Computation of Characteristic Sets of Radical Differential Ideals

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Abstract. The problem of computing a characteristic set of a radical differential ideal in Kolchin's sense is studied. A special class of radical differential ideals is introduced. These ideals satisfy a special *property of consistency*. For this class of ideals a theorem establishing a "distribution" of leaders of a characteristic set among the leaders of a characteristic decomposition is proved. In connection to this problem the usefulness of a characteristic decomposition is discussed. For the class of ideals satisfying the property of consistency an algorithm for computing a characteristic set in Kolchin's sense w.r.t. an orderly ranking is presented.

1 Introduction

This paper is devoted to the study of radical differential ideals and their characteristic sets. The concept of a characteristic set introduced by Ritt and Kolchin is one of the most important notions in differential algebra. The problem of computing a characteristic set of a radical differential ideal represented by a finite set of its generators is not completely solved yet especially in the partial differential case. In the case of ideals in rings of polynomials in a finite number of variables this problem was studied and completely solved by Gallo and Mishra [6],[7], and [8]. It was also investigated by Aubry, Lazard, and Moreno Maza [2].

So, it is very natural to study this problem in rings of differential polynomials. The most important contribution of our article is an *algorithm* for computing characteristic sets of radical differential ideals satisfying *the property of consistency* (see Definition 4) w.r.t. orderly rankings in the partial differential case. We use other technique and methods than that used by Gallo and Mishra. However, their algorithm for computing a characteristic set of an algebraic ideal plays an important role in Algorithm 5 of this paper.

Summarizing, we have proved that there exists an *algorithm* computing a characteristic set of a radical differential ideal. Due to Ritt and Kolchin everybody knows that a characteristic set exists but as to my knowledge the existence of an algorithm was not proved before.

Ten years ago a technique for effective and factorization-free computations in the radical differential ideal theory was developed by Boulier, Lazard, Ollivier, and Petitot (see [4] and [5]). In [9] and [10] Hubert continued to study this problem and introduced the notions of *characterizable* ideal and *characteristic decomposition* of a radical differential ideal. This decomposition of the ideal helps us to solve many problems concerning the system of differential equations associated with the ideal and to test the radical membership.

It should be emphasized that a characteristic decomposition of a radical differential ideal does not give us full information about the ideal. In some important cases a representation of this ideal by characteristic components cannot replace a representation of the ideal by its generators as a radical differential ideal. For example, at this moment one cannot check the inclusion of a radical differential ideal to another radical differential ideal knowing only a characteristic decomposition of the first one (see [12, 14, 15]). This problem is closely related to the well-known Ritt problem (see [16]). In this case it is necessary to know generators of the ideal and characteristic decomposition is partially useless.

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Nevertheless, in this paper we show that a characteristic decomposition of a radical differential ideal knows a lot about the ideal. Indeed, characteristic decomposition allows us not only to test membership to this ideal but also to compute its characteristic set in Kolchin's sense. Hence, the main contribution of this paper can also be considered as another application of a characteristic decomposition. Such a decomposition tells us a lot about a system of partial differential equations and we study what else one can do using a characteristic decomposition.

In Algorithm 5 we show how this work can be done. Theorem 4 in Section 3 provides a theoretical basis for it. Radical differential ideals, for which we propose the algorithm, satisfy the property of consistency (see Definition 4) w.r.t. an orderly ranking. Our aim is not to present an efficient algorithm for computing a characteristic set. Algorithm 5 works approximately as slow as such algorithms as Rosenfeld_Groebner (see [4, 5]) or χ-Decomposition (see [9]). So, the primary purpose of the article is to constructively prove that there exists an algorithm computing characteristic sets of radical differential ideals.

In summary, although a characteristic decomposition cannot replace the ideal in computational sense, this decomposition allows us to compute such an important subset of a radical differential ideal as its characteristic set.

