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COMPLEXITY OF A STANDARD BASIS OF A D-MODULE

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Abstract. A double-exponential upper bound is obtained for the degree and for the complexity of constructing a standard basis of a D-module. This generalizes a well-known bound for the complexity of a Gröbner basis of a module over the algebra of polynomials. It should be emphasized that the bound obtained cannot be deduced immediately from the commutative case. To get the bound in question, a new technique is developed for constructing all the solutions of a linear system over a homogeneous version of a Weyl algebra.

Introduction

Let A be the Weyl algebra $F[X_1, \dots, X_n, \frac{\partial}{\partial X_1}, \dots, \frac{\partial}{\partial X_n}]$ (or the algebra of differential operators $F(X_1, \dots, X_n)[\frac{\partial}{\partial X_1}, \dots, \frac{\partial}{\partial X_n}]$). For brevity, we denote $D_i = \frac{\partial}{\partial X_i}$, $1 \le i \le n$. Any A-module is called a D-module. It is well known that an A-module that is a submodule of a free finitely generated A-module has a Janet basis (if A is a Weyl algebra, it is often called a standard basis, but in this paper it is natural and convenient to call it a Janet basis also in that case). Historically, it was first introduced in [9]. In the more recent time of developing computer algebra, Janet bases were studied in [5, 14, 10]. The Janet bases generalize the Gröbner bases, which were widely used in the algebra of polynomials (see, e.g., [3]). For the Gröbner bases, a double-exponential complexity bound was obtained in [12, 6] with the help of [1]. Later, sharper results on the same subject (with independent and self-contained proofs) were obtained in [4].

Surprisingly, no complexity bound for Janet bases has been established so far. The reason is clear: the problem is not easy. In the present paper we fill this very essential gap and prove a double-exponential upper bound for complexity. On the other hand, a double-exponential complexity lower bound for Gröbner bases [12, 15] provides by the same token a bound for Janet bases.

Notice also that there has been a folklore opinion that the problem of constructing a Janet basis reduces easily to the commutative case by considering the associated graded module, and, on the other hand, in the commutative case [6, 12, 4], the double-exponential upper bound is well known. *But this turns out to be a fallacy*! *From a known system of generators of a D-module, no system of generators* (*even not necessarily a Gröbner basis*) *of the associated graded module can be obtained immediately.* The main problem here is to construct such a system of generators of the graded module. It may have elements of degrees $(dl)^{2^{O(n)}}$; see the notation below. Then, indeed, to the last system of generators of large degrees, one can apply the result known in the commutative case and get the bound $((dl)^{2^{O(n)}})^{2^{O(n)}} = (dl)^{2^{O(n)}}$. Thus, some new ideas specific to the noncommutative case are needed.

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We are interested in estimates for Janet bases of A-submodules of A^l. A Janet basis depends on the choice of a linear order on the monomials (we define them also for $l > 1$). In this paper we consider the most general linear orders on the monomials in A^l. They satisfy conditions (a) and (b) in §1 and are said to be *admissible*. If additionally a linear order satisfies condition (c) in §1, then it is said to be *degree-compatible*. For any admissible linear order, *the reduced Janet basis* is chosen canonically and it is defined uniquely; see §1. We prove the following result.

Theorem 1. *For any real number $d \geq 2$ and any admissible linear order on the monomials in A^l, any left A-submodule I of A^l generated by elements of degree less than d* (*with respect to the filtration in the corresponding algebra; see* §§1 *and* 9) *has a Janet basis with the degrees and the number of its elements less than*

$$(dl)^{2^{O(n)}}. \tag{1}$$

The same upper estimate (1) *is valid for the number of elements of the reduced Janet basis of the module I with respect to the linear order in question on the monomials.*

If, moreover, this linear order is degree-compatible or it is an arbitrary admissible order, but $l = 1$, then also the degrees of all the elements of the reduced Janet basis of the module I are bounded from above by (1).

We prove this theorem in detail for the case of the Weyl algebra A. The proof for the case of the algebra of differential operators is similar. It is sketched in §9. Theorem 1 implies that the Hilbert function $H(I, m)$, see §1, of the A-submodule from this theorem is stable for $m \geq (dl)^{2^{O(n)}}$ and that the absolute values of all coefficients of the Hilbert polynomial of I are bounded from above by $(dl)^{2^{O(n)}}$; cf., e.g., [12]. This fact follows directly from (11), Lemma 12 in Appendix 1, Lemma 2, and Theorem 2. We mention that, in [7], a similar bound was established for the leading coefficient of the Hilbert polynomial.

Now we outline the plan of our proof of Theorem 1. Below, the first occurrences of some terms introduced in the paper are italicized. The main tool in the proof is a *homogenized Weyl algebra* ${}^hA$ (or respectively, a *homogenized algebra of differential operators* ${}^hB$). It is introduced in §3 (respectively, in §9). The algebra ${}^hA$ (respectively, ${}^hB$) is generated over the ground field F by $X_0, \dots, X_n$, $D_1, \dots, D_n$ (respectively, over the field $F(X_1, \dots, X_n)$ by $X_0, D_1, \dots, D_n$). Here X_0 is a new *homogenizing variable*. In the algebra ${}^hA$ (respectively, ${}^hB$), relations (13) in §3 (respectively, (54) in §9) hold true for these generators.

We define the *homogenization* ${}^hI$ of the module I. It is an ${}^hA$-submodule of ${}^hA^l$. The main problem is to estimate the degrees of a system of generators of ${}^hI$. These estimates are central to the paper. They are deduced from Theorem 2 in §7. That theorem is devoted to the problem of solving systems of linear equations over the ring ${}^hA$; we discuss this below in more detail.

The system of generators of ${}^hI$ gives a system of generators of the graded $\mathrm{gr}(A)$-module $\mathrm{gr}(I)$ corresponding to I. But $\mathrm{gr}(A)$ is a polynomial ring. Hence, using Lemma 12 in Appendix 1, we get a double-exponential bound $(dl)^{2^{O(n)}}$ for the stabilization of the Hilbert function of $\mathrm{gr}(I)$ and for the absolute values of the coefficients of the Hilbert polynomial of $\mathrm{gr}(I)$. Therefore, there is a similar bound for the stabilization of the Hilbert functions of I and the coefficients of the Hilbert polynomial of I; see §2.

But the Hilbert functions of the modules I and ${}^hI$ coincide; see §3. Hence, the last bound serves also for the stabilization of the Hilbert functions of ${}^hI$ and the coefficients of the Hilbert polynomial of ${}^hI$. In §5 we introduce the linear order on the monomials in ${}^hA^l$ induced by the initial linear order on the monomials in A^l (the homogenizing variable X_0 is the least possible in this ordering). Next, we define the Janet basis of ${}^hI$ with respect to the induced linear order on the monomials. Such a basis can be obtained by *homogenization of the elements* of a Janet basis of I with respect to the initial linear order; see Lemma 3 (iii).

For every element $f \in {}^hA$, denote by $\mathrm{Hdt}(f) \in {}^hA$ the greatest monomial of the element f; i.e., each monomial of $f - \mathrm{Hdt}(f)$ is less than $\mathrm{Hdt}(f)$ with respect to the induced linear order on the monomials in ${}^hA$. Let $\mathrm{Hdt}({}^hI) = \{\mathrm{Hdt}(f) : f \in {}^hI\}$ be the set of all the greatest monomials of the elements of the module ${}^hI$; see §4. Let ${}^cI \subset {}^cA^l$, see §4, be the module over the polynomial ring ${}^cA = F[X_0, \dots, X_n, D_1, \dots, D_n]$ generated by all monomials in $\mathrm{Hdt}({}^hI)$ (they are viewed now as elements of ${}^cA^l$). Then the Hilbert functions of the modules ${}^hI$ and ${}^cI$ coincide. Thus, we have the same double-exponential estimate as above for the stabilization of the Hilbert function of ${}^cI$ and the coefficients of the Hilbert polynomial of ${}^cI$. Now, using Lemma 13, we get the estimate $(dl)^{2^{O(n)}}$ for the monomial system of generators of ${}^cI$. This gives a bound for the degrees of the elements of the reduced Janet basis of ${}^hI$, and hence, by Lemma 11, also the bound from Theorem 1 for the required Janet basis (respectively, in the case where the initial order is degree-compatible, for the reduced Janet basis) of I. Estimation of the degrees of the elements of the reduced Janet basis in the case where $l = 1$ requires special considerations; see §8.

Remark 1. The question as to whether there is a double-exponential upper bound for the degrees of the elements of the reduced Janet basis with respect to an arbitrary admissible linear order on monomials in the case where $l > 1$ remains open. Note the following description of all admissible linear orders on the monomials in A^l: each linear order corresponds to a rooted tree. But we do not need this description in the present paper.

The problem of solving systems of linear equations over the homogenized Weyl algebra is central to this paper; see Theorem 2. It is studied in §§5–7. A similar problem over the Weyl algebra (without homogenization) was considered in [7]. The principal idea is to try to extend the well-known method of [8], which was developed for the algebra of polynomials, to the homogenized Weyl algebra. There are two principal difficulties with this approach. The first is that in the method from [8] the use of determinants is essential, which should be avoided when we deal with noncommutative algebras. The second is that a kind of Noether normalization theorem is needed in the current situation. Therefore, the analog of the method of [8] requires choosing the leading elements with the smallest possible order ord_{X_0}, where X_0 is a homogenizing variable; see §3.

The bound obtained for the degree of a Janet basis implies a similar bound for the complexity of its construction. Indeed, by Corollary 1 (it is formulated for the case of a Weyl algebra, but a similar statement is true for the case of the algebra of differential operators), one can compute the linear space of all the elements $z \in I$ of degrees bounded from above by $(dl)^{2^{O(n)}}$. Hence, by Theorem 1, a Janet basis of I can be computed by solving linear systems over F of size bounded from above by $(dl)^{2^{O(n)}}$ (merely with the help of enumeration of all monomials of degrees at most $(dl)^{2^{O(n)}}$ that are possible elements of $\mathrm{Hdt}(I)$). After that, within time polynomial in $(dl)^{2^{O(n)}}$ and in the size of the input, by solving linear systems over F one can obtain the reduced Janet basis of I, provided that the upper bound $(dl)^{2^{O(n)}}$ for the degrees of its elements is known; see Theorem 1.

To make our text self-contained, in Appendix 1 (see Lemma 12) we give a short proof of the double-exponential estimate for stabilization of the Hilbert function of a graded module over a graded polynomial ring. The converse of Lemma 12 is also true; see Lemma 13 in Appendix 1. This fact is essential for us. The proof of Lemma 13 involves the classical description of the Hilbert function of a homogeneous ideal in $F[X_0, \dots, X_n]$ in terms of the Macaulay constants $b_{n+2}, \dots, b_1$ and the constant b_0 introduced in [4]. In Appendix 2, we give an independent and instructive proof of Proposition 1, which is similar to Lemma 13. In a sense, Proposition 1 is even stronger than Lemma 13, because to apply it one does not need a bound for the stabilization of the Hilbert function. Of course, a reference to Proposition 1 can be used in place of Lemma 13 in our paper.

§1. Definition of a Janet basis

Let $A = F[X_1, \dots, X_n, D_1, \dots, D_n]$, $n \geq 1$, be a Weyl algebra over a field F. So, A is determined by the following relations:

$$X_v X_w = X_w X_v,\ D_v D_w = D_w D_v,\ D_v X_v - X_v D_v = 1,\ X_v D_w = D_w X_v, \quad v \neq w. \tag{2}$$

By (2), any element $f \in A$ admits a unique representation in the form

$$f = \sum_{i_1,\dots,i_n,j_1,\dots,j_n \geq 0} f_{i_1,\dots,i_n,j_1,\dots,j_n} X_1^{i_1} \cdots X_n^{i_n} D_1^{j_1} \cdots D_n^{j_n}, \tag{3}$$

where all $f_{i_1,\dots,i_n,j_1,\dots,j_n}$ belong to F and only a finite number of the $f_{i_1,\dots,i_n,j_1,\dots,j_n}$ are nonzero. For brevity, we denote $\mathbb{Z}_+ = \{z \in \mathbb{Z} : z \geq 0\}$ and

$$\begin{gathered} i = (i_1, \dots, i_n), \quad j = (j_1, \dots, j_n), \quad f_{i,j} = f_{i_1,\dots,i_n,j_1,\dots,j_n}, \\ X^i = X_1^{i_1} \cdots X_n^{i_n}, \quad D^j = D_1^{j_1} \cdots D_n^{j_n}, \quad f = \sum_{i,j} f_{i,j} X^i D^j, \\ |i| = i_1 + \cdots + i_n, \quad i + j = (i_1 + j_1, \dots, i_n + j_n). \end{gathered} \tag{4}$$

Thus, $i, j \in \mathbb{Z}_+^n$ are multi-indices. By definition, the degree of f is

$$\deg f = \deg_{X_1,\dots,X_n,D_1,\dots,D_n} f = \max\{|i| + |j| \,:\, f_{i,j} \neq 0\}.$$

Let M be a left A-module given by its generators $m_1, \dots, m_l$, $l \geq 0$, and relations

$$\sum_{1 \leq w \leq l} a_{v,w} m_w, \quad 1 \leq v \leq k, \tag{5}$$

where $k \geq 0$ and all $a_{v,w}$ are in A. We assume that $\deg a_{v,w} < d$ for all v, w, where $d \geq 2$. By (5), we have the exact sequence

$$A^k \xrightarrow{\iota} A^l \xrightarrow{\pi} M \to 0 \tag{6}$$

of left A-modules. Denote $I = \iota(A^k) \subset A^l$. If $l = 1$, then I is a left ideal of A and $M = A/I$. In the general case, I is generated by the elements

$$(a_{v,1}, \dots, a_{v,l}) \in A^l, \quad 1 \leq v \leq k.$$

For an integer $m \geq 0$, put

$$A_m = \{a \,:\, \deg a \leq m\}, \quad M_m = \pi(A_m^l), \quad I_m = I \cap A_m^l. \tag{7}$$

So, now A, M, I are filtered modules with filtrations A_m, M_m, I_m, $m \geq 0$, respectively, and the sequence of homomorphisms of vector spaces

$$0 \to I_m \to A_m^l \to M_m \to 0$$

induced by (6) is exact for every $m \geq 0$. The Hilbert function $H(M, m)$ of the module M is defined by the formula

$$H(M, m) = \dim_F M_m, \quad m \geq 0.$$

Each element of A^l can be uniquely represented as an F-linear combination of elements $e_{v,i,j} = (0, \dots, 0, X^i D^j, 0, \dots, 0)$, where $i, j \in \mathbb{Z}_+^n$ are multi-indices, see (4), and the nonzero monomial X^iD^j is at the position v, $1 \le v \le l$. Thus, every element $f \in A^l$ can be represented in the form

$$f = \sum_{v,i,j} f_{v,i,j} e_{v,i,j}, \quad f_{v,i,j} \in F. \tag{8}$$

The elements $e_{v,i,j}$ will be called monomials.

