrally closed in E. In the complex case this can be immediately deduced from Liouville's Theorem. Real case follows as well as in 2. from complex case by complexification.
2. E is integrally closed (in its field of quotients $Q(E)$). This follows from Riemann's Theorem on extension of holomorphic functions.
3. As a consequence of 1. and 2. we see that R is integrally closed in $Q(E)$.
4. From 3. it follows that $Q(R)$ is algebraically closed in $Q(E)$.

The paper is organized as follows. First of all, we show that E is a faithfully flat extension of R using Gröbner basis techniques in Sections 2 and 3. Moreover, in Section 3 it is proved that prime (resp. primary) ideals of R generate prime (resp. primary) ideals in E and that an irredundant primary decomposition of a polynomial ideal $I \subset R$ extends componentwise to an irredundant primary decomposition of the extension ideal IE. Finally, these results are applied in order to give an elementary proof of a generalization of Hilbert's Nullstellensatz for entire functions in Section 4.

2 Flatness of E via Gröbner Bases

Let $R' := R[Y]$, where Y is an additional indeterminate, and let E' be the ring of entire functions on $\mathbb{K}^{n+1}$.

Lemma 2 *Let I' be an ideal of R' and set $I := I' \cap R$. Then $\left(I'E'\right) \cap E = IE$.*

Proof. Let $\prec$ be an elimination order for $\{Y\}$. Consider an arbitrary $f \in I'E' \cap E$. $f_{IE,red}$ is irreducible modulo $I'E'$ since any polynomial $g \in I'$ such that $lt(g) \in T(X_1, \ldots, X_n)$ belongs to I by the elimination property of $\prec$. Hence, we have $f_{IE,red} = \left(f_{IE,red}\right)_{I'E',red} = f_{I'E',red} = 0$ and it follows $f \in IE$ which implies $\left(I'E'\right) \cap E \subseteq IE$. The converse inclusion is obviously true. □

The next lemma is a straightforward extension of the well known formula

$$I \cap J = (YIA[Y] + (1-Y)JA[Y]) \cap A \tag{1}$$

allowing the computation of the intersection of ideals I, J of an arbitrary ring A by elimination of a new indeterminate Y from a certain ideal sum[1].

Lemma 3 *Let $\tilde{I}$, $\tilde{J}$ be ideals of E. Then*

$$\tilde{I} \cap \tilde{J} = \left(Y\tilde{I}E' + (1-Y)\tilde{J}E'\right) \cap E\ .$$

Proof. For $a \in \tilde{I} \cap \tilde{J}$ we have

$$Ya + (1-Y)a = a \in \left(Y\tilde{I}E' + (1-Y)\tilde{J}E'\right) \cap E\ .$$

Now, let $a \in \left(Y\tilde{I}E' + (1-Y)\tilde{J}E'\right) \cap E$. The function a does not depend on Y and can be presented in the form $a = Y\sum_{i=0}^{r} a_i f_i + (1-Y)\sum_{j=0}^{s} b_j g_j$, where $a_i \in \tilde{I}, b_j \in \tilde{J}, f_i \in E'$, and $g_j \in E'$ for $i = 0, \ldots, r$ and $j = 0, \ldots, s$. The f_i and g_j have well defined values $f_i(0), g_j(0), f_i(1), g_j(1) \in E$, $i = 0, \ldots, r$ and $j = 0, \ldots, s$, for $Y = 0$ and $Y = 1$, respectively. It follows $a = \sum_{j=0}^{s} b_j g_j(0) \in \tilde{J}$ and $a = \sum_{i=0}^{r} a_i f_i(1) \in \tilde{I}$. Hence, $a \in \tilde{I} \cap \tilde{J}$. □

In order to emphasize that the generalization of Equation (1) to the situation described in Lemma 3 is not self evident, note, that the equation

$$I \cap J = (YIA[[Y]] + (1-Y)JA[[Y]]) \cap A$$

is invalid in general since the right hand side is always equal to J.

[1] In the literature, e.g. [3] or [5], we found the formula only for polynomial rings A but the generalization to arbitrary rings is obvious.