2 Preliminaries

2.1 Basic Definitions

Differential algebra deals with differential rings and fields. These are commutative domains with 1 and a basic set of differentiations $\Delta = \{\delta_1, \ldots, \delta_n\}$ on a ring. The ring of differential polynomials was introduced to deal with algebraic differential equations.

Recent tutorials on constructive differential ideal theory are presented in [10] and [17]. We also use the Gröbner bases technique discussed in detail in [3]. The definition for the ring of *differential polynomials* in l variables over a differential field k is given in [11], [13], and [16]. This ring is denoted by $k\{y_1, \ldots, y_l\}$. We denote polynomials by $f, g, h, \ldots$ and use the notation I, J, P, Q for ideals.

We need the notion of reduction for algorithmic computations. First, we introduce a *ranking* on the set of differential variables of $k\{y_1, \ldots, y_l\}$. Construct the multiplicative monoid $\Theta = (\delta_1^{k_1}\delta_2^{k_2}\cdots\delta_n^{k_n}, k_i \geqslant 0)$. The ranking is a total ordering on the set $\{\theta y_i\}$ for each $\theta \in \Theta$ and $1 \leqslant i \leqslant l$ satisfying the following conditions:

$$\theta u \geqslant u\,, \quad u \geqslant v \Longrightarrow \theta u \geqslant \theta v.$$

In later discussions we suppose that a ranking is fixed.

Let u be a differential variable in $k\{y_1, \ldots, y_l\}$, that is $u = \theta y_j$ for a differential operator $\theta = \delta_1^{k_1}\delta_2^{k_2}\cdots\delta_n^{k_n} \in \Theta$ and $1 \leqslant j \leqslant l$. Set $\operatorname{ord}_w u = \sum_{i=1}^{n} w_i k_i$, where w_i are positive integers for $1 \leqslant i \leqslant n$. From now we suppose that some w is fixed and denote ord_w simply by ord. A ranking is said to be *orderly* iff $\operatorname{ord} u > \operatorname{ord} v$ implies $u > v$ for all differential variables u and v.

The highest ranked derivative θy_j appeared in a differential polynomial $f \in k\{y_1, \ldots, y_l\} \setminus k$ is called the leader of f. We denote the leader by u_f. Represent f as a univariate polynomial in u_f:

$$f = I_f u_f^n + a_1 u_f^{n-1} + \ldots + a_n\,.$$

The polynomial I_f is called the *initial* of f.

Apply any $\delta \in \Theta$ to f:

$$\delta f = \frac{\partial f}{\partial u_f}\delta u_f + \delta I_f u_f^n + \delta a_1 u_f^{n-1} + \ldots + \delta a_n\,.$$

The leading variable of δf is δu_f and the initial of δf is called the *separant* of f. We denote it by S_f. Note that for all $\theta \in \Theta$, $\theta \neq 1$, each θf has the initial equal to S_f.

Define the ranking on differential polynomials. We say that $f > g$ iff $u_f > u_g$ or in the case of $u_f = u_g$ we have $\deg_{u_g} f > \deg_{u_g} g$. Let $F \subset k\{y_1, \ldots, y_l\}$ be a set of differential polynomials. For the differential and radical differential ideal generated by F in $k\{y_1, \dots, y_l\}$, we use the notation $[F]$ and $\{F\}$, respectively.

We say that a differential polynomial f is *partially reduced* w.r.t. g iff no proper derivative of u_g appears in f. A differential polynomial f is *reduced* w.r.t. g iff f is partially reduced w.r.t. g and $\deg_{u_g} f < \deg_{u_g} g$. Consider any subset $\mathbb{A} \subset k\{y_1, \dots, y_l\}$. We say that $\mathbb{A}$ is autoreduced iff $\mathbb{A} \cap k = \varnothing$ and each element of $\mathbb{A}$ is reduced w.r.t. all the others. Every autoreduced set is finite. For autoreduced sets we use capital letters $\mathbb{A}, \mathbb{B}, \mathbb{C}, \dots$.