Consider a linear order $<$ on the set of all the monomials $e_{v,i,j}$ or, what is the same, on the set of triples (v, i, j), $1 \le v \le l$, $i, j \in \mathbb{Z}_+^n$. If $f \ne 0$, we put

$$o(f) = \max\{(v, i, j) : f_{v,i,j} \ne 0\}; \tag{9}$$

see (8). Set

$$o(0) = -\infty < o(f)$$

for every $0 \ne f \in A$. The *leading monomial* of an element $0 \ne f \in A^l$ is defined by the formula

$$\mathrm{Hdt}(f) = f_{v,i,j} e_{v,i,j},$$

where $o(f) = (v, i, j)$. Put $\mathrm{Hdt}(0) = 0$. Hence, $o(f - \mathrm{Hdt}(f)) < o(f)$ if $f \ne 0$. For $f_1, f_2 \in A^l$, if $o(f_1) < o(f_2)$, we shall write $f_1 < f_2$. We shall require additionally that

(a) for all multi-indices i, j, i', j' and all $1 \le v \le l$, if $i_1 \le i'_1, \dots, i_n \le i'_n$ and $j_1 \le j'_1, \dots, j_n \le j'_n$, then $(v, i, j) \le (v, i', j')$;

(b) for all multi-indices i, j, i', j', i'', j'' and all $1 \le v, v' \le l$, if $(v, i, j) < (v', i', j')$, then $(v, i + i'', j + j'') < (v', i' + i'', j' + j'')$.

Conditions (a) and (b) imply that, for all $f_1, f_2 \in A^l$ and every nonzero $a \in A$, if $f_1 < f_2$, then $af_1 < af_2$; i.e., the linear order under consideration is compatible with the product. Any linear order on the monomials $e_{v,i,j}$ satisfying (a) and (b) will be called *admissible*. Consider additionally the condition

(c) for all multi-indices i, j, i', j' and all $1 \le v, v' \le l$, if $|i| + |j| < |i'| + |j'|$, then $(v, i, j) < (v', i', j')$.

Any linear order on the monomials $e_{v,i,j}$ satisfying (c) will be called a *degree-compatible* order (in what follows all the degree-compatible linear orders that we consider will also be admissible, i.e., satisfying (a) and (b)).

For every subset $E \subset A^l$ we put $\mathrm{Hdt}(E) = \{\mathrm{Hdt}(f) : f \in E\}$. In particular,

$$\mathrm{Hdt}(I) = \{\mathrm{Hdt}(f) : f \in I\}.$$

Thus, $\mathrm{Hdt}(I)$ is a subset of A^l. By definition, a family $f_1, \dots, f_m$ of elements of I is a *Janet basis* of the module I if and only if

1) $\mathrm{Hdt}(I) = \mathrm{Hdt}(Af_1) \cup \dots \cup \mathrm{Hdt}(Af_m)$.

Next, the Janet basis $f_1, \dots, f_m$ of I is *reduced* if and only if the following conditions are fulfilled.

2) $f_1, \dots, f_m$ does not contain a smaller Janet basis of I.
3) $\mathrm{Hdt}(f_1) > \dots > \mathrm{Hdt}(f_m)$.
4) The coefficient from F of every monomial $\mathrm{Hdt}(f_v)$, $1 \le \alpha \le m$, is 1.
5) Let $f_\alpha = \sum_{v,i,j} f_{\alpha,v,i,j} e_{v,i,j}$ be the representation (3) for f_α, $1 \le \alpha \le m$. Then for all $1 \le \alpha < \beta \le m$, all $1 \le v \le l$, and all multi-indices i, j, the monomial $f_{\alpha,v,i,j} e_{v,i,j}$ does not belong to $\mathrm{Hdt}(Af_\beta \setminus \{0\})$.

Let C denote the ring of polynomials in $X_1, \dots, X_n$, $D_1, \dots, D_n$ with coefficients in F (we can take $C = \mathrm{gr}(A)$; see the next section). For every $f \in A^l$, the monomial $\mathrm{Hdt}(f)$ can be viewed as an element of C^l. To avoid ambiguity, we denote it by $\mathrm{Hdtc}(f) \in C^l$. Now, $f_1, \dots, f_m$ is a Janet basis of the module I if and only if the C-submodule of C^l generated by $\mathrm{Hdtc}(f_\alpha)$, $1 \le \alpha \le m$, contains all the elements $\mathrm{Hdtc}(f)$, $f \in A$. Since the ring C is Noetherian, the module I under consideration admits a Janet basis. Moreover, the reduced Janet basis of I is uniquely determined.

§2. The graded module corresponding to a D-module

Put $A_v = I_v = M_v = 0$ for $v < 0$ and

$$\mathrm{gr}(A) = \bigoplus_{m\geq 0} A_m/A_{m-1}, \quad \mathrm{gr}(I) = \bigoplus_{m\geq 0} I_m/I_{m-1}, \quad \mathrm{gr}(M) = \bigoplus_{m\geq 0} M_m/M_{m-1}.$$

The structure algebra on A induces the structure of a graded algebra on $\mathrm{gr}(A)$. Thus, $\mathrm{gr}(A) = F[X_1, \dots, X_n, D_1, \dots, D_n]$ is an algebra of polynomials with respect to the variables $X_1, \dots, X_n$, $D_1, \dots, D_n$. Next, $\mathrm{gr}(I)$ and $\mathrm{gr}(M)$ are graded $\mathrm{gr}(A)$-modules. Using (7), we get the exact sequences

$$0 \to I_m/I_{m-1} \to (A_m/A_{m-1})^l \to M_m/M_{m-1} \to 0, \quad m \geq 0. \tag{10}$$

The Hilbert function of the module $\mathrm{gr}(M)$ is defined as follows:

$$H(\mathrm{gr}(M), m) = \dim_F M_m/M_{m-1}, \quad m \geq 0.$$

Obviously,

$$H(M,m) = \sum_{0\leq v\leq m} H(\mathrm{gr}(M), v), \quad H(\mathrm{gr}(M), m) = H(M,m) - H(M, m-1) \tag{11}$$

for every $m \geq 0$.

For arbitrary $a \in M$, we denote by $\mathrm{gr}(a) \in \mathrm{gr}(M)$ the image of a in $\mathrm{gr}(M)$.

Lemma 1. *Assume that $b_1, \dots, b_s$ is a system of generators of I. Let $\nu_i = \deg b_i$, $1 \leq i \leq s$. Suppose that*

$$I_m = \Big\{ \sum_{1\leq v\leq \mu} c_v b_v : c_v \in A,\ \deg c_v \leq m - \nu_v,\ 1 \leq i \leq s \Big\} \tag{12}$$

for every $m \geq 0$. Then $\mathrm{gr}(b_1), \dots, \mathrm{gr}(b_s)$ is a system of generators of the $\mathrm{gr}(A)$-module $\mathrm{gr}(I)$.

Proof. This is straightforward. □

§3. Homogenization of the Weyl algebra

Let X_0 be a new variable. Consider the algebra ${}^hA = F[X_0, X_1, \dots, X_n, D_1, \dots, D_n]$ given by the relations

$$\begin{gathered} X_vX_w = X_wX_v, \quad D_vD_w = D_wD_v \quad \text{for all} \quad v, w, \\ D_vX_v - X_vD_v = X_0^2, \quad 1 \leq v \leq n, \quad X_vD_w = D_wX_v \quad \text{for all} \quad v \neq w. \end{gathered} \tag{13}$$

The algebra ${}^hA$ is Noetherian, like the Weyl algebra A. By (13), any element $f \in {}^hA$ can be uniquely represented in the form

$$f = \sum_{i_0, i_1, \dots, i_n, j_1, \dots, j_n \geq 0} f_{i_0,\dots,i_n,j_1,\dots,j_n} X_0^{i_0} \cdots X_n^{i_n} D_1^{j_1} \cdots D_n^{j_n}, \tag{14}$$

where all $f_{i_0,\dots,i_n,j_1,\dots,j_n}$ are in F and only finitely many of $f_{i_0,\dots,i_n,j_1,\dots,j_n}$ are nonzero. Let i, j be multi-indices; see (4). Denote for brevity

$$\begin{gathered} i = (i_1, \dots, i_n), \quad j = (j_1, \dots, j_n), \quad f_{i_0,i,j} = f_{i_0,\dots,i_n,j_1,\dots,j_n}, \\ f = \sum_{i_0,i,j} f_{i_0,i,j} X_0^{i_0} X^i D^j. \end{gathered} \tag{15}$$

By definition,

$$\begin{aligned}
&\deg f = \deg_{X_0,\dots,X_n,D_1,\dots,D_n} f = \max\{i_0 + |i| + |j| \,:\, f_{i_0,i,j} \neq 0\},\\
&\deg_{D_1,\dots,D_n} f = \max\{|j| \,:\, f_{i_0,i,j} \neq 0\},\\
&\deg_{D_\alpha} f = \max\{j_\alpha \,:\, f_{i_0,i,j} \neq 0\}, \quad 1 \le \alpha \le n,\\
&\deg_{X_\alpha} f = \max\{i_\alpha \,:\, f_{i_0,i,j} \neq 0\}, \quad 1 \le \alpha \le n.
\end{aligned}$$

Set $\operatorname{ord} 0 = \operatorname{ord}_{X_0} 0 = +\infty$. If $0 \neq f \in {}^hA$, then we put

$$\operatorname{ord} f = \operatorname{ord}_{X_0} f = \mu \iff f \in X_0^{\mu}({}^hA) \setminus X_0^{\mu+1}({}^hA), \quad \mu \ge 0. \tag{16}$$

For every $z = (z_1, \dots, z_l) \in {}^hA^l$, put

$$\operatorname{ord} z = \min_{1\le i\le l}\{\operatorname{ord} z_i\}, \quad \deg z = \max_{1\le i\le l}\{\deg z_i\}.$$

The quantities $\operatorname{ord} b$ and $\deg b$ are defined similarly for an arbitrary $(k \times l)$-matrix b with coefficients in ${}^hA$. More precisely, here b is viewed as a vector with kl entries.

An element $f \in {}^hA$ is homogeneous if and only if $f_{i_0,i,j} \neq 0$ implies $i_0+|i|+|j| = \deg f$, i.e., f is a sum of monomials of the same degree $\deg f$. The homogeneous degree of a nonzero homogeneous element f is its degree. The homogeneous degree of 0 is not defined (0 belongs to all the homogeneous components of ${}^hA$; see below).

Next, for every integer m, the mth homogeneous component of ${}^hA$ is the F-linear space

$$({}^hA)_m = \left\{\, z \in {}^hA \,:\, z \text{ is homogeneous and } \deg z = m \text{ or } z = 0 \,\right\}.$$

Now ${}^hA$ is a graded ring with respect to the homogeneous degree. By definition, the ring ${}^hA$ is a homogenization of the Weyl algebra A.

We shall consider the category of finitely generated graded modules G over the ring ${}^hA$. Such a module $G = \bigoplus_{m\ge m_0} G_m$ is a direct sum of its homogeneous components G_m, where m, m_0 are integers. Every G_m is a finite-dimensional F-linear space, and $({}^hA)_p G_m \subset G_{p+m}$ for all integers p, m. Let G and G' be two finitely generated graded ${}^hA$-modules; then $\varphi : G \to G'$ is a morphism (of degree 0) of graded modules if and only if φ is a morphism of ${}^hA$-modules and $\varphi(G_m) \subset G'_m$ for every integer m.

An element $z \in {}^hA$ (respectively, $z \in A$) is called a *term* if and only if $z = \lambda z_1 \cdots z_\nu$ for some $0 \neq \lambda \in F$, some integer $\nu \ge 0$, and $z_w \in \{X_0, \dots, X_n, D_1, \dots, D_n\}$ (respectively, $z_w \in \{X_1, \dots, X_n, D_1, \dots, D_n\}$), $1 \le w \le \nu$.

Let $z = \sum_j z_j \in A$ be an arbitrary element of the Weyl algebra A represented as a sum of terms z_j, and let $\deg z = \max_j \deg z_j$. For example, here we can take the representation (3) for z. Then we define the homogenization ${}^hz \in {}^hA$ by the formula

$${}^hz = \sum_j z_j X_0^{\deg z - \deg z_j}.$$

By (2) and (13), the right-hand side of this relation does not depend on the choice of a representation of z as a sum of terms. Hence, ${}^hz$ is well defined. If $z \in {}^hA$, then ${}^az \in A$ is obtained by substituting $X_0 = 1$ in z. Hence, for every $z \in A$ we have ${}^{ah}z = z$, and for every $z \in {}^hA$ we have $z = {}^{ha}z X_0^{\mu}$, where $\mu = \operatorname{ord} z$.

For an element $z = (z_1, \dots, z_l) \in A^l$, we put $\deg z = \max_{1\le i\le l}\{\deg z_i\}$ and

$${}^hz = \left({}^hz_1 X_0^{\deg z - \deg z_1}, \dots, {}^hz_l X_0^{\deg z - \deg z_l}\right) \in {}^hA^l.$$

The degree $\deg a$ and the homogenization ${}^ha$ can be defined similarly for an arbitrary $(k \times l)$-matrix $a = (a_{v,w})_{1\le v\le k,\,1\le w\le l}$ with coefficients in A. More precisely, here a is viewed as a vector with kl entries. Hence, if $b = (b_{v,w})_{1\le v\le k,\,1\le w\le l} = {}^ha$, then $b_{v,w} = {}^ha_{v,w} X_0^{\deg a - \deg a_{v,w}}$ for all v, w.

Next, the mth homogeneous component of ${}^hA^l$ is

$$({}^hA^l)_m = \left\{ {}^hz \,:\, z \in A^l \text{ and } \deg z = m \ \text{ or } z = 0 \right\}.$$

For an F-linear subspace $X \subset A^l$, we let ${}^hX$ be the smallest linear subspace of ${}^hA^l$ containing the set $\{{}^hz \,:\, z \in X\}$. If X is an A-submodule of A^l, then ${}^hX$ is a graded submodule of ${}^hA^l$. The graduation on ${}^hX$ is induced by that of ${}^hA^l$.

For an element $z = (z_1, \dots, z_l) \in {}^hA^l$, put ${}^az = ({}^az_1, \dots, {}^az_l) \in A^l$. For a subset $X \subset {}^hA^l$, put ${}^aX = \{{}^az \,:\, z \in X\} \subset A^l$. If X is an F-linear space, then ${}^aX$ is also an F-linear space. If X is a graded submodule of ${}^hA^l$, then ${}^aX$ is a submodule of A^l.