Lemma 4 *Let I, J be ideals of R. Then $IE \cap JE = (I \cap J)E$.*

Proof. Equation (1) and Lemmata 2 and 3 yield:

$$\begin{aligned} IE \cap JE &= \big(Y(IE)E' + (1-Y)(JE)E'\big) \cap E \\ &= \big(YIE' + (1-Y)JE'\big) \cap E \\ &= \big(\big(YIR' + (1-Y)JR'\big)\, E'\big) \cap E \\ &= \big(\big(YIR' + (1-Y)JR'\big) \cap R\big)\, E \\ &= (I \cap J)E \end{aligned}$$

□

Corollary 5 *Let I be an ideal of R. Then for all $f \in R$ we have*

$$IE :_E f = (I :_R f)E\ .$$

Proof. This follows from

$$f \cdot (IE :_E f) = IE \cap fE = (I \cap fR)E = (f \cdot (I :_R f))E = f \cdot ((I :_R f)E)$$

□

Lemma 6 *For all $f_1, \dots, f_k \in R$ it holds*

$$\mathrm{Syz}_E(f_1, \dots, f_k) = \mathrm{Syz}_R(f_1, \dots, f_k)E\ .$$

Proof. The inclusion $\mathrm{Syz}_R(f_1, \dots, f_k)E \subseteq \mathrm{Syz}_E(f_1, \dots, f_k)$ is obvious.

Now, consider $s \in \mathrm{Syz}_E(f_1, \dots, f_k)$. The ideal of the first components of the elements of $\mathrm{Syz}_E(f_1, \dots, f_k)$ is $(f_2, \dots, f_k)E :_E f_1$ which is equal, by Corollary 5, to $((f_2, \dots, f_k)R :_R f_1)\, E$. But this is the extension of the ideal of the first components of the elements of $\mathrm{Syz}_R(f_1, \dots, f_k)$ in E. Therefore, modulo $\mathrm{Syz}_R(f_1, \dots, f_k)E$ we can arrange that the first component of s becomes zero. The statement now follows by induction on k. □

Corollary 7 *The ring E is flat over R.*

3 Extension of Prime and Primary Ideals

Let d be an integer with $1 \le d < n$ and set $\tilde{R} := \mathbb{K}[X_1, \dots, X_d]$. By $\tilde{E}$ we denote the ring of everywhere convergent power series in the variables $X_1, \dots, X_d$ with coefficients from $\mathbb{K}$.

Lemma 8 *Let $I \subset R$ be an ideal such that R/I is a finitely generated $\tilde{R}$-module. Then*

$$E/IE \simeq R/I \ \otimes_{\tilde{R}} \tilde{E}. \tag{2}$$

Proof. Since R/I is finitely generated as an $\tilde{R}$-module for each $i = d+1, \ldots, n$ there are polynomials in $I \cap R[X_i]$ which are monic with respect to X_i. Fix an arbitrary elimination order $\prec$ for $\{X_{d+1}, \ldots, X_n\}$. Then for all $i = d+1, \ldots, n$ there are $\alpha_i \in \mathbb{N}$ such that $X_i^{\alpha_i} \in lt(I)$.

Consider now the map $\varphi : R/I \ \otimes_{\tilde{R}} \tilde{E} \to E/IE$ given by $(r + I) \otimes \tilde{e} \mapsto r\tilde{e} + IE$. Let $f \in E$. Then $f_{IE,red} \in \tilde{E}[X_{d+1}, \ldots, X_n]$ and therefore φ is an epimorphism.

Let $m_1, \ldots m_s$ be elements of R such that their residue classes modulo I generate R/I as an $\tilde{R}$-module. Then we have an exact sequence

$$\tilde{R}^t \xrightarrow{\psi} \tilde{R}^s \xrightarrow{\pi} R/I \longrightarrow 0$$

for suitable $t \in \mathbb{N}$. We note that $\mathrm{Im}\psi$ consists of the syzygies of R/I as an $\tilde{R}$-module. Tensoring with $\tilde{E}$ we get a commutative diagram with exact top row

$$\begin{array}{ccccccc}
\tilde{E}^t & \xrightarrow{\tilde{\psi}} & \tilde{E}^s & \xrightarrow{\tilde{\pi}} & R/I \otimes_{\tilde{R}} \tilde{E} & \longrightarrow & 0 \\
\big\downarrow{=} & & \big\downarrow{=} & & \big\downarrow{\varphi} & & \\
\tilde{E}^t & \xrightarrow{\tilde{\psi}} & \tilde{E}^s & \xrightarrow{\rho} & E/IE & \longrightarrow & 0
\end{array} \quad , \tag{3}$$

where $\tilde{\psi} := \psi \otimes \mathrm{id}_{\tilde{E}}, \tilde{\pi} := \pi \otimes \mathrm{id}_{\tilde{E}}$ and $\rho := \varphi \circ \tilde{\pi}$. It is clear that the bottom row of (3) is a complex and that ρ is an epimorphism.