We denote the product of the initials and the separants of $\mathbb{A}$ by $I_\mathbb{A}$ and $S_\mathbb{A}$ respectively. Denote $I_\mathbb{A} \cdot S_\mathbb{A}$ by $H_\mathbb{A}$. Let S be a finite set of differential polynomials. Denote by S^∞ the multiplicative set containing 1 and generated by S. Let I be an ideal in a commutative ring R. Let $I : S^\infty = \{a \in R \mid \exists s \in S^\infty : sa \in I\}$. If I is a differential ideal then $I : S^\infty$ is also a differential ideal.

If we want to enumerate the elements of $\mathbb{A}$ we write the following: $\mathbb{A} = A_1, A_2, \dots, A_p$. Let $\mathbb{A} = A_1, \dots, A_r$ and $\mathbb{B} = B_1, \dots, B_s$ be autoreduced sets. Let the elements of $\mathbb{A}$ and $\mathbb{B}$ be arranged in order of increasing rank. We say that $\mathbb{A}$ has lower rank than $\mathbb{B}$ iff there exists $k \leqslant r, s$ such that $\operatorname{rank} A_i = \operatorname{rank} B_i$ for $1 \leqslant i < k$ and $\operatorname{rank} A_k < \operatorname{rank} B_k$, or if $r > s$ and $\operatorname{rank} A_i = \operatorname{rank} B_i$ for $1 \leqslant i \leqslant s$. We say that $\operatorname{rank} \mathbb{A} = \operatorname{rank} \mathbb{B}$ iff $r = s$ and $\operatorname{rank} A_i = \operatorname{rank} B_i$ for $1 \leqslant i \leqslant r$.

Consider two differential polynomials f and g in $R = k\{y_1, \dots, y_l\}$. Let I be the differential ideal in R generated by g. Applying a finite number of differentiations and pseudo-divisions one can compute a *differential partial remainder* f_1 and a *differential remainder* f_2 of f w.r.t. g such that there exist $s \in S_g^\infty$ and $h \in H_g^\infty$ satisfying $sf \equiv f_1$ and $hf \equiv f_2 \bmod I$ with f_1 and f_2 partially reduced and reduced w.r.t. g, respectively. If $\mathbb{A}$ is an autoreduced set then the reduction w.r.t. $\mathbb{A}$ is defined as it was done by Ritt [16, pp. 5-7].

Let $\mathbb{A}$ be an autoreduced set in $k\{y_1, \dots, y_l\}$. Consider the polynomial ring $k[x_1, \dots, x_n]$ with $x_1, \dots, x_n$ belong to ΘY for $Y = y_1, \dots, y_l$. Let $U, V \subset \{x_1, \dots, x_n\}$ be the sets of "leaders" and "non-leaders" appearing in the autoreduced set $\mathbb{A}$, respectively. We denote $k[x_1, \dots, x_n]$ by $k[V][U]$ and the leader of A_i by u_{A_i} or u_i for each $1 \leqslant i \leqslant p$.

Example 1. Let $\mathbb{A} = A_1, A_2 \subset k\{v, u_1, u_2\}$, where $A_1 = vu_1^2 + u_1 + v^2, A_2 = u_1u_2^3 + v$ and $v < u_1 < u_2$. We have $U = u_1, u_2$ and $V = v$.

The notion of a characteristic set in *Kolchin's sense* in characteristic zero is important in our further discussions.

Definition 1. *[11, page 82] An autoreduced set of the lowest rank in an ideal I is called a characteristic set of I.*

As it is mentioned in [11, Lemma 8, page 82], in characteristic zero $\mathbb{A}$ is a characteristic set of a proper differential ideal I iff each element of I reduces to zero w.r.t. $\mathbb{A}$. Consider the definition of a characterizable radical differential ideal.