Now, ${}^hI$ is a graded submodule of ${}^hA^l$, and ${}^{ah}I = I$. Let $({}^hI)_m$ be the mth homogeneous component of ${}^hI$. Then

$$ {}^h(I_m) = \bigoplus_{0 \le j \le m} ({}^hI)_j, \quad m \ge 0, \tag{17}$$

$$ {}^a(({}^hI)_m) = I_m, \quad m \ge 0, \tag{18}$$

and (18) induces an isomorphism $\iota \,:\, ({}^hI)_m \to I_m$ of linear spaces over F. Set ${}^hM = {}^hA^l/{}^hI$. Then ${}^hM$ is a graded ${}^hA$-module, and we have the exact sequence

$$ 0 \to {}^hI \to {}^hA^l \to {}^hM \to 0. \tag{19}$$

Now, for the mth homogeneous component $({}^hM)_m$ of ${}^hM$ we have

$$ ({}^hM)_m = ({}^hA^l)_m/({}^hI)_m \simeq A^l_m/I_m, \tag{20}$$

by the isomorphism ι. We have the exact sequences

$$ 0 \to ({}^hI)_m \to ({}^hA^l)_m \to ({}^hM)_m \to 0, \quad m \ge 0. \tag{21}$$

By definition, the Hilbert function of the module ${}^hM$ is

$$H({}^hM, m) = \dim_F ({}^hM)_m, \quad m \ge 0.$$

By (20), we have $H(M, m) = H({}^hM, m)$ for every $m \ge 0$; i.e., the Hilbert functions of M and ${}^hM$ coincide.

Lemma 2. *Let $b_1, \dots, b_s$ be a system of homogeneous generators of the ${}^hA$-module ${}^hI$. Then*

$$\mathrm{gr}({}^ab_1), \dots, \mathrm{gr}({}^ab_s) \in \mathrm{gr}(A)^l$$

is a system of generators of the $\mathrm{gr}(A)$*-module* $\mathrm{gr}(I)$.

Proof. By (18), we have ${}^a(({}^hI)_m) = I_m$. Now the claim follows from Lemma 1. The lemma is proved. □

§4. The Janet bases of a module and of its homogenization

Each element of ${}^hA^l$ can be uniquely represented as an F-linear combination of elements $e_{v,i_0,i,j} = (0, \dots, 0, X_0^{i_0} X^i D^j, 0, \dots, 0)$, where $0 \le i_0 \in \mathbb{Z}$, $i, j \in \mathbb{Z}_+^n$ are multi-indices, see (4), and the nonzero monomial $X_0^{i_0} X^i D^j$ is at the position v, $1 \le v \le l$. Therefore, every element $f \in {}^hA^l$ can be written in the form

$$ f = \sum_{v,i_0,i,j} f_{v,i_0,i,j} e_{v,i_0,i,j}, \quad f_{v,i_0,i,j} \in F, \tag{22}$$

and only a finite number of $f_{v,i_0,i,j}$ are nonzero. The elements $e_{v,i_0,i,j}$ will be called *monomials*.

In §1, everywhere after the definition of the Hilbert function, we can replace the ring A, the monomials $e_{v,i,j}$, the multi-indices i, i', i'', the triples (v, i, j) and (v, i', j'), the module I, and so on by the ring ${}^hA$, the monomials $e_{v,i_0,i,j}$, the pairs (i_0, i), (i'_0, i'),

(i''_0, i'') (they are used without parentheses), the quadruples (v, i_0, i, j), (v, i'_0, i', j'), the homogenization ${}^hI$, and so on, respectively. This gives us the definitions of $o(f)$, $\mathrm{Hdt}(f)$ for $f \in {}^hA^l$, new conditions (a) and (b) that define admissible linear orders on the monomials of ${}^hA^l$, a new condition (c) and the definition of the degree-compatible linear order, new conditions 1)–5), and the definitions of the set $\mathrm{Hdt}({}^hI)$, the Janet basis, and the reduced Janet basis of ${}^hI$. For example, $o(0) = +\infty$, $\mathrm{Hdt}(0) = 0$, and if $0 \neq f \in {}^hA^l$, then

$$\begin{aligned} o(f) &= \max\{(v, i_0, i, j) \,:\, f_{v,i_0,i,j} \neq 0\}, \\ \mathrm{Hdt}(f) &= f_{v,i_0,i,j} e_{v,i_0,i,j}, \quad \text{where} \quad o(f) = (v, i_0, i, j), \\ \mathrm{Hdt}({}^hI) &= \{\mathrm{Hdt}(f) \,:\, f \in {}^hI\}, \end{aligned}$$

while the new conditions (a) and (b) look like this:

(a) for all indices i_0, i'_0, all multi-indices i, j, i', j', and all $1 \leq v \leq l$, if $i_0 \leq i'_0$, $i_1 \leq i'_1, \dots, i_n \leq i'_n$, and $j_1 \leq j'_1, \dots, j_n \leq j'_n$, then $(v, i_0, i, j) \leq (v, i'_0, i', j')$;

(b) for all indices i_0, i'_0, i''_0, all multi-indices i, j, i', j', i'', j'', and all $1 \leq v, v' \leq l$, if $(v, i_0, i, j) < (v', i'_0, i', j')$, then $(v, i_0 + i''_0, i + i'', j + j'') < (v', i'_0 + i''_0, i' + i'', j' + j'')$.

The existence of a Janet basis of ${}^hI$ and the uniqueness of the reduced Janet basis with respect to an admissible linear order are proved much as the existence of a Janet basis of I and the uniqueness of the reduced Janet basis of I; see §1. The Janet basis of ${}^hI$ is homogeneous if and only if it consists of homogeneous elements of ${}^hA^l$. Since the module ${}^hI$ is homogeneous, the family of homogeneous components of any Janet basis of ${}^hI$ is a homogeneous Janet basis of ${}^hI$. Hence, the reduced Janet basis of ${}^hI$ is homogeneous (here we leave the details to the reader).

Let $<$ be an admissible linear order on the monomials in A^l, or, what is the same, on the triples (v, i, j); see §1. Thus, this order satisfies conditions (a) and (b). We define a linear order on the monomials $e_{v,i_0,i,j}$, or, what is the same, on the quadruples (v, i_0, i, j). This linear order is induced by $<$ on the triples (v, i, j) and will be denoted again by $<$. Namely, for two quadruples (v, i_0, i, j) and (v', i'_0, i', j') we put $(v, i_0, i, j) < (v', i'_0, i', j')$ if and only if $(v, i, j) < (v', i', j')$, or $(v, i, j) = (v', i', j')$ but $i_0 < i'_0$. Observe that this induced linear order satisfies conditions (a) and (b) (in the new sense).

Remark 2. If $f_1, \dots, f_m$ is a Janet basis of I (respectively, a homogeneous Janet basis of ${}^hI$) satisfying 1)–4), then there are unique $c_{\alpha,\beta} \in A$ (respectively, homogeneous $c_{\alpha,\beta} \in {}^hA$), $1 \leq \alpha < \beta \leq m$, such that the elements

$$f_\alpha + \sum_{\alpha < \beta \leq m} c_{\alpha,\beta} f_\beta, \quad 1 \leq \alpha \leq m,$$

form a reduced Janet basis of I (respectively, a reduced homogeneous Janet basis of ${}^hI$); cf. [3].

Obviously, an admissible linear order $<$ on the monomials in A^l (respectively, in ${}^hA^l$) is degree-compatible if and only if for any two monomials z_1, z_2 the inequality $\deg z_1 < \deg z_2$ implies $z_1 < z_2$.

Lemma 3. *The following assertions are true.*

(i) *Let $f_1, \dots, f_m$ be a* (*reduced*) *Janet basis of I with respect to the linear order $<$ and suppose that the order $<$ is degree-compatible. Then ${}^hf_1, \dots, {}^hf_m$ is a* (*reduced*) *homogeneous Janet basis of the module ${}^hI$ with respect to the induced linear order $<$.*

(ii) *Conversely, suppose that the initial order* $<$ *is degree-compatible, and* $g_1, \dots, g_m$ *is a (reduced) homogeneous Janet basis of the module* ${}^hI$ *with respect to the induced linear order* $<$. *Then* ${}^ag_1, \dots, {}^ag_m$ *is a (reduced) Janet basis of* I *with respect to the linear order* $<$.

(iii) *Suppose that the initial order* $<$ *is arbitrary admissible. Let* $g_1, \dots, g_m$ *be a homogeneous Janet basis of the module* ${}^hI$ *with respect to the induced linear order* $<$. *Then* ${}^ag_1, \dots, {}^ag_m$ *is a Janet basis of* I *with respect to the linear order* $<$. *Moreover,* ${}^{ha}g_w = g_w$ *for all* $1 \le w \le m$.

Proof. This follows immediately from the definitions. □

Let $f \in {}^hA^l$, and let the module ${}^hI$ be as above. We show that there is a unique element $g \in {}^hA^l$ such that

$$g = \sum_{v,i_0,i,j} g_{v,i_0,i,j} e_{v,i_0,i,j}, \quad g_{v,i_0,i,j} \in F, \tag{23}$$

$f - g \in {}^hI$, and if $g_{v,i_0,i,j} \neq 0$, then $e_{v,i_0,i,j} \notin \mathrm{Hdt}({}^hI)$. Indeed, if there are two such elements $g \neq g'$, then $0 \neq g - g' \in {}^hI$, but $\mathrm{Hdt}(g - g') \notin \mathrm{Hdt}({}^hI)$, and we get a contradiction. To prove the existence of g, we may assume without loss of generality that f is homogeneous and show additionally that the sum on the left in (23) is taken over (v, i_0, i, j) such that $i_0 + |i| + |j| = \deg f$. We can write

$$f = \sum_{v,i_0,i,j} f_{v,i_0,i,j} e_{v,i_0,i,j}, \quad f_{v,i_0,i,j} \in F, \quad i_0 + |i| + |j| = \deg f.$$

We use induction on the number $\nu(f)$ of quadruples (v, i_0, i, j) in the last sum such that $e_{v,i_0,i,j} \in \mathrm{Hdt}({}^hI)$ and $e_{v,i_0,i,j} \le \mathrm{Hdt}(f)$. If $\nu(f) > 0$, then there is a homogeneous $z \in {}^hI$ such that $\mathrm{Hdt}(z) = \mathrm{Hdt}(f)$, $\deg z = \deg f$. Then $\nu(f - z) < \nu(f)$. The required assertion is proved.

The element g as in (23) is called the *normal form of* f with respect to the module ${}^hI$. We denote $g = \mathrm{nf}({}^hI, f)$. Obviously, $\mathrm{nf}({}^hI, ({}^hA^l)_m) \subset ({}^hA^l)_m$ is a linear subspace, and

$$\dim_F \mathrm{nf}({}^hI, ({}^hA^l)_m) = l\binom{m+2n}{2n} - H({}^hI, m) = H({}^hA^l/{}^hI, m).$$

Let ${}^cA = F[X_0, \dots, X_n, D_1, \dots, D_n]$ denote the polynomial ring in the variables $X_0, \dots, X_n, D_1, \dots, D_n$. Each monomial $e_{v,i_0,i,j}$ can also be viewed as an element of ${}^cA^l$. Hence, $\mathrm{Hdt}(f)$ can be viewed as an element of ${}^cA^l$ for every $f \in {}^hA^l$. To avoid ambiguity, we shall denote it by $\mathrm{Hdtc}(f) \in {}^cA^l$. Put $\mathrm{Hdtc}({}^hI) = \{\mathrm{Hdtc}(f) : f \in {}^hI\}$. So, the sets $\mathrm{Hdt}({}^hI)$ and $\mathrm{Hdtc}({}^hI)$ are in one-to-one correspondence.

We denote by ${}^cI \subset {}^cA^l$ the graded submodule of ${}^cA^l$ generated by $\mathrm{Hdtc}({}^hI)$. It is easily seen that the set of monomials from the module ${}^cI$ coincides with $\mathrm{Hdtc}({}^hI) \setminus \{0\}$. Next, for every $m \ge 0$, the F-linear space ${}^cI_m$ of homogeneous elements is generated by the monomials $e_{v,i_0,i,j}$ such that there is $0 \neq f \in {}^hI_m$ with $o(f) = (v, i_0, i, j)$. For the Hilbert function, we have

$$H({}^cI, m) = \dim_F\{(z_1, \dots, z_l) \in {}^cI : \forall i\, (\deg z_i = m \text{ or } z_i = 0)\},$$
$$H({}^cA^l/{}^cI, m) = l\binom{m+2n}{2n} - H({}^cI, m).$$

Let $f \in {}^cA^l$, and let the module ${}^cI$ be as above. Then there is a unique element $g \in {}^cA^l$ such that

$$g = \sum_{v,i_0,i,j} g_{v,i_0,i,j} e_{v,i_0,i,j}, \quad g_{v,i_0,i,j} \in F,$$

$f - g \in {}^hI$, and if $g_{v,i_0,i,j} \neq 0$, then $e_{v,i_0,i,j} \notin \mathrm{Hdtc}({}^hI)$ (the proof is similar to that of the existence and uniqueness of g in (23)). The element g is called the *normal form* of f with respect to the module ${}^cI$; see [4]. We denote $g = \mathrm{nf}({}^cI, f)$. Obviously, $\mathrm{nf}({}^cI, ({}^cA^l)_m) \subset ({}^cA^l)_m$ is a linear subspace, and

$$\dim_F \mathrm{nf}({}^cI, ({}^cA^l)_m) = l\binom{m+2n}{2n} - H({}^cI, m) = H({}^cA^l/{}^cI, m).$$

Since, by the definitions given above, the F-linear spaces $\mathrm{nf}({}^cI, ({}^cA^l)_m)$ and $\mathrm{nf}({}^hI, ({}^hA^l)_m)$ are generated by the same monomials, for every $m \geq 0$ we have

$$\dim_F \mathrm{nf}({}^cI, ({}^cA^l)_m) = \dim_F \mathrm{nf}({}^hI, ({}^hA^l)_m),$$
$$H({}^hA^l/{}^hI, m) = H({}^cA^l/{}^cI, m), \quad H({}^hI, m) = H({}^cI, m).$$

Therefore, see §3,

$$H(I, m) = H({}^cI, m), \quad m \geq 0. \tag{24}$$

§5. Bound for the kernel of a matrix over the homogenized Weyl algebra

Lemma 4. *Let $k \geq 1$ and $l \geq 1$ be integers. Let $b = (b_{i,j})_{1\leq i\leq k,\, 1\leq j\leq l}$ be a matrix, where $b_{i,j} \in {}^hA$ are homogeneous elements for all i, j. Suppose $\deg b_{i,j} < d$, $d \geq 2$, for all i, j. Assume that there are integers $d_j \geq 0$, $1 \leq i \leq k$, and $d'_i \geq 0$, $1 \leq j \leq l$, such that*

$$\deg b_{i,j} = d_i - d'_j \tag{25}$$

for all nonzero $b_{i,j}$, and that, moreover, the d'_j are chosen to be minimal possible (this means that there are no integers $\widetilde{d}_i, \widetilde{d}'_j$ similar to d_i, d'_j such that $\widetilde{d}'_j \leq d'_j$ for all $1 \leq j \leq l$ and at least one of the inequalities is strict). Then $d_i < \min\{k+1, l\}d$ and $d'_j < \min\{k, l-1\}d$ for all i, j.