Take $(\tilde{e}_1, \ldots, \tilde{e}_s) \in \ker\rho$, i.e. $m_1\tilde{e}_1 + \ldots + m_s\tilde{e}_s \in IE$. Assume $I = (p_1, \ldots, p_r)R$. Then there are $e_1, \ldots, e_r \in E$ with

$$m_1\tilde{e}_1 + \ldots + m_s\tilde{e}_s + p_1e_1 + \ldots + p_re_r = 0,$$

i.e. $(\tilde{e}_1, \ldots, \tilde{e}_s, e_1, \ldots, e_r) \in E^{s+r}$ is a syzygy of the ideal $J := (m_1, \ldots, m_s, p_1, \ldots, p_r)R$ of R.

By Lemma 6, $(\tilde{e}_1, \ldots, \tilde{e}_s, e_1, \ldots, e_r) = \sum_{i=1}^{q} \sigma_i e_i^*$ with polynomial syzygies $\sigma_1, \ldots, \sigma_q$ of J and $e_1^*, \ldots, e_q^* \in E$. Let now $\tilde{\sigma}_i := (a_{i,1}, \ldots, a_{i,s}) \in R^s$ be the tuple consisting of the first s components of σ_i, $i = 1, \ldots, q$. Then $\sum_{j=1}^{s} a_{i,j}m_j \in I$ for all $i = 1, \ldots, q$. Since $\tilde{R} \subset R$ is a flat extension, $\tilde{\sigma}_i$ is a linear combination over R of syzygies of R/I as an $\tilde{R}$-module, i.e. we can assume without loss of generality that $\tilde{\sigma}_i \in \tilde{R}^s$, $i = 1, \ldots, q$. By substituting $X_{d+1} = \ldots = X_n = 0$ in $(\tilde{e}_1, \ldots, \tilde{e}_s) = \sum_{i=1}^{q} \tilde{\sigma}_i e_i^*$ we obtain $(\tilde{e}_1, \ldots, \tilde{e}_s) = \sum_{i=1}^{q} \tilde{\sigma}_i f_i$ with $f_i \in \tilde{E}$, $i = 1, \ldots, q$. Therefore the bottom row of (3) is exact, too, and hence φ is an isomorphism. □

Let $P \subset R = \mathbb{K}[X_1, \ldots, X_n]$ be a prime ideal such that R/P is integral over $\tilde{R}$ and $P \cap \tilde{R} = \{0\}$. If I is a P-primary ideal then R/I is integral over $\tilde{R}$ as well.

Consider the rings $L := R/P \otimes_{\tilde{R}} Q(\tilde{R})$ and $M := R/I \otimes_{\tilde{R}} Q(\tilde{R})$. Since tensoring with $Q(\tilde{R})$ is nothing else then localizing at the nonzero elements of $\tilde{R}$, L is an integral domain and the zero ideal of M is a primary ideal in M. Moreover $\tilde{R} \subset R/P$ implies $Q(\tilde{R}) \subset L$. Since L is generated as a $Q(\tilde{R})$-algebra by the residue classes of $X_{d+1}, \ldots, X_n$, L is a field because L is now finite and integral over $Q(\tilde{R})$. Hence $L \simeq Q(\tilde{R})[\eta]$ for some primitive element $\eta \in L$, i.e. $L \simeq Q(\tilde{R})[Z] \;/\; FQ(\tilde{R})[Z]$ for some irreducible polynomial $F \in Q(\tilde{R})[Z]$. Moreover M is an artinian local $Q(\tilde{R})$-algebra (since L is a field, M has only one maximal ideal).
Now we have

Theorem 2

1. *For any prime (primary) ideal $P \subset R$ the extension ideal PE is a prime (primary) ideal of E.*
2. *E is faithfully flat over R.*