Definition 2. *[9, Definition 2.6] A radical differential ideal I in $k\{y_1, \dots, y_l\}$ is said to be characterizable iff there exists a characteristic set $\mathbb{A}$ of I in Kolchin's sense such that $I = [\mathbb{A}] : H_\mathbb{A}^\infty$.*

The following definition makes a bridge between differential and commutative algebra. Let v be a derivative in $k\{y_1, \dots, y_l\}$. $\mathbb{A}_v$ is the set of the elements of $\mathbb{A}$ and their derivatives that have a leader ranking strictly lower than v.

Definition 3. *[11, III.8] $\mathbb{A}$ is coherent iff whenever $A, B \in \mathbb{A}$ are such that u_A and u_B have a common derivative: $v = \psi u_A = \phi u_B$, then $S_B \psi A - S_A \phi B \in (\mathbb{A}_v) : H_\mathbb{A}^\infty$.*

We emphasize that a characteristic set of a differential ideal is a coherent autoreduced set.

2.2 Important Assertions

Consider several important results concerning radical differential ideals in rings of differential polynomials. The technique described in [9] and [11] helps us to cover some properties of these ideals.

Theorem 1. *[11, III.8, Lemma 5] Let $\mathbb{A}$ be a coherent autoreduced set in $k\{y_1,\ldots,y_l\}$. Suppose that a differential polynomial g is partially reduced w.r.t. $\mathbb{A}$. Then $g \in [\mathbb{A}] : H_{\mathbb{A}}^{\infty}$ iff $g \in (\mathbb{A}) : H_{\mathbb{A}}^{\infty}$.*

Note that Theorem 1 is also known as *Rosenfeld's lemma.*

Theorem 2. *[9, Theorem 3.2] Let $\mathbb{A}$ be an autoreduced set of $k[V][U]$. If $1 \notin (\mathbb{A}) : S_{\mathbb{A}}^{\infty}$ then any minimal prime of $(\mathbb{A}) : S_{\mathbb{A}}^{\infty}$ admits the set of non-leaders of $\mathbb{A}$, V, as a transcendence basis. More specially, any characteristic set of a minimal prime of $(\mathbb{A}) : S_{\mathbb{A}}^{\infty}$ has the same set of leaders as $\mathbb{A}$.*

Theorem 3. *[9, Theorem 4.5] Let $\mathbb{A}$ be a coherent autoreduced set of $R = k\{y_1,\ldots,y_l\}$ such that $1 \notin [\mathbb{A}] : H_{\mathbb{A}}^{\infty}$. There is a one-to-one correspondence between the minimal primes of $(\mathbb{A}) : H_{\mathbb{A}}^{\infty}$ in $k[V][U]$ and the essential prime components of $[\mathbb{A}] : H_{\mathbb{A}}^{\infty}$ in R. Assume $\mathbb{C}_i$ to be a characteristic set of a minimal prime of $(\mathbb{A}) : H_{\mathbb{A}}^{\infty}$ then $\mathbb{C}_i$ is the characteristic set of a single essential prime component of $[\mathbb{A}] : H_{\mathbb{A}}^{\infty}$ (and vice versa).*

Lemma 1. *Let $\mathbb{A} = A_1,\ldots,A_p$ be an autoreduced set in $k[x_1,\ldots,x_m] = R$ and a characteristic set of $(\mathbb{A}) : I_{\mathbb{A}}^{\infty}$. Suppose that a polynomial $f = a_m x_t^m + \ldots + a_0 \in R$ is reducible to zero w.r.t. $\mathbb{A}$ and the indeterminate x_t does not appear in A_i for each $1 \leqslant i \leqslant p$. Then a_j is reducible to zero w.r.t. $\mathbb{A}$ for all $0 \leqslant j \leqslant m$.*