Next, assume that $k = l - 1$. Then there are homogeneous elements $z_1, \ldots, z_l \in {}^hA$ such that $(z_1, \ldots, z_l) \neq (0, \ldots, 0)$ and

$$\sum_{1\leq j\leq l} b_{i,j} z_j = 0, \quad 1 \leq i \leq l-1. \tag{26}$$

There is an integer $\mu \geq 0$ such that, for all $1 \leq j \leq l-1$, if $z_j \neq 0$, then $\deg z_j = \mu + d'_j$, and hence, all nonzero $b_{i,j}z_j$ have one and the same degree depending only on i. Furthermore,

$$\deg z_j \leq (2n+1)l \max_{1\leq i\leq k}\{d_i\} < (2n+1)l^2 d, \quad 1 \leq j \leq l. \tag{27}$$

Moreover, if all $b_{i,j}$ do not depend on X_n (i.e., they can be represented as sums of monomials that do not contain X_n), then the elements $z_1, \ldots, z_l$ can be chosen so as to satisfy additionally the same property. Finally, dividing by an appropriate power of X_0, we can assume without loss of generality that $\min\{\mathrm{ord}\, z_i : 1 \leq i \leq l\} = 0$.

Proof. First, we prove that $d_i < \min\{k+1, l\}d$ and $d'_j < \min\{k, l-1\}d$ for all i, j and arbitrary $k, l \geq 1$. We define an equivalence relation on the set of pairs $P = \{(v, w) : 1 \leq v \leq k,\ 1 \leq w \leq l, \text{ and } b_{v,w} \neq 0\}$ as follows. Put $(v, w) \sim (v', w')$ if and only if in P there is a sequence of pairs $(v_1, w_1), \ldots, (v_\nu, w_\nu)$, $\nu \geq 1$, such that

1) $(v, w) = (v_1, w_1)$, $(v', w') = (v_\nu, w_\nu)$,
2) $v_\alpha = v_{\alpha+1}$ or $w_\alpha = w_{\alpha+1}$ for every $1 \leq \alpha \leq \nu - 1$.

Let $\pi \subset P$ be an equivalence class with respect to $\sim$. Then there is a pair $(p, q) \in \pi$ such that $d'_q = 0$, because the numbers d'_j are chosen to be minimal possible. Moreover, for all $(v, w), (v', w') \in \pi$, a sequence $(v_1, w_1), \ldots, (v_\nu, w_\nu)$ as above can always be chosen so as to possess the following five properties:

3) $(v_\alpha, w_\alpha) \neq (v_{\alpha+1}, w_{\alpha+1})$ for every $1 \le \alpha \le \nu - 1$;
4) if $v_\alpha = v_{\alpha+1}$, then $w_{\alpha+1} = w_{\alpha+2}$ for every $1 \le \alpha \le \nu - 2$;
5) if $w_\alpha = w_{\alpha+1}$, then $v_{\alpha+1} = v_{\alpha+2}$ for every $1 \le \alpha \le \nu - 2$;
6) for all $1 \le \alpha, \beta \le \nu$, if $\beta \notin \{\alpha - 1, \alpha, \alpha + 1\}$, then $v_\beta \neq v_\alpha$;
7) for all $1 \le \alpha, \beta \le \nu$, if $\beta \notin \{\alpha - 1, \alpha, \alpha + 1\}$, then $w_\beta \neq w_\alpha$

(we leave the details to the reader). Now conditions 1)–7) imply that the number of pairs satisfies

$$\#\{(w_\alpha, w_{\alpha+1}) : w_\alpha \neq w_{\alpha+1} \,\&\, 1 \le \alpha \le \nu - 1\} \le \min\{k, l - 1\}.$$

Next, if $w_\alpha \neq w_{\alpha+1}$, then $v_{\alpha+1} = v_\alpha$ and $|d'_{w_{\alpha+1}} - d'_{w_\alpha}| = |\deg b_{v_{\alpha+1}, w_{\alpha+1}} - \deg b_{v_\alpha, w_\alpha}| < d$. Hence, $d'_{w_\nu} < \min\{k, l-1\}d + d'_{w_1}$. For $(v_1, w_1) = (p, q)$ and an arbitrary $(v, w) = (v_\nu, w_\nu) \in \pi$ we get $d'_w < \min\{k, l - 1\}d$. Finally, $\deg b_{v,w} = d_v - d'_w < d$ implies $d_v < \min\{k + 1, l\}d$. The required inequalities are proved. □

Now, suppose that $\deg b_{i,j} = \deg b$ for all nonzero $b_{i,j}$ and $k = l - 1$. We prove the existence of $z_1, \dots, z_l$ and obtain an estimate for $\deg z_j$ in this case. Consider the linear mapping

$$\begin{gathered} ({}^hA)^l_{m-\deg b} \longrightarrow ({}^hA)^{l-1}_m, \\ (\, z_1, \dots, z_l \,) \mapsto \Big(\sum_{1 \le j \le l} b_{i,j} z_j \Big)_{1 \le i \le l-1}. \end{gathered} \tag{28}$$

If

$$l \binom{m - \deg b + 2n}{2n} > (l - 1) \binom{m + 2n}{2n}, \tag{29}$$

then the kernel of (28) is nonzero. But (29) is true provided

$$\prod_{1 \le w \le 2n} \Big(1 + \frac{\deg b}{m + w - \deg b} \Big) < \frac{l}{l - 1}. \tag{30}$$

Next, (30) is true if $(1 + \deg b/(m - \deg b))^{2n} < l/(l - 1)$. The last inequality follows from $m \ge (2n + 1) \deg b / \log(l/(l - 1))$, and hence, also from $m \ge (2n + 1)l \deg b$. Thus, the existence of $z_1, \dots, z_l$ is proved, and moreover, all the nonzero z_j have one and the same degree $((2n + 1)l - 1) \deg b$, which does not depend on j. Observe that, in the case under consideration we have proved a stronger inequality: $\deg z_j < ((2n + 1)l - 1)d$ for all $1 \le j \le l$.

Finally, let $k = l - 1$ and suppose that the degrees $\deg b_{i,j}$ are arbitrary but satisfy (25). Multiplying the ith equation in (26) by $X_0^{\max_w\{d_w\} - d_i}$, we may assume without loss of generality that all d_i are equal. We substitute $z_j X_0^{d'_j}$ for z_j in (26). Now the degrees of all the nonzero coefficients of the resulting system are equal to $\max_{1 \le i \le k}\{d_i\}$ and are less than ld. If, in the case of $\deg b_{i,j} = \deg b$ considered above, we replace the bound d by $\max_{1 \le i \le k}\{d_i\} < ld$, we get the required $z_1, \dots, z_l$ such that $\deg z_j = ((2n + 1)l - 1) \max_{1 \le i \le k}\{d_i\} + d'_j$ or $z_j = 0$ for all $1 \le j \le l$, together with the estimate

$$\deg z_j \le ((2n + 1)l - 1) \max_{1 \le i \le k}\{d_i\} + d'_j \le (2n + 1)l \max_{1 \le i \le k}\{d_i\} < (2n + 1)l^2 d$$

for all j.

Suppose that $a_1, \dots, a_l$ do not depend on X_n. We represent $z_i = \sum_j z_{i,j} X_n^j$, $1 \le i \le l$, where all $z_{i,j}$ do not depend on X_n. Let $\alpha = \max_i\{\deg_{X_n} z_i\}$. Obviously, in this case we can replace $(z_1, \dots, z_l)$ by $(z_{1,\alpha}, \dots, z_{l,\alpha})$. The lemma is proved. □

Remark 3. Lemma 4 remains true if condition (26) in its statement is replaced by

$$\sum_{1\le j\le l} z_j b_{i,j} = 0, \quad 1\le i\le l-1. \tag{31}$$

The proof is similar.

Remark 4. Let the elements $b_{i,j}$ be as in Lemma 4. Notice that there are integers $\delta'_i \ge 0$, $1\le i\le k$, and $\delta_j\ge 0$, $1\le j\le l$, such that

$$\deg b_{i,j} = \delta_j - \delta'_i$$

for all nonzero $b_{i,j}$, and $\min_{1\le i\le k}\{\delta'_i\} = 0$. Namely, $\delta'_i = -d_i + \max_{1\le i\le k}\{d_i\}$, $\delta_j = -d'_j + \max_{1\le i\le k}\{d_i\}$.

Remark 5. Let $b_{i,j}\in {}^hA$, $1\le i\le k$, $1\le j\le l$, be homogeneous elements. Suppose there are integers $\widetilde{d}_i$, $1\le i\le k$, and $\widetilde{d}'_j$, $1\le j\le l$, such that $\deg b_{i,j} = \widetilde{d}_i - \widetilde{d}'_j$ for all nonzero $b_{i,j}$. Then there are integers $d_i\ge 0$, $1\le i\le k$, and $d'_j\ge 0$, $1\le j\le l$, such that (25) is fulfilled for all nonzero $b_{i,j}$.

§6. Reducing a matrix with coefficients in ${}^hA$ to a trapezoidal form

Let b be a matrix as in Lemma 4, and let integers $k,l\ge 1$ be arbitrary. Thus, (25) is true. Let $b=(b_1,\dots,b_l)$, where $b_1,\dots,b_l\in {}^hA^k$, be the columns of the matrix b (note that in Lemmas 1 and 2 all b_i are rows of size l, so that now we change the notation). By definition, $b_1,\dots,b_l$ are linearly independent over ${}^hA$ from the right (or linearly independent if this will not lead to ambiguity; in what follows in this paper, if it is not stated otherwise, "linearly independent" will mean "linearly independent from the right") if and only if for all $z_1,\dots,z_l\in {}^hA$ the relation $b_1z_1+\dots+b_lz_l=0$ implies $z_1=\dots=z_l=0$. By (25), in this definition we may consider only homogeneous $z_1,\dots,z_l$. From an arbitrary family $b_1,\dots,b_l$ as in Lemma 4 (with arbitrary k,l) we can choose a maximal subfamily $b_{i_1},\dots,b_{i_r}$, linearly independent from the right. By Lemma 4, we have $r\le k$. It turns out that r does not depend on the choice of a subfamily. More precisely, the following statement is true.

Lemma 5. *Let $c_j=\sum_{1\le i\le l} b_i z_{i,j}$, $1\le j\le r_1$, where $z_{i,j}\in {}^hA$ are homogeneous elements. Suppose that there are integers d''_j, $1\le j\le r_1$, such that, for all i,j, $\deg z_{i,j} = d'_i - d''_j$ if $z_{i,j}\ne 0$. Assume that c_j, $1\le j\le r_1$, are linearly independent over ${}^hA$ from the right. Then $r_1\le r$, and if $r_1<r$, then there are $c_{r_1+1},\dots,c_r\in\{b_{i_1},\dots,b_{i_r}\}$ such that the c_j, $1\le j\le r$, are linearly independent over ${}^hA$ from the right.*

Proof. The proof is similar to the case of vector spaces over a field, and we leave it to the reader. □

We denote $r=\operatorname{rankr}\{b_1,\dots,b_l\}$ and call this number the rank from the right of $b_1,\dots,b_l$. In a similar way we can define the rank from the left of $b_1,\dots,b_l$, denoting it by $\operatorname{rankl}\{b_1,\dots,b_l\}$. It is not difficult to construct examples when $\operatorname{rankr}\{b_1,\dots,b_l\}$ $\ne \operatorname{rankl}\{b_1,\dots,b_l\}$. Our aim in this section is to prove the following result.

Lemma 6. *Let b be a matrix with homogeneous entries in ${}^hA$ and satisfying* (25)*; see above. Suppose that $d\ge 2$ and $\deg b_{i,j}<d$ for all i,j. Let $l_1=\operatorname{rankr}\{b_1,\dots,b_l\}$, and let $b_1,\dots,b_{l_1}$ be linearly independent. Hence, $0\le l_1\le l$ and $k\ge l_1$. Then there is a matrix $(z_{j,r})_{1\le j,r\le l_1}$ (if $l_1=0$, then this matrix is empty) with homogeneous entries $z_{j,r}\in {}^hA$,*

and a square permutation matrix σ of size k, with the following properties:

(i) *There are integers d''_r, $1 \le r \le l_1$, such that for all $1 \le j, r \le l_1$ we have $\deg z_{j,r} = d'_j - d''_r$ or $z_{j,r} = 0$, and hence, all the nonzero elements $b_{i,j} z_{j,r}$, $1 \le j \le l_1$, have one and the same degree $d_i - d''_r$ depending only on i, r. Next,*

$$\deg z_{j,r} \le (2n+1) l_1 \max_{1 \le i \le k} \{d_i\} < (2n+1) l_1^2 d. \tag{32}$$

(ii) *Introduce the matrix $e = (e_{i,j})_{1 \le i \le k,\, 1 \le j \le l_1} = \sigma(b_1, \dots, b_{l_1}) z$, where $(b_1, \dots, b_{l_1})$ is the matrix formed by the first l_1 columns of the matrix b. Then*

$$e = \begin{pmatrix} e' \\ e'' \end{pmatrix},$$

where $e' = \mathrm{diag}(e'_{1,1}, \dots, e'_{l_1,l_1})$ is a diagonal matrix with l_1 columns, and each $e'_{j,j}$, $1 \le j \le l_1$, is nonzero.

(iii) $\mathrm{ord}\, e_{i,j} \ge \mathrm{ord}\, e'_{j,j}$ *for all $1 \le i \le k$, $1 \le j \le l_1$.*

Moreover, if all $a_{i,j}$ (and hence, all $b_{i,j}$) do not depend on X_n (i.e., they can be represented as sums of monomials that do not contain X_n), then the $z_{j,r}$ can be chosen so as to possess the same property. Finally, dividing by an appropriate power of X_0, we may assume without loss of generality that $\min\{\mathrm{ord}\, z_{j,r} : 1 \le j \le l_1\} = 0$ *for every $1 \le r \le l_1$.*

Proof. First, we show how to construct z, e, and σ satisfying (ii) and (iii). We shall use a kind of Gauss elimination and Lemma 4. Namely, we transform the matrix e. To start with, we put

$$e = (e_1, \dots, e_{l_1}) = (b_1, \dots, b_{l_1}).$$

We shall perform some ${}^hA$-linear transformations of columns and permutations of rows of the matrix e and replace e each time by the resulting matrix. These transformations will not change the rank from the right of the family of columns of e. At the end, we get a matrix e satisfying the required properties (ii), (iii).