Proof. Since the statement of the Theorem is invariant under linear transformations, we may assume that we have with $d := \dim R/P$, $\quad P \cap \tilde{R} = \{0\}$ and R/P is integral over $\tilde{R}$ (use Noetherian normalization). Therefore, we can use the notations introduced above. Now $E/IE \subseteq M \otimes_{\tilde{R}} \tilde{E}$ (Lemma 8) and $\tilde{E}$ is flat over $\tilde{R}$ (Corollary 7). Since $Q(\tilde{R}) \otimes_{\tilde{R}} \tilde{E} \subseteq Q(\tilde{E})$ we get $M \otimes_{\tilde{R}} \tilde{E} \simeq \left(M \otimes_{Q(\tilde{R})} Q(\tilde{R})\right) \otimes_{\tilde{R}} \tilde{E} \simeq M \otimes_{Q(\tilde{R})} \left(Q(\tilde{R}) \otimes_{\tilde{R}} \tilde{E}\right) \subseteq M \otimes_{Q(\tilde{R})} Q(\tilde{E})$. In particular, we get $E/PE \subseteq L \otimes_{Q(\tilde{R})} Q(\tilde{E}) \simeq Q(\tilde{E})[Z] \;/\; FQ(\tilde{E})[Z]$. But F is irreducible over $Q(\tilde{E})$, too (cf. Remark 1., 4.) and therefore $L \otimes_{Q(\tilde{R})} Q(\tilde{E})$ is a field. Since $PE \cap R = P \neq R$ (cf. Corollary 1), PE is a proper ideal in E and therefore E/PE is an integral domain, i.e. PE is a prime ideal in E. Moreover, since M is artinian, M is a finite dimensional $Q(\tilde{R})$-vector space. Therefore $M \otimes_{Q(\tilde{R})} Q(\tilde{E})$ is a finite dimensional $Q(\tilde{E})$-vector space and thus a local artinian $Q(\tilde{E})$-algebra (since $L \otimes_{Q(\tilde{R})} Q(\tilde{E})$ is a field, $M \otimes_{Q(\tilde{R})} Q(\tilde{E})$ has only one maximal ideal). Therefore the zero ideal of $E/IE \subseteq M \otimes_{Q(\tilde{R})} Q(\tilde{E})$ is a primary ideal, i.e. IE is a primary ideal in E which finishes the proof of 1.

Finally, 2. follows from Corollary 7 and 1. □

Proposition 9 *Let I be an ideal of R.*

1. *Assume that* $I = Q_1 \cap \ldots \cap Q_m$ *is an (irredundant) primary decomposition of* I *in* R *with* P_i*-primary ideals* Q_i, $i = 1, \ldots, m$ *and* $P_1, \ldots, P_m \in \operatorname{Spec} R$. *Then* $IE = Q_1 E \cap \ldots \cap Q_m E$ *is an (irredundant) primary decomposition of* IE *in* E *with* $P_i E$*-primary ideals* $Q_i E$, $i = 1, \ldots, m$ *and* $P_1 E, \ldots, P_m E \in \operatorname{Spec} E$.
2. $\operatorname{rad}(\mathrm{I} \cdot \mathrm{E}) = (\operatorname{rad} \mathrm{I}) \cdot \mathrm{E}$

Proof. Part 1 follows from Theorem 2 and Lemma 4. Using the notation of part 1 we obtain

$$\begin{aligned}
\operatorname{rad}(I \cdot E) &= \operatorname{rad}(Q_1 E \cap \ldots \cap Q_m E) \\
&= \operatorname{rad}(Q_1 E) \cap \ldots \cap \operatorname{rad}(Q_m E) \\
&= (P_1 E) \cap \ldots \cap (P_m E) = (P_1 \cap \ldots \cap P_m) \cdot E \\
&= (\operatorname{rad} I) \cdot E
\end{aligned}$$