Proof. Since f is reducible to zero w.r.t. $\mathbb{A}$, there exists $I \in I_{\mathbb{A}}^{\infty}$ such that

$$I \cdot f = \sum_{i=1}^{p} g_i A_i \,.$$

Let $g_i = \sum\limits_{j=1}^{t_i} h_j x_t^j$ for each $1 \leqslant i \leqslant p$. Thus, we have $I \cdot \sum\limits_{k=0}^{m} a_k x_t^k = \sum\limits_{k=0}^{q} d_k x_t^k$ with $d_k \in (A_1,\ldots,A_p)$. Hence, $I \cdot a_i \in (\mathbb{A})$ for each $0 \leqslant i \leqslant m$, that is, $a_i \in (\mathbb{A}) : I_{\mathbb{A}}^{\infty}$. Since $\mathbb{A}$ is a characteristic set of $(\mathbb{A}) : I_{\mathbb{A}}^{\infty}$, we have all a_i are reducible to zero w.r.t. $\mathbb{A}$.

3 The Principal Result

Denote [9, Algorithm 7.1] by χ-Decomposition. An input of this algorithm is a finite set of differential polynomials F and its output is a set $\mathfrak{C} = \mathbb{C}_1,\ldots,\mathbb{C}_n$ of characteristic sets $\mathbb{C}_i$ of characterizable ideals $[\mathbb{C}_i] : H_{\mathbb{C}_i}^{\infty}$ forming a characteristic decomposition of the radical differential ideal $\{F\}$ in $k\{y_1,\ldots,y_l\}$:

$$\{F\} = [\mathbb{C}_1] : H_{\mathbb{C}_1}^{\infty} \cap \ldots \cap [\mathbb{C}_n] : H_{\mathbb{C}_n}^{\infty} \,.$$

Introduce a special class of radical differential ideals.

Definition 4. *We say that a radical differential ideal I satisfies the property of consistency iff there exists a characteristic set $\mathbb{C} \subset I$ such that $1 \notin [\mathbb{C}] : H_{\mathbb{C}}^{\infty}$.*

It is clear that any characterizable radical differential ideal satisfies the property of consistency. Moreover, it follows from Theorem 2 and Theorem 3 that any proper regular differential ideal (an ideal of the form $[\mathbb{A}] : H_{\mathbb{A}}^{\infty}$ for a coherent autoreduced set $\mathbb{A}$) satisfies this property. Consider an example of non-regular radical differential ideal satisfying the property of consistency.

Example 2. Let $\mathbb{A} = x(x-1), xy, xz$ in $k\{x,y,z\}$ with $x < y < z$. We have $1 \notin [\mathbb{A}] : H_{\mathbb{A}}^{\infty}$. Consider the minimal prime decomposition:

$$\{\mathbb{A}\} = [x] \cap [x-1, y, z] \,.$$

We see $\mathbb{A}$ is a characteristic set of $\{\mathbb{A}\}$. Then the radical differential ideal $\{\mathbb{A}\}$ satisfies the property of consistency. Nevertheless, since minimal primes of $\{\mathbb{A}\}$ have different sets of leader, $\{\mathbb{A}\}$ is not a regular ideal due to Theorem 2 and Theorem 3.

So, we are ready to prove Theorem 4. Let θy_i be a differential variable in $k\{y_1, \ldots, y_l\}$. Then, by definition its order equals $\operatorname{ord}\theta$. Characteristic sets of polynomial ideals (in rings of polynomials in a finite number of variables) are called *algebraic* characteristic sets.

Theorem 4. *Let I be a radical differential ideal in $k\{y_1, \ldots, y_l\}$ satisfying the property of consistency and a characteristic set $\mathbb{C} \subset I$ with $1 \notin [\mathbb{C}] : H_{\mathbb{C}}^{\infty}$.*