We have $\mathrm{rankr}(e) = l_1$. If $l_1 = 0$, i.e., if e is an empty matrix, then this is the end of the construction: z is an empty matrix. Suppose that $l_1 > 0$. We choose indices $1 \le i_0 \le k$ and $1 \le j_0 \le l_1$ such that $\mathrm{ord}\, e_{i_0,j_0} = \min_{1 \le j \le l_1}\{\mathrm{ord}\, e_j\}$. Permuting rows and columns of e, we may assume without loss of generality that $(i_0, j_0) = (1, 1)$.

By Lemma 4, we get elements $w_{i,1}, w_{i,i} \in {}^hA$ of degree at most $(2n+1)4d$ such that $e_{1,1} w_{1,i} = e_{1,i} w_{i,i}$, $1 \le i \le l_1$, and $\mathrm{ord}\, w_{i,i} = 0$ for every $1 \le i \le l_1$. Set $w' = (-w_{1,2}, \dots, -w_{1,l_1})$, and let $w'' = \mathrm{diag}(w_{2,2}, \dots, w_{l_1,l_1})$ be the diagonal matrix. Let

$$w = \begin{pmatrix} 1 & w' \\ 0 & w'' \end{pmatrix}$$

be the square matrix with l_1 rows. We replace e by ew. Now

$$e = \begin{pmatrix} e_{1,1} & 0 \\ E_{2,1} & E_{2,2} \end{pmatrix},$$

where $E_{2,2}$ has $l_1 - 1$ columns, and

$$\min_{1 \le j \le l_1}\{\mathrm{ord}\, b_j\} = \mathrm{ord}\, e_{1,1} = \min_{1 \le j \le l_1}\{\mathrm{ord}\, e_j\} \tag{33}$$

(for the new matrix e).

We apply recursively the described construction to the matrix $E_{2,2}$ in place of e. So, using only linear transformations of columns with indices $2, \dots, l_1$ and permutation of rows with indices $2, \dots, k$, we transform e to

$$\sigma e \tau = \begin{pmatrix} e_{1,1} & 0 \\ E'_{2,1} & E'_{2,2} \\ E''_{2,1} & E''_{2,2} \end{pmatrix}, \quad \tau = \begin{pmatrix} 1 & 0 \\ 0 & \tau' \end{pmatrix},$$

where σ is a permutation matrix and τ' is a square matrix with l_1-1 rows (it transforms $E_{2,2}$), the matrix $E'_{2,2}=\operatorname{diag}(e_{2,2},\dots,e_{l_1,l_1})$ is a diagonal matrix with $l_1-1\geq 0$ columns, and all the elements $e_{2,2},\dots,e_{l_1,l_1}\in {}^hA$ are nonzero. We shall assume without loss of generality that $\sigma=1$ is the identity matrix. We replace e by $e\tau$. Condition (iii) is fulfilled for the resulting matrix e, and moreover, by (iii) applied recursively to $(E_{2,2},E'_{2,2},E''_{2,2})$ in place of (e,e',e''), and by (33), the same equalities in (33) are satisfied for the new matrix e.

Let $E'_{2,1}=(e_{2,1},\dots,e_{l_1,1})^t$, where t denotes transposition. By Lemma 4, there are nonzero elements $v_{1,1},\dots,v_{l_1,1}\in {}^hA$ of degrees at most

$$(2n+1)(\max\{\deg e_{i,j}\,:\,1\leq i\leq l_1,\ j=1,i\}+1)l_1^2 \tag{34}$$

such that $e_{i,1}v_{1,1}=e_{i,i}v_{i,1}$ and $\min\{\operatorname{ord} v_{1,i}\,:\,1\leq i\leq l_1\}=0$ for all $1\leq i\leq l_1-1$. Let $v'=(-v_{2,1},\dots,-v_{l_1,1})^t$, and let v'' be the identity matrix of size l_1-1. Put

$$v=\begin{pmatrix}v_{1,1} & 0\\ v' & v''\end{pmatrix}.$$

We replace e by ev and put $z=w\tau v$. Recall that, without loss of generality, $\sigma=1$ is the identity permutation. We have $e=(b_1,\dots,b_{l_1})z$. These Gauss elimination transformations of e do not change the rank from the right of the family of columns of e. This can be proved easily by using recursion on l; cf. Lemma 8 below. Now the matrix e satisfies the required conditions (ii) and (iii), and $\sigma=1$.

Now we change the notation. We denote the matrix z obtained so far by z'. Let $z'=(z'_1,\dots,z'_{l_1})$, where z'_j is the jth column of z'. Our aim now is to prove the existence of a matrix z satisfying (i)–(iii). By Lemma 4, for every $1\leq r\leq l_1$, there are homogeneous elements $z_{j,r}\in {}^hA$, $1\leq j\leq l_1$, such that $(z_{1,r},\dots,z_{l_1,r})\neq(0,\dots,0)$, $\deg z_{j,r}=d'_r+\mu_r$ or $z_{j,r}=0$ for all $1\leq j\leq l_1$,

$$\sum_{1\leq j\leq l_1} b_{i,j}z_{j,r}=0 \quad\text{for every}\quad 1\leq i\leq l_1,\ i\neq r, \tag{35}$$

and estimates (32) for the degrees are true. Put $z=(z_{j,r})_{1\leq j,r\leq l_1}$ and $d''_r=-\mu_r$. Let $z=(z_1,\dots,z_{l_1})$, where z_j is the jth column of z. Hence, $z_j=(z_{1,j},\dots,z_{l_1,j})^t$.

Lemma 7. *For every $1\leq r\leq l_1$ we have*

$$\sum_{1\leq j\leq l_1} b_{r,j}z_{j,r}\neq 0, \tag{36}$$

and for every $1\leq r\leq l_1$ there are nonzero homogeneous elements $g'_r,g_r\in {}^hA$ such that $z'_rg'_r=z_rg_r$.

Proof. Consider the matrix (z',z_r) with l_1 rows and l_1+1 columns. Using Lemma 4, we see that there are homogeneous elements $h_1,\dots,h_{l_1+1}\in {}^hA$ (depending on r) such that $(h_1,\dots,h_{l_1+1})\neq(0,\dots,0)$ and the following is fulfilled. Denote $h=(h_1,\dots,h_{l_1+1})^t$ and $h'=(h_1,\dots,h_{l_1})^t$. Then

$$z'h'+z_rh_{l_1+1}=0 \tag{37}$$

(at present, we do not need any estimate on degrees from Lemma 4; we only need the existence of h). Denote by b'' the submatrix formed by the first l_1 rows of the matrix $(b_1,\dots,b_{l_1})$. Multiplying (37) by b'' from the left, we get

$$b''z'h'+b''z_rh_{l_1+1}=0. \tag{38}$$

But $b''z'$ is a diagonal matrix with nonzero elements on the diagonal; see (ii) (with z' in place of z). Hence, by (35) and (38), $h_j=0$ for every $j\neq r$.

Now suppose that $h_r = 0$. Then $h' = 0$. Since $z_r \neq 0$, we have $h_{l_1+1} = 0$ by (3.7). Hence, $h = (0, \dots, 0)^t$, a contradiction.

Suppose that $h_{l_1+1} = 0$. Then by (38) we have $h_r = 0$. Hence $h = (0, \dots, 0)^t$ and again we get a contradiction.

Thus, $h_r \neq 0$ and $h_{l_1+1} \neq 0$. Now (38) implies (36). Put $g'_r = h_r$ and $g_r = -h_{l_1+1}$. We have $z'_r g'_r = z_r g_r$ by (37). The lemma is proved. □

We return to the proof of Lemma 6. Now (i)–(iii) are satisfied by Lemma 7. The last assertions of Lemma 6 are proved much as those in Lemma 4. Lemma 6 is proved. □

§7. An algorithm for solving linear systems with coefficients in ${}^hA$

Let $u = (u_1, \dots, u_l) \in {}^hA^l$. Suppose that all nonzero u_j are homogeneous elements of degree $-d'_j + \rho$ for an integer ρ, and that $-d'_j + \rho < d'$ for an integer $d' > 1$. Let $b = (b_{i,j})_{1\le i\le k,\, 1\le j\le l}$ be a matrix as in Lemma 6, having k rows and l columns (but now k and l are arbitrary). So, $\deg b_{i,j} = d_i - d'_j < d$ for all i, j and $d \geq 2$. Let $Z = (Z_1, \dots, Z_k)$ be unknowns. Consider the linear system

$$\sum_{1\le i\le k} Z_i b_{i,j} = u_j, \quad 1 \le j \le l, \tag{39}$$

or, what is the same,

$$Zb = u.$$

Denote

$$\operatorname{ord} u = \min_{1\le i\le l} \{\operatorname{ord} u_i\}. \tag{40}$$

Similar notation will be used for other vectors and matrices. In this section we describe an algorithm for solving linear systems over ${}^hA$ and *prove the following theorem for an infinite field* F (for a finite field F this theorem reduces easily to the case where F is infinite, but we shall not use this theorem for a finite field F in this paper).

Theorem 2. *Suppose that system* (39) *has a solution over* ${}^hA$. *Then the set of all solutions of* (39) *over* ${}^hA$ *can be represented in the form*

$$J + z^*,$$

where $J \subset {}^hA^l$ *is an* ${}^hA$*-submodule of all the solutions of the homogeneous system corresponding to* (39) (*i.e., system* (39) *with all* $u_j = 0$) *and* z^* *is a particular solution of* (39). *Moreover, the following assertions hold true.*

- (A) *The solution* z^* *can be chosen so that* $\operatorname{ord} z^* \geq \operatorname{ord} u - \nu$, *where* $\nu \geq 0$ *is an integer bounded from above by* $(dl)^{2^{O(n)}}$. *The degree* $\deg z^*$ *is bounded from above by* $d' + (dl)^{2^{O(n)}}$.
- (B) J *admits a system of generators of degrees bounded from above by* $(dl)^{2^{O(n)}}$. *The number of elements of this system of generators is bounded from above by* $k(dl)^{2^{O(n)}}$.

The constants in $O(n)$ *in* (A) *and* (B) *are absolute. Moreover, if all* $b_{i,j}$ *and* u_j *do not depend on* X_n (*i.e., they can be represented as sums of monomials that do not contain* X_n), *then* z^* *and all the generators of the module* J *also possess this property.*

Proof. Let $l_1 = \operatorname{rankr}(b_1, \dots, b_l)$. If $l_1 = 0$, then $u = (0, \dots, 0)$, $J = {}^hA^k$, and we can take $z^* = (0, \dots, 0)$. So, in what follows we shall assume that $l_1 > 0$. Then $1 \le l_1 \le k$ by Lemma 4. Permuting equations of (39), we may assume without loss of generality that $(b_1, \dots, b_{l_1})$ are linearly independent from the right over ${}^hA$. Let σ, z, e, e', e'' be the matrices occurring in Lemma 6. As in the proof of Lemma 6, we shall assume without loss of generality that $\sigma = 1$. Denote by b' the submatrix of b formed by the first l_1 columns of b, i.e., $b' = (b_1, \dots, b_{l_1})$. By Lemma 4, there are nonzero homogeneous elements $q_{1,1}, \dots, q_{l_1,l_1}$ of degrees at most

$$(2n+1)(\max\{\deg e_{i,i} \,:\, 1 \le i \le l_1\} + 1)l_1^2$$

such that $e_{1,1}q_{1,1} = e_{i,i}q_{i,i}$ and $\min\{\operatorname{ord} q_{i,i} \,:\, 1 \le i \le l_1\} = 0$. We introduce the diagonal matrix $q = \operatorname{diag}(q_{1,1}, \dots, q_{l_1,l_1})$. Let $\nu_0 = \operatorname{ord} e_{1,1}q_{1,1}$. Then $\operatorname{ord}(b'zq) \ge \nu_0$ by Lemma 6 (iii). Let $X_0^{\nu_0}\delta = b'zq$. Then δ is a matrix with coefficients in ${}^hA$, and

$$\delta = (\delta_{i,j})_{1\le i\le k,\, 1\le j\le l_1} = \begin{pmatrix} \delta' \\ \delta'' \end{pmatrix},$$

where $\delta' = \operatorname{diag}(\delta_{1,1}, \dots, \delta_{l_1,l_1})$ is a diagonal matrix with homogeneous coefficients from ${}^hA$ and such that all the elements on the diagonal are nonzero and equal, i.e., $\delta_{j,j} = \delta_{1,1}$ for every $1 \le j \le l_1$. Also, $\operatorname{ord} \delta_{1,1} = 0$. Next, $\delta'' = (\delta_{i,j})_{l_1+1\le i\le k,\, 1\le j\le l_1}$. We have $\operatorname{ord}(uzq) \ge \nu_0$ because, otherwise, system (39) has no solutions. Obviously, $\operatorname{ord} u \le \operatorname{ord}(uzq)$. Denote $u' = (u'_1, \dots, u'_{l_1}) = X_0^{-\nu_0}uzq \in {}^hA^l$. Then $\operatorname{ord} u' \ge \operatorname{ord}(u) - \nu_0$.

By Lemma 6 (i), and since q is a diagonal matrix with nonzero homogeneous entries on the diagonal, there are integers d''_j, $1 \le j \le l_1$, such that for all i, j we have

$$\deg \delta_{i,j} = d_i - d''_j \tag{41}$$

or $\delta_{i,j} = 0$. Besides that, for the same reason there is an integer ρ' such that $\deg u'_j = -d''_j + \rho'$ or $u'_j = 0$ for all $1 \le j \le l_1$ (here we leave the details to the reader).

Consider the linear system

$$Z\delta = u'. \tag{42}$$

Lemma 8. *Suppose that system* (39) *has a solution over* ${}^hA$. *Then the linear system* (42) *is equivalent to* (39), *i.e., the sets of solutions of systems* (42) *and* (39) *over* ${}^hA$ *coincide.*

Proof. The system $Zb'z = uz$ is equivalent to (39) by Lemma 5. System (42) is equivalent to $Zb'z = uz$ because the ring ${}^hA$ has no zero divisors. The lemma is proved. □

Remark 6. We have $\operatorname{rankr}(b_1, \dots, b_l) = l_1$. Hence, by Lemma 6, for every $l_1 + 1 \le j \le l$ there are homogeneous $z_{j,j}, z_{1,j}, \dots, z_{l_1,j} \in {}^hA$ such that $z_{j,j} \ne 0$, $b_jz_{j,j} + \sum_{1\le r\le l_1} b_rz_{r,j} = 0$, and all $\deg z_{j,j}$, $\deg z_{r,j}$ are bounded from above by $(2n+1)(l_1+1)^2d$. Put $u'_j = u_jz_{j,j} + \sum_{1\le r\le l_1} u_rz_{r,j}$, $l_1 + 1 \le j \le l$. Then system (39) has a solution if and only if system (42) has a solution and $u'_j = 0$ for all $l_1 + 1 \le j \le l$. This follows from Lemmas 8 and 5. But in what follows for our aims it suffices to use only Lemma 8.