which proves part 2. □

4 Hilbert's Nullstellensatz for Entire Functions

In this section we restrict ourself to the field $\mathbb{C}$ of complex numbers. As in Section 3 let d be an integer with $1 \leq d < n$ and set $\tilde{R} := \mathbb{C}[X_1, \ldots, X_d]$ and $R^* := \mathbb{C}[X_{d+1}, \ldots, X_n]$. By $\tilde{E}$ (resp. E^*) we denote the ring of entire functions in the variables $X_1, \ldots, X_d$ (resp. $X_{d+1}, \ldots, X_n$) with coefficients in $\mathbb{C}$. Let $y \in \mathbb{C}^d$. For $f \in E$ we denote by $f(y) \in E^*$ the entire function which is obtained from f by substituting $X_1, \ldots, X_d$ by the corresponding coordinates $y_1, \ldots, y_d$ of y. For an ideal I of R we let $I(y) := \{f(y) \mid f \in I\}$ and $I_y := I + (X_1 - y_1, \ldots, X_d - y_d)R$ which are ideals of R^* and R, resp. One easily observes the equality $I_y = I(y)R + (X_1 - y_1, \ldots, X_d - y_d)R$. If R/I is a finitely generated $\tilde{R}$-module, then $\dim R^*/I(y) = \dim R/I_y = 0$, i.e. $V(I(y)) \subset \mathbb{C}^{n-d}$ and $V(I_y) \subset \mathbb{C}^n$ are non empty and consist only of finitely many points. As already mentioned in the proof of Lemma 8, in this situation I contains for all $i = d+1, \ldots, n$ polynomials of the form $X_i^{\alpha_i} + g_i$ with $\alpha_i \in \mathbb{N}^+$ and $g_i \in \tilde{R}$.

The next proposition extends a result due to Jarnicki, O'Carroll and Winiarski in the polynomial case (see [8]) to entire functions.

Proposition 10 *Let* $Q \subset R$ *be a primary ideal such that* R/Q *is a finitely generated* $\tilde{R}$*-module with* $Q \cap \tilde{R} = \{0\}$. *Then for any non-empty Zariski open subset* U *of* $\mathbb{C}^d$ *we have*

$$\bigcap_{y \in U} Q_y E = QE,$$

in particular,

$$\bigcap_{y\in\mathbb{C}^d} Q_y E = QE.$$

Proof. It is clear, that $\bigcap_{y\in U} Q_y E \supseteq QE$ (since $Q \subseteq Q_y$ for all $y \in U$). Let $B = \{b_1, \dots, b_m\}$ be a Gröbner basis of Q with respect to some elimination order $\prec$ for $\{X_{d+1}, \dots, X_n\}$. For $f \in R$ let $lt_d f \in T(X_{d+1}, \dots, X_n)$ denote the leading term of f considered as an element of $\mathbb{C}[X_1, \dots, X_d][X_{d+1}, \dots, X_n]$. We note that in this context the corresponding leading coefficient $lc_d f$ is a polynomial in $X_1, \dots, X_d$. By our assumption there are $\alpha_i \in \mathbb{N}$ such that $X_i^{\alpha_i} \in lt(Q)$ for all $i = d+1, \dots, n$. The set $U' := \{y \in U \mid (lc_d b_i)(y) \neq 0$ for all $i = d+1, \dots, n\}$ is a non-empty Zariski open set such that $B(y) := \{b_1(y), \dots, b_m(y)\}$ is a Gröbner basis of $Q(y)$ for all $y \in U'$ (cf. [2]). Moreover, the trace of a final reduction of $h(y)$ ($h \in R$) modulo $B(y)$ is the same for each $y \in U'$. Let now $f \in \bigcap_{y\in U} Q_y E$. Replacing f by $f_{QE,red}$ we can assume without loss of generality that $f \in \tilde{E}[X_{d+1}, \dots, X_n]$, i.e. $f(y) \in R^*$ for all $y \in U$. In addition, for all $y \in U$ we have $f(y) \in Q(y)$ since $f \in Q_y E$. Therefore $f(y)_{Q(y)E^*,red} = 0$, i.e. $f(y) = \sum_{i=1}^m \tilde{a}_i(y) b_i(y)$, where for all $y \in U'$ the $\tilde{a}_i$ are polynomials in $X_{d+1}, \dots, X_n$ with coefficients which are quotients of an entire function by a polynomial (both in the variables $X_1, \dots, X_d$). Multiplying this equation by $c(y)$, where $c \in \tilde{R}$ is a common denominator of $\tilde{a}_1, \dots, \tilde{a}_m$, for suitable $a_1, \dots, a_m \in \tilde{E}[X_{d+1}, \dots, X_n]$ we obtain $\left(cf - \sum_{i=1}^m a_i b_i\right)(y) = 0$ for all $y \in U'$. Since U' is dense in $\mathbb{C}^d$, this implies $cf - \sum_{i=1}^m a_i b_i = 0$ and, hence, $cf \in QE$. Since Q is primary, QE is primary, too (cf. Theorem 2). Since $Q \cap \tilde{R} = \{0\}$ and $c \in \tilde{R}$ we can deduce $f \in QE$, i.e. $\bigcap_{y\in U} Q_y E \subseteq QE$ and therefore $\bigcap_{y\in U} Q_y E = QE$. □