1. *Let U be the set of leaders of $\mathbb{C}$ and U' be the set of leaders of any characteristic decomposition of I. Then $U \subset U'$.*
2. *Let $I = \bigcap_{i=1}^{n} [\mathbb{C}_i] : H_{\mathbb{C}_i}^{\infty}$ be a characteristic decomposition w.r.t. an orderly ranking with $\mathbb{C}_i = C_i^1, \ldots, C_i^{p_i}$. Let h be the maximal order of differential variables appearing in the elements of $\mathbb{C}_i$ for $1 \leqslant i \leqslant n$. Then the lowest differentially autoreduced subset of an algebraic characteristic set of the ideal $\bigcap_{i=1}^{n} (\theta_i C_i^j, \operatorname{ord}\theta_i u_{C_i^j} \leqslant h) : H_{\mathbb{C}_i}^{\infty}$ is a characteristic set of I in Kolchin's sense w.r.t. the orderly ranking.*

Proof. We have $[\mathbb{C}] \subset I \subset [\mathbb{C}] : H_{\mathbb{C}}^{\infty}$. Consider the minimal prime decomposition $I = \bigcap_{i=1}^{m} P_i$. We have

$$I = \bigcap_{i=1}^{n} [\mathbb{C}_i] : H_{\mathbb{C}_i}^{\infty} = \bigcap_{i=1}^{m} P_i \,.$$

Some components of the characteristic decomposition may appear to be unnecessary. Let $I = \bigcap_{i=1}^{k} [\mathbb{C}_i] : H_{\mathbb{C}_i}^{\infty}$ be a minimal characteristic decomposition, that is, $I \neq \bigcap_{i=1, i\neq j}^{k} [\mathbb{C}_i] : H_{\mathbb{C}_i}^{\infty}$ for all $1 \leqslant j \leqslant k$. Let U' be the union of leaders of $\mathbb{C}_i$ for $1 \leqslant i \leqslant k$. If P and P_j are prime ideals for each $1 \leqslant j \leqslant t$ and $P \supset \bigcap_{i=1}^{t} P_i$ then $P \supset P_i$ for some $1 \leqslant j \leqslant t$ (see [1, Proposition 1.11]). Thus, if P is a minimal prime of I then P is a minimal prime of $[\mathbb{C}_i] : H_{\mathbb{C}_i}^{\infty}$ for some $1 \leqslant i \leqslant k$.

We obtain that the set of leaders of any characteristic set of P is equal to those of $\mathbb{C}_i$ by Theorem 2 and Theorem 3 . Hence, the union of leaders of characteristic sets of minimal primes of I is equal to U'. Include $[\mathbb{C}] : H_{\mathbb{C}}^{\infty}$ into a characteristic decomposition of I. For this purpose output $\mathbb{C}$ instead of the first usage of Coherent-Autoreduced algorithm in χ-Decomposition algorithm and then continue the algorithm computing a characteristic decomposition. Thus, we have $I = [\mathbb{C}] : H_{\mathbb{C}}^{\infty} \cap [\mathbb{B}_2] : H_{\mathbb{B}_2}^{\infty} \cap \ldots \cap [\mathbb{B}_r] : H_{\mathbb{B}_r}^{\infty}$. Denote the set of leaders of $\mathbb{C}$ by U.

Let P be a minimal prime of $[\mathbb{C}] : H_{\mathbb{C}}^{\infty}$. Then P is a minimal prime of $\{\mathbb{C}\}$ (see [9, page 644]). Thus, P is a minimal prime of I, because $\{\mathbb{C}\} \subset I \subset [\mathbb{C}] : H_{\mathbb{C}}^{\infty}$. Since P is a minimal prime of $[\mathbb{C}_i] : H_{\mathbb{C}_i}^{\infty}$ for some $1 \leqslant i \leqslant k$, then the set of leaders of $\mathbb{C}_i$ is equal to U and $U \subset U'$. So, we know an upper bound for the order of a characteristic set of I.