Remark 7. Assume that $\deg_{X_n} b_{i,j} \le 0$ for all i, j, i.e., the elements of the matrix b do not depend on X_n. Then by Lemmas 4 and 6 and by our construction, all the elements of the matrices $b, z, q, \delta, \delta', \delta''$ also do not depend on X_n.

By Lemma 4 and Remark 3, for every $l_1 + 1 \le i \le k$, there are homogeneous elements $g_{i,i}, g_{i,j} \in {}^hA$, $1 \le j \le l_1$, such that

$$g_{i,i}\delta_{i,j} = g_{i,j}\delta_{1,1}, \quad 1 \le j \le l_1,$$

all the degrees $\deg g_{i,i}$, $\deg g_{i,j}$, $1 \le j \le l_1$, are bounded from above by

$$(2n+1)(l_1+1)^2(\max\{\deg \delta_{i,j} \,:\, 1 \le j \le k\} + 1),$$

and $\min_{1\le j\le l_1}\{\operatorname{ord} g_{i,i}, \operatorname{ord} g_{i,j}\} = 0$. Therefore, $\operatorname{ord} g_{i,i} = 0$ for every $l_1 + 1 \le i \le k$, because $\operatorname{ord} \delta_{1,1} = 0$.

We need an analog of the Noether normalization theorem from commutative algebra; cf. also Lemma 3.1 in [7].

Lemma 9. *Let $h \in {}^hA$ be an arbitrary nonzero element, and let $\deg h = \varepsilon$. There is a linear automorphism α of the algebra ${}^hA$,*

$$\alpha : {}^hA \to {}^hA, \quad \alpha(X_i) = \sum_{1\le j\le n} (\alpha_{1,i,j}X_j + \alpha_{2,i,j}D_j),$$
$$\alpha(D_i) = \sum_{1\le j\le n} (\alpha_{3,i,j}X_j + \alpha_{4,i,j}D_j), \quad \alpha(X_0) = X_0, \quad 1 \le i \le n,$$

such that all $\alpha_{s,i,j}$ are in F and $\deg_{D_n} \alpha(h) = \varepsilon$. Moreover, one can choose α so that, additionally, for every $H \in {}^hA$, if $\deg_{X_n} H = 0$, then $\deg_{X_n} \alpha(H) = 0$.

Proof. To start with, it is not difficult to construct a linear automorphism β such that $\beta(X_0) = X_0$, $\beta(X_n) = X_n$, $\beta(D_n) = D_n$,

$$\beta(X_i) = \beta_{1,i}X_i + \beta_{2,i}D_i, \quad \beta(D_i) = \beta_{3,i}X_i + \beta_{4,i}D_i, \quad 1 \le i \le n,$$

and $\beta(h)$ contains a monomial $a_{i_1,\dots,i_n}D_1^{i_1},\dots,D_n^{i_n}$ with $a_{i_1,\dots,i_n} \ne 0$ and $i_1+\dots+i_n = \varepsilon$, i.e., $\varepsilon = \deg_{D_1,\dots,D_n} \beta(h)$. After that, we can find an automorphism γ such that $\gamma(X_0) = X_0$,

$$\gamma(X_i) = X_i, \quad \gamma(D_i) = D_i + \gamma_i D_n, \quad 1 \le i \le n-1,$$
$$\gamma(X_n) = X_n - \sum_{1\le i\le n-1} \gamma_i X_i, \gamma(D_n) = D_n,$$

where $\gamma_i \in F$ for all $1 \le i \le n-1$, and $(\gamma\circ\beta)(h)$ contains a monomial aD_n^ε with $0 \ne a \in F$. Put $\alpha = \gamma \circ \beta$. Obviously, if $H \in {}^hA$ and $\deg_{X_n} H = 0$, then $\deg_{X_n} \alpha(H) = 0$. The lemma is proved. □

Put $h = \delta_{1,1}g_{l_1+1,l_1+1}g_{l_1+2,l_1+2}\cdots g_{k,k}$. Then $h \in {}^hA$ is a nonzero homogeneous element and $\operatorname{ord} h = 0$. By Lemma 9, we obtain an automorphism α. Applying α to the coefficients of system (42), we get a new linear system. Again by Lemma 9, if all the coefficients of system (42) do not depend on X_n, then all the coefficients of the new system also do not depend on X_n. In what follows, to simplify the notation, we shall assume without loss of generality that $\alpha = 1$. Thus, h contains a monomial aD_n^ε with $0 \ne a \in F$, where $\varepsilon = \deg h$. Then

$$(43) \qquad \deg_{D_n} \delta_{1,1} = \deg \delta_{1,1}, \quad \deg_{D_n} g_{i,i} = \deg g_{i,i}, \quad l_1 + 1 \le i \le k.$$

Let $z = (z_1,\dots,z_k) \in {}^hA^k$ be a solution of (42). Then (43) implies that we can uniquely represent z_i in the form

$$(44) \qquad z_i = z_i'g_{i,i} + \sum_{0\le r<\deg g_{i,i}} z_{i,r}D_n^r, \quad l_1 + 1 \le i \le k,$$

where $z_i', z_{i,r} \in {}^hA$, and $\deg_{D_n} z_{i,r} \le 0$ for all $l_1 + 1 \le i \le k$, $0 \le r < \deg_{D_1} g_{i,i}$. Again by (43), we can uniquely represent u_j' in the form

$$u_j' = u_j''\delta_{1,1} + \sum_{0\le s<\deg \delta_{1,1}} u_{j,s}'D_n^s, \quad 1 \le j \le l_1,$$

where $u_j'', u_{j,s}' \in {}^hA$, and $\deg_{D_n} u_{j,s}' \le 0$ for all $1 \le j \le l_1$, $0 \le s < \deg_{D_1} g_{i,i}$. Finally, by (43), for all $l_1 + 1 \le i \le k$, $1 \le j \le l_1$, and $0 \le r < \deg_{D_1} g_{i,i}$, there is a unique representation

$$D_n^r\delta_{i,j} = \delta_{i,r,j}\delta_{1,1} + \sum_{0\le s<\deg \delta_{1,1}} \delta_{i,r,j,s}D_n^s,$$

where $\delta_{i,r,j}, \delta_{i,r,j,s} \in {}^hA$, and $\deg_{D_n} \delta_{i,r,j,s} \le 0$ for all the i,r,j,s involved. Put

$$\mathcal{I} = \{ (i,r) : l_1+1 \le i \le k \,\&\, 0 \le r < \deg g_{i,i} \},$$
$$\mathcal{J} = \{ (j,s) : 1 \le j \le l_1 \,\&\, 1 \le s < \deg \delta_{1,1} \}.$$

Therefore,

$$z_j = -\sum_{l_1+1 \le i \le k} z'_i g_{i,j} - \sum_{(i,r)\in\mathcal{I}} z_{i,r}\delta_{i,r,j} + u''_j, \quad 1 \le j \le l_1, \tag{45}$$

$$\sum_{(i,r)\in\mathcal{I}} z_{i,r}\delta_{i,r,j,s} = u'_{j,s}, \quad (j,s)\in\mathcal{J}. \tag{46}$$

We introduce new unknowns $Z_{i,r}$, $(i,r)\in\mathcal{I}$. By (44)–(46), system (39) reduces to the linear system

$$\sum_{(i,r)\in\mathcal{I}} Z_{i,r}\delta_{i,r,j,s} = u'_{j,s}, \quad (j,s)\in\mathcal{J}. \tag{47}$$

More precisely, any solution of system (39) is given by (44), (45), where the $z'_i \in {}^hA$ are arbitrary and $z_{i,r}$ is a solution of system (46) over ${}^hA$ (we emphasize that this solution $z_{i,r}$ may depend on D_n, although we can restrict ourselves to solutions $z_{i,r}$ that do not depend on D_n). Note that all $\delta_{i,r,j,s}$ and $u'_{j,s}$ are homogeneous elements of ${}^hA$. Put $d_{i,r} = d_i + r$, $(i,r)\in\mathcal{I}$ and $d'_{j,s} = d''_j + s$, $(j,s)\in\mathcal{J}$, $\widetilde{\rho} = \rho'$, where d_j, d''_i, ρ' are as introduced above; see (41). Then $\deg \delta_{i,r,j,s} = d_{i,r} - d'_{j,s}$ or $\delta_{i,r,j,s} = 0$, and $\deg u'_{j,s} = -d'_{j,s} + \widetilde{\rho}$ or $u'_{j,s} = 0$, for all $(i,r)\in\mathcal{I}$, $(j,s)\in\mathcal{J}$. This follows immediately from our construction (we leave the details to the reader).

Now all the coefficients of system (47) do not depend on D_n. As we have proved, if the coefficients of (39) do not depend on X_n, then the coefficients of (47) also do not depend on X_n, and hence, in this case they do not depend on X_n, D_n.

If the coefficients of (47) depend on X_n, we perform an automorphism $X_n \mapsto D_n$ $D_n \mapsto -X_n$, $X_i \mapsto X_i$, $D_i \mapsto D_i$, $1 \le i \le n-1$. Now the coefficients of system (47) do not depend on X_n (but depend on D_n).

After that, we apply our construction recursively to system (47). Here, we need to lean upon Remark 5, because the integers $d'_{j,s}$ are not necessarily positive.

More precisely, denote ${}^hA = A^{(n)}$ for brevity. As the input of the step in question we have system (39) over the ring $A^{(n)}$. Now, as the input of the next recursive step we have system (47) over $A^{(n')}$, where $n' = n$ if at least one of the coefficients of system (39) depends on X_n, and $n' = n-1$ if all the coefficients of system (39) do not depend on X_n (thus, n is replaced by n', and it reduces after each two steps of recursion; n' is a new value of n for the input of the recursive step). Let J' be the module of solutions of the homogeneous system corresponding to (47) over the ring $A^{(n')}$. Then, obviously, ${}^hAJ'$ is the module of solutions of the same homogeneous system over ${}^hA$. Each system of generators of J' over $A^{(n')}$ is a system of generators of ${}^hAJ'$ over ${}^hA$. Similarly, a particular solution of system (47) over $A^{(n')}$ is a particular solution of system (47) over ${}^hA$.

At the final step of the recursion, at least one of the following conditions is fulfilled:

- $l_1 = 0$ for the newly obtained system (in place of (39));
- $n = 0$ (although in the statement of the theorem we have $n \ge 1$, see §1, we are interested only in Weyl algebras).

If $l_1 = 0$, we get the required z^* and J at this recursive step immediately; see above. If $n = 0$, then $\mathcal{I} = \mathcal{J} = \varnothing$. Hence, using (45) for $n = 0$, we get the required z^* and J for $n = 0$.

Thus, using the recursive assumption, we get a particular solution $Z_{i,r} = z^*_{i,r}$, $(i,r) \in \mathcal{I}$, of system (47), an integer ν_1 (in place of ν from assertion (A)) satisfying the inequality

$$\min_{(i,r)\in\mathcal{I}}\{\operatorname{ord} z^*_{i,r}\} \geq \min_{(j,s)\in\mathcal{J}}\{\operatorname{ord} u'_{j,s}\} - \nu_1, \tag{48}$$

and a system of generators

$$(\, z_{\alpha,i,r}\,)_{(i,r)\in\mathcal{I}}\,, \quad 1 \leq \alpha \leq \beta, \tag{49}$$

of the module J' of solutions of the homogeneous system corresponding to (47). Put

$$\begin{aligned} z^*_j &= -\sum_{(i,r)\in\mathcal{I}} z^*_{i,r}\delta_{i,r,j} + u''_j, \quad 1 \leq j \leq l_1, \\ z^*_i &= \sum_{0\leq r<\deg g_{i,i}} z^*_{i,r} D^r_n, \qquad l_1+1 \leq i \leq k, \\ z^* &= (z^*_1,\dots,z^*_k). \end{aligned}$$

Then z^* is a particular solution of (39). Put

$$\begin{aligned} z_{\alpha,j} &= -\sum_{(i,r)\in\mathcal{I}} z_{\alpha,i,r}\delta_{i,r,j}, \quad 1 \leq j \leq l_1,\ 1 \leq \alpha \leq \beta, \\ z_{\alpha,i} &= \sum_{0\leq s<\deg g_{i,i}} z_{\alpha,i,s} D^s_n, \quad l_1+1 \leq i \leq k,\ 1 \leq \alpha \leq \beta, \\ z_{\beta-l_1+i,j} &= 0, \qquad l_1+1 \leq i,j \leq k,\ j \neq i, \\ z_{\beta-l_1+i,i} &= g_{i,i}, \qquad l_1+1 \leq i \leq k, \\ z_{\beta-l_1+i,j} &= -g_{i,j}, \quad 1 \leq j \leq l_1,\ l_1+1 \leq i \leq k. \end{aligned}$$

Then $J = \sum_{1\leq\alpha\leq\beta+k-l_1} {}^hA(z_{\alpha,1},\dots,z_{\alpha,k})$. Hence, $(z_{\alpha,1},\dots,z_{\alpha,k})$, $1 \leq \alpha \leq \beta+k-l_1$, is a system of generators of the module J. By (48) and the definitions of u', u''_j, and $u'_{j,s}$, we have $\operatorname{ord} z^* \geq \operatorname{ord}(u) - \nu_0 - \nu_1$. Put $\nu = \nu_0 + \nu_1$.

Lemma 10. *All the degrees* $\deg \delta_{i,j}$, $\deg g_{i,i}$, $\deg g_{i,j}$, $\deg \delta_{i,r,j}$, $\deg \delta_{i,r,j,s}$ *and the number* ν_0*, see above, are bounded from above by* $(nld)^{O(1)}$*; the degrees* $\deg u'_j$, $\deg u''_j$, $\deg u'_{j,s}$ *are bounded from above by* $d' + (nld)^{O(1)}$*. Next,* $\operatorname{ord} u'_j$, $\operatorname{ord} u''_j$*, and* $\operatorname{ord} u'_{j,s}$ *are bounded from below by* $\operatorname{ord} u - \nu_0$*. Finally, in system* (47) *the number* $\#\mathcal{J}$ *of equations is bounded from above by* $(nld)^{O(1)}$*, and the number* $\#\mathcal{I}$ *of unknowns is bounded from above by* $k(nld)^{O(1)}$.