Corollary 11 *Let $I \subset R$ be an ideal. Then IE is an intersection of ideals of the form JE, where the J are zero dimensional ideals of R (i.e.* $\dim R/J = 0$*).*

Proof. By Proposition 9 we can assume without loss of generality that I is a primary ideal with associated prime ideal P. Take now a noetherian normalization of R/P, i.e. up to linear change of coordinates we assume that R/P and hence R/I is integral over $\mathbb{C}[X_1, \dots, X_d]$, where $d = \dim R/P$. Now apply Proposition 10. □

The next result generalizes Hilbert's Nullstellensatz for entire functions. A proof using analytic methods was given by W. Rudin (cf. [10]).

Corollary 12 *Let $V \subseteq \mathbb{C}^n$ be an algebraic subset given by an ideal $I \subset R$. Let $f \in E$ be an entire function vanishing everywhere on V. Then $f \in$* $(\mathrm{rad}\, \mathrm{I}) \cdot \mathrm{E}$.

Proof. Again we can assume without loss of generality that I is a primary ideal. First consider the case that $\dim R/I = 0$, i.e. $V = \{z\}$. Expanding f at z into a power series shows that $f \in PE$, where $P \subset R$ is the vanishing ideal of z. Since I is P-primary, there is a $t \in \mathbb{N}^+$ with $P^t \subseteq I$, whence $f^t \in IE$.

Now let I be an arbitrary primary ideal. As in the proof of the previous Corollary we choose a noetherian normalization of R/I. Using the same notation as in Proposition 10 we choose $y \in \mathbb{C}^d$. By the previous considerations there is a $t(y) \in \mathbb{N}^+$ such that $f^{t(y)} \in Q_y E$. Since $t(y) \le \operatorname{rank}_{\mathbb{C}} R/Q_y = \operatorname{rank}_{\mathbb{C}} \mathbb{C}[X_{d+1}, \dots, X_n]/Q(y) \le \alpha_{d+1} \cdot \ldots \cdot \alpha_n$ we have $f^t \in \bigcap_{y \in \mathbb{C}^d} Q_y E$ for sufficiently high $t \in \mathbb{N}^+$. Now the claim follows by Propositions 9 and 10. $\square$

Finally, we remark that Proposition 10 could be also applied in order to prove a version of the effective Nullstellensatz for entire functions in a similar way as it was done in the polynomial case in [8].

References

1. Apel, J., Stückrad, J., Tworzewski, P., Winiarski, T.: Reduction of everywhere convergent power series with respect to Gröbner bases. J. Pure Appl. Algebra **110**(2), 113–129 (1996)
2. Bayer, D., Galligo, A., Stillman, M.: Gröbner bases and extension of scalars. In: Sympos. Math. XXXIV, Ed. D. Eisenbud et.al., Cambridge University Press, Cambridge, 1993, pp. 198–215
3. Becker, T., Weispfenning, V.: in cooperation with H. Kredel. Gröbner Bases, A Computational Approach to Commutative Algebra. Springer, New York, Berlin, Heidelberg, 1993
4. Buchberger, B.: Gröbner Bases: An algorithmic method in polynomial ideal theory. In: Multidimensional System Theory, Ed. N.K. Bose, D. Reidel Publishing Co., Dordrecht, 1985 pp. 184–232
5. Cox, D., Little, J., O'Shea, D.: Ideals, Varieties, and Algorithms. An introduction to computational algebraic geometry and commutative algebra. Springer, New York, 1992
6. Eisenbud, D.: Commutative Algebra. With a view toward algebraic geometry. Springer, New York, 1995
7. Gianni, P., Trager, B., Zacharias, G.: Gröbner Bases and Primary Decomposition of Polynomial Ideals. J. Symb. Comp. **6**(2/3), 149–167 (1988)
8. Jarnicki, W., O'Carroll, L., Winiarski, T.: Ideal as an intersection of zero-dimensional ideals and the Noether exponent. (IMUJ Preprint 2000/25) To appear in: Univ. Iag. Acta Math. 2000
9. Matsumura, H.: Commutative Algebra. 2nd edn, Mathematics Lecture Note Series **56**, The Benjamin/Cummings Publishing Company, Inc. 1980
10. Rudin, W.: A geometric criterion for algebraic varieties. J. Math. Mech. **17**, 671–683 (1968)