Remember that the characteristic decompostion of the ideal I has been computed w.r.t. the orderly ranking. Then, due to Theorem 1 and Lemma 1 we have

$$[\mathbb{C}_i] : H_{\mathbb{C}_i}^{\infty} \cap k[\theta_i y_i, \operatorname{ord}\theta_i y_i \leqslant h] = (\theta_i C_i^j, \operatorname{ord}\theta_i u_{C_i^j} \leqslant h) : H_{\mathbb{C}_i}^{\infty}$$

and $\operatorname{ord} u_{C_i^j} \leqslant h$ for each $1 \leqslant i \leqslant n$. Hence, our problem is purely commutative algebraic now.

In order to get a characteristic set of the ideal I it is sufficient to compute an algebraic characteristic set $\mathbb{C}'$ of I' and then find in $\mathbb{C}'$ the lowest differentially autoreduced subset $\mathbb{C}$. This set differentially reduces the ideal I' to zero. Thus, $\mathbb{C}$ is a characteristic set of I.

The main contribution of Theorem 4 is that our computations are moved into the ring of commutative polynomials in a *finite* number of variables. This is a crucial point in Algorithm 5.

Remark 1. We see that the set of leaders U of $\mathbb{C}$ is not only a subset of U'. We have that U is equal to the set of leaders of some characteristic component in any characteristic decomposition of I. Thus, U is "concentrated" in some characteristic component. We call it the "localization" property.

Remark 2. The fact that the set of leaders of $\mathbb{C}$ is equal to that of some characteristic component holds true for *any* differential ranking. This follows from the proof of Theorem 4.

Note that the condition that an ideal satifies the property of consistency cannot be omitted in Theorem 4. To support this fact we give Example 5.

4 Algorithm

In conclusion, we obtain the following algorithm. If f is a differential polynomial then $\operatorname{ord} f$ denotes the maximal order of differential variables appeared in f. Let some orderly ranking be fixed.

Algorithm 5 Characteristic Set Computation

INPUT: *a finite set F of differential polynomials such that $\{F\}$ satisfies the property of consistency.*

OUTPUT: *characteristic set of $\{F\}$ in Kolchin's sense.*

- *Let* $\mathfrak{C} = \chi\text{-}\mathsf{Decomposition}(F)$ *and* $\mathfrak{C} = \mathbb{C}_1, \ldots, \mathbb{C}_n$ *with* $\mathbb{C}_i = C_i^1, \ldots, C_i^{p_i}$.
- *Let* $\mathbb{C}_i = C_1^i, \ldots, C_{p_i}^i$ *for each* $1 \leqslant i \leqslant n$.
- *Let* $h = \max\limits_{1 \leqslant i \leqslant n} \max\limits_{1 \leqslant j \leqslant p_i} \operatorname{ord} C_i^j$.
- *Compute* $I' = \bigcap_{i=1}^n (\theta_i C_i^j, \operatorname{ord} \theta_i u_{C_i^j} \leqslant h) : H_{\mathbb{C}_i}^\infty$.
- $\mathbb{C}'$:= *an algebraic characteristic set of the ideal* I'.
- *Return the differentially autoreduced subset of* $\mathbb{C}'$ *with the lowest rank.*

The last steps of Algorithm 5 can be performed by means of computations discussed in [3] and [6], [7], and [8]. More precisely, one can compute each $I_i = (\theta_i C_i^j, \operatorname{ord} \theta_i u_{C_i^j} \leqslant h) : H_{\mathbb{C}_i}^\infty$ using the Rabinovich trick and the elimination technique. Then, the intersection of the ideals $I' = \bigcap_{i=1}^n I_i$ has to be computed. The solutions to these two problems are presented in [3]. Finally, an algorithm for computing a characteristic set of I' is given in [6], [7], and [8].

5 Examples

We show how to apply Algorithm 5 to a particular radical differential ideal.