Proof. This follows immediately from the construction. □

We return to the proof of Theorem 2. Applying Lemma 10 and, recursively, assertions (A) and (B) for the formulas giving z^* and J, we get (A) and (B) from the statement of the theorem. The last claim (related to the case where all $b_{i,j}$ and u_j do not depend on D_n) has already been proved. The theorem is proved. □

§8. Proof of Theorem 1 for Weyl algebras

We start with showing that it suffices to prove the theorem for an infinite field F. Indeed, let F_1 be an infinite field such that $F_1 \supset F$. Let $f_1,\dots,f_m$ be a Janet basis of the module $I \otimes_F F_1$ with all the degrees $\deg f_w$, $1 \leq w \leq m$, bounded from above by $d^{2^{O(n)}}$. There is a finite extension $F_2 \supset F$ such that for all v,i,j and all $1 \leq w \leq m$ the coefficient in f_w of the monomial $e_{v,i,j}$ belongs to the field F_2. Let a_α, $1 \leq \alpha \leq \mu$, be the basis of the field F_2 over F. Then we can write $f_w = \sum_{1\leq\alpha\leq\mu} a_\alpha f_{\alpha,w}$, where all $f_{\alpha,w}$ belong to I. Now $\deg f_{\alpha,w} \leq \deg f_w$ and $f_{\alpha,w}$, $1 \leq w \leq m$, $1 \leq \alpha \leq \mu$, is a Janet basis of the module I. Moreover, the reduced Janet basis of the module I remains the same after an arbitrary extension of scalars. The required assertion is proved. Thus, extending the ground field F, we may assume without loss of generality that F is infinite.

Let a be a matrix as in §1. There is no loss of generality in assuming that the vectors $(a_{i,1}, \ldots, a_{i,l})$, $1 \le i \le k$, are linearly independent over F. We have $\deg a_{i,j} < d$. This implies $k \le l\binom{d+2n}{2n}$.

Put $b = {}^h a$. We define graded submodules of ${}^h I$:

$$J_0 = {}^h A(b_{1,1}, \ldots, b_{1,l}) + \cdots + {}^h A(b_{k,1}, \ldots, b_{k,l}),$$
$$J_\gamma = J_0 : (X_0^\gamma) = \{z \in {}^h A^l \,:\, zX_0^\gamma \in J_0\}, \quad \gamma \ge 1.$$

We have the following exact sequence of graded ${}^h A$-modules:

$$ {}^h A^k \to J_0 \to 0.$$

Next, we have $J_\gamma \subset J_{\gamma+1} \subset {}^h I$ for every $\gamma \ge 0$, and ${}^h I = \bigcup_{\gamma \ge 0} J_\gamma$. Since ${}^h A$ is Noetherian, there exists $N \ge 0$ such that ${}^h I = J_N$. Therefore, to construct a system of generators of ${}^h I$, it suffices to compute the smallest N such that ${}^h I = J_N$ and to find a system of generators of J_N.

Lemma 11. *${}^h I = J_N$ for some N bounded from above by $(dl)^{2^{O(n)}}$. There is a system of generators $b_1, \ldots, b_s$ of the module J_N such that s and all the degrees $\deg b_v$, $1 \le v \le s$, are bounded from above by $(dl)^{2^{O(n)}}$.*

Proof. We show that $J_{N+1} \subset J_N$ for $N \ge \nu$. Let $u \in J_{N+1}$. Consider system (39). By assertion (A) of Theorem 2, there is a particular solution z^* of (39) such that $\operatorname{ord} z^* \ge 1$. Hence, $u \in X_0 J_N \subset J_N$. The claim is proved. Thus, ${}^h I = J_\nu$.

We replace $(u_1, \ldots, u_l)$ in (39) by $(U_1 X_0^\nu, \ldots, U_l X_0^\nu)$, where $U_1, \ldots, U_l$ are new unknowns. Then, applying statement (B) of Theorem 2 to this new homogeneous linear system with respect to the unknowns $U_1, \ldots, U_l$, $Z_1, \ldots, Z_k$, we get the required estimates for the number of generators of J_ν and for the degrees of these generators. The lemma is proved. □

Corollary 1. *Let $(a_{i,1}, \ldots, a_{i,l})$, $1 \le i \le l$, be as at the beginning of the section, and let the integer N be as in Lemma 11. Then, for every integer $m \ge 0$, the F-linear space*

$$A_{m+N}(a_{1,1}, \ldots, a_{1,l}) + \cdots + A_{m+N}(a_{k,1}, \ldots, a_{k,l}) \quad \text{includes} \quad I_m. \tag{50}$$

Proof. By Lemma 11, we have $(J_0)_{m+N} \supset X_0^N (J_N)_m = X_0^N ({}^h I)_m$. Taking the affine parts yields (50). The corollary is proved. □

Now everything is ready for the proof of Theorem 1. By Lemmas 11 and 1, there is a system of generators of the module $\operatorname{gr}(I)$ with degrees bounded from above by $(dl)^{2^{O(n)}}$. By Lemma 12 (see Appendix 1), the Hilbert function $H(\operatorname{gr}(I), m)$ is stable for $m \ge (dl)^{2^{O(n)}}$. By (11) (see §2), the Hilbert function $H(I, m)$ is stable for all $m \ge (dl)^{2^{O(n)}}$.

Consider the linear order $<$ on the monomials in ${}^h A^l$ that is induced by the linear order $<$ on the monomials in A^l; see §4. Then the monomial (i.e., generated by monomials) submodule ${}^c I \subset {}^c A^l$ is well defined, see §4, where ${}^c A = F[X_0, \ldots, X_n, D_1, \ldots, D_n]$ is the polynomial ring. By (24), the Hilbert function $H({}^c I, m)$ is stable for all $m \ge (dl)^{2^{O(n)}}$. Hence, all the coefficients of the Hilbert polynomial of ${}^c I$ are bounded from above by $(dl)^{2^{O(n)}}$. Therefore, by Lemma 13, the module ${}^c I$ has a system of generators with degrees $(dl)^{2^{O(n)}}$. We can assume without loss of generality that this system of generators of ${}^c I$ consists of monomials. The sets of monomials in ${}^c I$ and in $\operatorname{Hdt}({}^h I)$ are in a natural degree-preserving one-to-one correspondence; see §4. Therefore, see §4, the degrees of all the elements of a Janet basis of ${}^hI$ with respect to the induced linear order $<$ are bounded from above by $(dl)^{2^{O(n)}}$. Since the ideal ${}^hI$ is homogeneous, the same bound is valid for the degrees of all the elements (they are homogeneous) of the reduced Janet basis of ${}^hI$. Hence, by Lemma 3 (iii) (see §4), the same is true for some Janet basis $f_1, \dots, f_m$ (respectively, by Lemma 3 (ii), for the reduced Janet basis in the case where the initial order $<$ is degree-compatible) of the module I with respect to the linear order $<$ on the monomials in A^l.

It remains to consider the case where $l = 1$ and an admissible linear order $<$ is arbitrary. We need to obtain estimates for the reduced Janet basis of I in this case. In the case in question, the linear order $<$ is given on the set of pairs of multi-indices (i, j), $i, j \in \mathbb{Z}_+^n$. Now (see, e.g., [13, p. 58]), there is a real ordered field R and a linear form $L \in R[Y_1, \dots, Y_n, Z_1, \dots, Z_n]$ with positive coefficients such that, for all pairs (i, j), (i', j') of multi-indices, $(i', j') < (i, j)$ if and only if

$$L(i - i', j - j') = L(i_1 - i'_1, \dots, i_n - i'_n, j_1 - j'_1, \dots, j_n - j'_n) > 0$$

in the real ordered field R.

Let $\psi_1 < \cdots < \psi_a$ be all the monomials in $X_1, \dots, X_n, D_1, \dots, D_n$ with nonzero coefficients in the elements $f_1, \dots, f_m$, and let $(i^{(1)}, j^{(1)}) < \cdots < (i^{(a)}, j^{(a)})$ be the corresponding pairs of multi-indices. Let $\varepsilon > 0$ be an infinitesimal with respect to the field R. Now

$$L(i^{(s+1)} - i^{(s)}, j^{(s+1)} - j^{(s)}) \geq \varepsilon, \quad 1 \leq s \leq a - 1, \tag{51}$$

in the field $R(\varepsilon)$. Let $U = \sum_{1 \leq w \leq n} (u_w Y_w + v_w Z_w)$ be a generic linear form in the variables $Y_1, \dots, Y_n, Z_1, \dots, Z_n$; i.e., the family $\{u_w, v_w\}_{1 \leq w \leq n}$ of coefficients of U has the transcendency degree $2n$ over $R(\varepsilon)$. Consider the following system of linear inequalities with coefficients in $\mathbb{Q}[\varepsilon]$ with respect to u_w, v_w, $1 \leq w \leq n$,

$$\begin{cases} U(i^{(s+1)} - i^{(s)}, j^{(s+1)} - j^{(s)}) \geq \varepsilon, & 1 \leq s \leq a - 1, \\ u_w \geq \varepsilon, & 1 \leq w \leq n, \\ v_w \geq \varepsilon, & 1 \leq w \leq n. \end{cases} \tag{52}$$

Let K_ε be the set of solutions of system (52) in $R(\varepsilon)^{2n}$. By (51), and since all the coefficients of the linear form L are positive, system (52) has a solution in $R(\varepsilon)^{2n}$. The left-hand sides of the inequalities in (52) are linear forms in u_w, v_w, $1 \leq w \leq n$, with integral coefficients. We denote them by $Q_1, \dots, Q_\mu$, $\mu = a - 1 + 2n$. Observe that the absolute values of the coefficients of the linear forms $Q_1, \dots, Q_\mu$ are bounded from above by $d^{2^{O(n)}}$.

We show that there are indices $1 \leq w_1 < \cdots < w_s \leq \mu$ and $s \leq 2n$ such that $\mathcal{Z}(Q_{w_1} - \varepsilon, \dots, Q_{w_s} - \varepsilon) \subset K_\varepsilon$ (here $\mathcal{Z}(Q_{w_1} - \varepsilon, \dots, Q_{w_s} - \varepsilon)$ is the set of all common zeros of the polynomials $Q_{w_1} - \varepsilon, \dots, Q_{w_s} - \varepsilon$ in $R(\varepsilon)^{2n}$) and the linear forms $Q_{w_1}, \dots, Q_{w_s}$ are linearly independent over $\mathbb{Q}$. Indeed, we can construct $Q_{w_1}, \dots, Q_{w_s}$ recursively, by choosing subsequently Q_{w_α}, $\alpha \geq 1$, such that $\mathcal{Z}(Q_{w_\alpha} - \varepsilon)$ has a nonempty intersection with the boundary of $\mathcal{Z}(Q_{w_1} - \varepsilon, \dots, Q_{w_{\alpha-1}} - \varepsilon) \cap K_\varepsilon$ (we leave the details to the reader).

Solving the linear system $Q_{w_1} - \varepsilon = \cdots = Q_{w_s} - \varepsilon = 0$, we see that there is a point $(u'_w, v'_w)_{1 \leq w \leq n} \in K_\varepsilon$ such that $u'_w = a_w \varepsilon / c$ and $v'_w = b_w \varepsilon / c$, where all a_w, b_w, c are positive integers with absolute values bounded from above by $d^{2^{O(n)}}$. Put $\varepsilon^* = 1$, $u^*_w = a_w / c$ and $v^*_w = b_w / c$, $1 \leq w \leq n$. We view (52) as a linear system with respect to all u_w, v_w and ε. Then u^*_w, v^*_w and $\varepsilon^* > 0$ is a solution of (52) in $\mathbb{Q}^{2n+1}$. Set $L^* = c \sum_{1 \leq w \leq n} (u^*_w Y_w + v^*_w Z_w)$.

Now $L^* \in \mathbb{Z}[Y_1, \dots, Y_n, Z_1, \dots, Z_n]$ is a linear form with positive integral coefficients bounded from above by $d^{2^{O(n)}}$ and such that

$$L^*(i^{(s+1)} - i^{(s)}, j^{(s+1)} - j^{(s)}) > 0, \quad 1 \le s \le a-1. \tag{53}$$

We assume without loss of generality that $\mathrm{Hdt}(f_1), \dots, \mathrm{Hdt}(f_m)$ is the family of leading monomials of the reduced Janet basis $f'_1, \dots, f'_m$ of the module I with respect to the linear order $<$, and $\mathrm{Hdt}(f_1) > \dots > \mathrm{Hdt}(f_m)$. For any $g \in A$, put $\lambda(g) = L^*(i, j)$, where $\mathrm{Hdt}(g) = g_{i,j} X^i D^j$, $0 \ne g_{i,j} \in F$. Then inequality (53) and the definitions show that $\lambda(f_w) = \lambda(f'_w)$ for all $1 \le w \le m$. Hence, all $\lambda(f'_w)$ are bounded from above by $d^{2^{O(n)}}$. But, obviously, $\deg f'_w \le \lambda(f'_w)$, $1 \le w \le m$. Theorem 1 is proved for Weyl algebras.

§9. The case of an algebra of differential operators

Extending the ground field F, we may suppose without loss of generality that the field F is infinite. We denote by $B = F(X_1, \dots, X_n)[D_1, \dots, D_n]$ the algebra of differential operators. Recall that $A \subset B$, so that relations (2) are satisfied. Next, each element $f \in B$ can be uniquely represented in the form

$$f = \sum_{j_1, \dots, j_n \ge 0} f_{j_1, \dots, j_n} D_1^{j_1} \cdots D_n^{j_n} = \sum_{j \in \mathbb{Z}_+^n} f_j D^j,$$

where all $f_{j_1, \dots, j_n} = f_j$ belong to $F(X_1, \dots, X_n)$ and $F(X_1, \dots, X_n)$ is the field of rational functions over F. Everywhere in §§1 and 2, we replace A, $X^i D^j$, $\deg f = \deg_{X_1, \dots, X_n, D_1, \dots, D_n} f$, $\dim_F M$, $e_{v,i,j}$, $f_{v,i,j} \in F$, and (v, i, j), (i, j), (i', j'), (i'', j'') by B, D^j, $\deg f = \deg_{D_1, \dots, D_n} f$, $\dim_{F(X_1, \dots, X_n)} M$, $e_{v,j}$, $f_{v,j} \in F(X_1, \dots, X_n)$, and (v, j), j, j', j'', respectively. This leads to the definition of the Janet basis and all other objects occurring in §1 for the case of the algebra of differential operators.

The definition of the homogenization ${}^hB$ of B is similar to that of ${}^hA$; see §3. Namely, ${}^hB = F(X_1, \dots, X_n)[X_0, D_1, \dots, D_n]$ is given by the relations

$$\begin{gathered} X_i X_j = X_j X_i,\ D_i D_j = D_j D_i, \quad \text{for all} \quad i, j, \\ D_i X_i - X_i D_i = X_0, \quad 1 \le i \le n, \quad X_i D_j = D_j X_i \quad \text{for all} \quad i \ne j. \end{gathered} \tag{54}$$

The further considerations are similar to the case of the Weyl algebra A with minor changes. We leave them to the reader. For example, Theorem 2 for the case of the algebra of differential operators is the same. One need only to replace A, ${}^hA$, and X_n by B, ${}^hB$ and D_n everywhere, respectively. Thus, Theorem 1 can be proved in the case where A is an algebra of differential operators (but now it is B). Theorem 1 is proved completely.