Example 3. Let $I = \{(x-t)x',\ x'y',\ (x-t)(z'+y')\}$ in $\mathbb{Q}(t)\{x, y, z\}$ with $t' = 1$ and an orderly ranking $x < y < z$. We have the following decomposition:

$$I = [x - t, y'] \cap [x', z' + y'] \,.$$

So, the maximal order of variables appearing in this decomposition is equal to 1. Hence, we need to compute the reduced Gröbner basis G of the ideal $I' = (x - t, x' - 1, y') \cap (x', z' + y')$. This can be done by the elimination technique: G equals the intersection of the reduced Gröbner basis w.r.t. the lexicographic ordering $x < x' < y' < z' < w$ of the ideal $(w(x-t),\ w(x'-1),\ wy',\ (1-w)x',\ (1-w)(z'+y'))$ and the ring $\mathbb{Q}(t)[x, x', y', z']$.

Finally, $G = (x-t)x',\ x'(x'-1),\ x'y',\ (x-t)(z'+y'),\ x'z' - (z'+y'),\ y'(z'+y')$. Then a characteristic set of I' equals $\mathbb{C} = (x-t)x',\ (x-t)(z'+y')$ and by Theorem 4 a characteristic set of the radical differential ideal I is also $\mathbb{C}$.

The following example shows the *difference* between radical differential ideals with the consistency property and an arbitrary radical differential ideal. This can be considered as a "counter-example" for Remark 1 and Remark 2.

Example 4. Let $\mathbb{A} = x(x-1),\ xy,\ (x-1)z$. We have $1 \in [\mathbb{A}] : H_{\mathbb{A}}^\infty$. Consider the minimal prime decomposition:

$$\{\mathbb{A}\} = [x, z] \cap [x - 1, y] \,.$$

We see $\mathbb{A}$ is a characteristic set of $\{\mathbb{A}\}$ with the set of leaders equals $U = x,\ y,\ z$. Thus, U does not necessarily correspond to the *unique* characteristic component. In this example $U \subset x, z \cup x, y$ and the localization property is not valid.

Note that Theorem 4 is true for Example 4: in this case we have a restriction to the orders of the elements of a characteristic set $\mathbb{A}$. Consider a "counter-example" for Theorem 4. This example is due to M.V. Kondratieva.

Example 5. Consider a radical differential ideal defined by its characteristic decomposition:

$$I = [x-1, y] \cap [x, y^{(n)}, z^{(m)} + y]$$

in $k\{x, y, z\}$ with $x < y < z$, an orderly ranking, and $n \leqslant m$. Both of these components are prime differential ideals, because they are generated by linear differential polynomials. In addition, since they are prime, these radical differential ideals are also characterizable (see [9, page 646]). One can show that a characteristic set of I is $\mathbb{C} = x(x-1)$, xy, $(x-1)z^{(m+n)}$. The radical differential ideal I does not satisfy the property of consistency and Theorem 4 is not true for I. Indeed, for $m, n > 0$ we have $m + n > \max\{m, n\}$.

So, we see that the upper bound established in Theorem 4 is *wrong* for some radical differential ideals *not satisfying* the property of consistency.

6 Conclusions

We presented a solution to the problem of computing a characteristic set of a radical differential ideal satisfying the special property of consistency that is new, and previously it was completely solved only in the non-differential case. The author hopes that the technique obtained in this paper can be generalized to any radical differential ideals using the ideas we presented.

Another natural way of generalizing these results is to investigate non-orderly rankings such as, for example, very important elimination ones. The main idea is to use a characteristic decomposition w.r.t. *orderly* rankings computing characteristic sets w.r.t. other rankings. The following conjecture can be suggested.

If a radical differential ideal I does not satisfy the property of consistency then the upper bound is supposed to increase rather in the ordinary case: most probably, we will need to replace h in Algorithm 5 by $h = \max \operatorname{ord} \mathbb{C}_i$, where $\mathbb{C}_i$ are characteristic components of the ideal I and $\operatorname{ord} \mathbb{C}_i = \operatorname{ord} C_i^1 + \ldots + \operatorname{ord} C_i^{p_i}$. The previous h is smaller than this new one in general.

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