One can consider a more general algebra of differential operators. Let $\mathcal{F}$ be a field with n derivatives $D_1, \dots, D_n$. Then $K_n = \mathcal{F}[D_1, \dots, D_n]$ is an algebra of differential operators, and its homogenization ${}^hK_n$ can be defind as before, by means of adding a variable X_0 satisfying the relations

$$D_i D_j = D_j D_i, \quad X_0 D_i = D_i X_0, \quad D_i f - f D_i = f_{D_i} X_0$$

for all i, j and all elements $f \in \mathcal{F}$, where $f_{D_i} \in \mathcal{F}$ denotes the result of an application of D_i to f. Following the proof of Theorem 1, we can deduce the statement below.

Remark 8. Bounds similar to those in Theorem 1 hold true for K_n (in place of the algebra of differential operators A).

Appendix 1: Degrees of generators of a graded module over a polynomial ring, and its Hilbert function

We give a short proof of the following result; cf. [1, 12, 6, 4]. Let $\mathcal{A} = F[X_0, \dots, X_n]$ be a graded polynomial ring. The homogeneous elements of $\mathcal{A}$ are homogeneous polynomials in $X_0, \dots, X_n$.

Lemma 12. *Let $I \subset \mathcal{A}^l$ be a graded $\mathcal{A}$-module with a system of generators $f_1, \dots, f_m$ of degrees less than d, where $d \geq 2$. Then the Hilbert function $H(\mathcal{A}^l/I, m) = \dim_F(\mathcal{A}^l/I)_m$ is stable for $m \geq (dl)^{2^{O(n+1)}}$. Furthermore, all the coefficients of the Hilbert polynomial of $\mathcal{A}^l/I$ are bounded from above by $(dl)^{2^{O(n+1)}}$.*

Proof. Extending the ground field F, we may suppose without loss of generality that the field F is infinite. Denote $M = \mathcal{A}^l/I$. Let $L \in F[X_0, \dots, X_n]$ be a linear form in general position. Let K stand for the kernel of the morphism $M \to M$ of multiplication by L. We have $K = \{z \in \mathcal{A}^l : Lz = \sum_{1 \leq i \leq m} f_i z_i,\, z_i \in \mathcal{A}\}$. Hence, solving a linear system over $\mathcal{A}$, we see that K has a system of generators $g_1, \dots, g_\mu$ with degrees bounded from above by $(dl)^{2^{O(n+1)}}$. Let $\mathfrak{P}$ be an arbitrary associated prime ideal of the module M such that $\mathfrak{P} \neq (X_0, \dots, X_n)$. Since L is in general position, we have $L \notin \mathfrak{P}$. Therefore, $\mathfrak{P}$ is not an associated prime ideal of K. Consequently, $K_N = 0$ for all sufficiently large N. So, $X_i^N g_j \in I$ for sufficiently large N and all i, j. Hence, $g_j = \sum_{1 \leq i \leq m} y_{j,i} f_i$, where $y_{j,i} \in F(X_i)[X_0, \dots, X_n]$. Solving a linear system over the ring $F(X_i)[X_0, \dots, X_n]$, we get a bound on the denominators from $F[X_i]$ of all $y_{j,i}$. Since all g_j and f_i are homogeneous, we may assume without loss of generality that all the denominators are X_i^N. Thus, we get an upper bound for N. Namely, N is bounded from above by $(dl)^{2^{O(n+1)}}$.

Therefore, the sequence

$$0 \to M_m \to M_{m+1} \to (M/LM)_{m+1} \to 0 \tag{55}$$

is exact for $m \geq (dl)^{2^{O(n+1)}}$. But $M/LM = \mathcal{A}^l/(I + L\mathcal{A}^l)$ is a module over a polynomial ring $F[X_0, \dots, X_n]/(L) \simeq F[X_0, \dots, X_{n-1}]$. Hence, by the inductive assumption, the Hilbert function $H(\mathcal{A}^l/(I + L\mathcal{A}^l), m)$ is stable for $m \geq (dl)^{2^{O(n)}}$. Now (55) implies that the Hilbert function $H(\mathcal{A}^l/I, m)$ is stable for $m \geq (dl)^{2^{O(n+1)}}$.

Obviously, for $m < (dl)^{2^{O(n+1)}}$ the values $H(\mathcal{A}^l/I, m)$ are bounded from above by $(dl)^{2^{O(n+1)}}$. Using the Newton interpolation, we conclude that all the coefficients of the Hilbert polynomial of $\mathcal{A}^l/I$ are bounded from above by $(dl)^{2^{O(n+1)}}$. The lemma is proved. □

We also need a converse to Lemma 12.

Lemma 13. *Let $I \subset \mathcal{A}^l$ be a graded $\mathcal{A}$-module. Assume that the Hilbert function $H(\mathcal{A}^l/I, m) = \dim_F(\mathcal{A}^l/I)_m$ is stable for $m \geq D$ and that all absolute values of the coefficients of the Hilbert polynomial of the module $\mathcal{A}^l/I$ are bounded from above by D, for some integer $D > 1$. Then I has a system of generators $f_1, \dots, f_m$ with degrees $D^{2^{O(n+1)}}$.*

Proof. Let $f_1, \dots, f_m$ be the reduced Gröbner basis of I with respect to an admissible linear order $<$ on the monomials in $\mathcal{A}^l$; cf. the definitions in §§1 and 4. The degree of a monomial from $\mathcal{A}^l$ is defined as in §§1 and 4. We assume additionally that the linear order under consideration is degree-compatible; i.e., for any two monomials z_1, z_2, if $\deg z_1 < \deg z_2$, then $z_1 < z_2$. For every $z \in \mathcal{A}^l$, the greatest monomial $\mathrm{Hdt}(z)$ is defined. The monomial module $\mathrm{Hdt}(I)$ is generated by all $\mathrm{Hdt}(z)$, $z \in I$. Now, $\mathrm{Hdt}(f_1), \dots, \mathrm{Hdt}(f_m)$ is a minimal system of generators of $\mathrm{Hdt}(I)$, and $\deg f_i = \deg \mathrm{Hdt}(f_i)$ for every $1 \leq i \leq m$. The values of the Hilbert functions $H(\mathcal{A}^l/\mathrm{Hdt}(I), m) = H(\mathcal{A}^l/I, m)$ coincide for all

$m \geq 0$; cf. §4. Thus, replacing I by $\mathrm{Hdt}(I)$, we shall assume in what follows in this proof that I is a monomial module.

For every $1 \leq i \leq l$, denote by $\mathcal{A}_i \subset \mathcal{A}^l$ the ith direct summand of $\mathcal{A}^l$. Put $I_i = I \cap \mathcal{A}_i$, $1 \leq i \leq l$. Then $I \simeq \bigoplus_{1\leq i\leq l} I_i$ because I is a monomial module. Next, for every $1 \leq \alpha \leq m$ there is $1 \leq i \leq l$ such that $f_\alpha \in I_i$. We identify $\mathcal{A}_i = \mathcal{A}$. Then $I_i \subset \mathcal{A}$ is a homogeneous monomial ideal. The case where $I_i = \mathcal{A}$ for some i is not excluded. For Hilbert functions, we have

$$H(\mathcal{A}^l/I, m) = \sum\nolimits_{1\leq i\leq l} H(\mathcal{A}/I_i, m), \quad m \geq 0. \tag{56}$$

If $(\mathcal{A}/I_i)_D = 0$ for some i, then $(\mathcal{A}/I_i)_m = 0$ for every $m \geq D$. In this case the ideal I_i is generated by $\sum_{0\leq m\leq D}(I_i)_m$. Hence for $m \geq D$ we can omit this index i in the sum on the right in (56). Therefore, in this case the proof reduces to a smaller l. So, we may assume without loss of generality that $(\mathcal{A}/I_i)_D \neq 0$, $1 \leq i \leq l$. Next, we use an exact description of the Hilbert function of a homogeneous ideal; see [4, §7]. Namely, there are unique integers $b_{i,0} \geq b_{i,1} \geq \cdots \geq b_{i,n+2} = 0$ such that

$$H(\mathcal{A}/I_i, m) = \binom{m+n+1}{n+1} - 1 - \sum_{1\leq j\leq n+1} \binom{m - b_{i,j} + j - 1}{j} \tag{57}$$

for all sufficiently large m and

$$b_{i,0} = \min\{d : d \geq b_{i,1} \text{ and for all } m \geq d,\ (57) \text{ is true}\}. \tag{58}$$

This description (without the constants $b_{i,0}$) dates back to the classical paper [11]. The integers $b_{i,0}, \ldots, b_{i,n+2}$ are called the Macaulay constants of the ideal I_i. We have

$$h(i,m) = H(\mathcal{A}/I_i, m) - \binom{m+n+1}{n+1} + 1 + \sum_{1\leq j\leq n+1} \binom{m - b_{i,j} + j - 1}{j} \geq 0 \tag{59}$$

for every $m \geq b_{i,1}$; see [4, §7]. By Lemma 7.2 in [4], for all $1 \leq \alpha \leq m$, if $f_\alpha \in I_i$, then $\deg f_\alpha \leq b_{i,0}$. Hence, it suffices to prove that all $b_{i,0}$, $1 \leq i \leq l$, are bounded from above by $D^{2^{O(n+1)}}$.

By (56) and (57), the coefficient of m^{n-j}, $0 \leq j \leq n$, in the Hilbert polynomial of $\mathcal{A}^l/I$ is

$$\frac{\mu_j}{(n+1-j)!} \sum_{1\leq i\leq l} b_{i,n+1-j} + \sum_{0\leq v\leq j-1} \sum_{1\leq i\leq l} \frac{1}{(n+1-v)!} \mu_{j,v}(b_{i,n+1-v}), \tag{60}$$

where $0 \neq \mu_j$ is an integer, and $\mu_{j,v} \in \mathbb{Z}[Z]$, $0 \leq v \leq j-1$, is a polynomial with integral coefficients and $\deg \mu_{j,v} = j - v + 1$. Moreover, $|\mu_j|$ and the absolute values of the coefficients of all the polynomials $\mu_{j,v}$ are bounded from above by, say, $2^{O(n^2)}$. Denote $b_j = \sum_{1\leq i\leq l} b_{i,j}$, $0 \leq j \leq n+2$. By the condition of the lemma, the coefficients of the Hilbert polynomial of $\mathcal{A}^l/I$ are bounded from above by D. Hence, via (60), we can recursively estimate $b_{n+1}, b_n, \ldots, b_1$. Namely, $b_{n+1-j} = (2^{n^2} lD)^{2^{O(j+1)}}$, $0 \leq j \leq n$. Consequently, $b_1 = (lD)^{2^{O(n+1)}}$. Observe that $b_{i,1} \leq \max_{1\leq i\leq l} b_{i,1} \leq b_1$ for every $1 \leq i \leq m$.

Now, let $m \geq \max_{1\leq i\leq l} b_{i,1}$. By (59), if $h(i,m) \neq 0$ for some $1 \leq i \leq l$, then $m < D$; i.e., m is less than the bound D for the stabilization of the Hilbert function of $\mathcal{A}^l/I$. Thus, $b_{i,0} \leq \max\{b_{i,1}, D\}$ by (58). Hence, $b_{i,0}$ is bounded from above by $(lD)^{2^{O(n+1)}}$.

We have $(\mathcal{A}/I_i)_D \neq 0$ for every $1 \leq i \leq l$. This implies $H(\mathcal{A}^l/I, D) \geq l$. Let c_j denote the jth coefficient of the Hilbert polynomial of the module $\mathcal{A}^l/I$. Now $|c_j|D^j \geq l/(n+1)$ for at least one j. Hence, $D^{n+1}(n+1) \geq l$ by the condition of the lemma. This implies that $l^{2^{O(n+1)}}$ is bounded from above by $D^{2^{O(n+1)}}$. Therefore, $b_{i,0}$ is bounded from above by $D^{2^{O(n+1)}}$. The lemma is proved. □

Appendix 2: Bound for the Gröbner basis of a monomial module in terms of the coefficients of its Hilbert polynomial

We denote by $C_l = \mathbb{Z}_+^n \cup \cdots \cup \mathbb{Z}_+^n$ the disjoint union of l copies of the semigroup $\mathbb{Z}_+^n = \{(i_1, \ldots, i_n) \in \mathbb{Z}^n : i_j \geq 0,\ 1 \leq j \leq n\}$. A subset of C_l that intersects each disjoint copy of $\mathbb{Z}_+^n$ by a semigroup closed with respect to addition in $\mathbb{Z}_+^n$ is called an *ideal* of C_l. Clearly, I corresponds to a monomial submodule M_I in the free module $(F[X_1, \ldots, X_n])^l$. Any ideal I in C_l has a unique finite Gröbner basis $V = V_I$ corresponding to the Gröbner basis of M_I. Denote $T = C_l \setminus I$. The degree of an element $u = (k; i_1, \ldots, i_n) \in C_l, 1 \leq k \leq l$, is defined as $|u| = i_1 + \cdots + i_n$. The degree of a subset in C_l is defined as the supremum of the degrees of its elements. The Hilbert function $H_T(z)$ is equal to the number of vectors $u \in T$ such that $|u| \leq z$. Hence, $H_T(z) = \sum_{0 \leq s \leq m} c_s z^s$, $z \geq z_0$, for a suitable z_0 and integers $c_0, \ldots, c_m$, where $m \leq n$. Let $c = \max_{0 \leq s \leq m} |c_s| s! + 1$.

Proposition 1 (cf. [6, 12, 4])**.** *The degree of V does not exceed $(cn)^{2^{O(m)}}$.*

Proof. By an *s-cone*, $0 \leq s \leq n$, we shall mean a subset of the kth copy of $\mathbb{Z}_+^n$ in C_l for some $1 \leq k \leq l$ of the form

$$P = \{X_{j_1} = i_1, \ldots, X_{j_{n-s}} = i_{n-s}\} \tag{61}$$

for suitable $1 \leq j_1, \ldots, j_{n-s} \leq n$. We define the degree of the s-cone (61) as $|P| = i_1 + \cdots + i_{n-s}$ (note that this definition is different from that in [4]). By a *predecessor* of (61) we mean each s-cone in the same kth copy of $\mathbb{Z}_+^n$ of the type

$$\{X_{j_1} = i_1, \ldots, X_{j_{p-1}} = i_{p-1}, X_{j_p} = i_p - 1, X_{j_{p+1}} = i_{p+1}, \ldots, X_{j_{n-s}} = i_{n-s}\} \tag{62}$$

for some $1 \leq p \leq n - s$, provided that $i_p \geq 1$. We fix an arbitrary linear order on s-cones compatible with the predecessor relation.

Using inverse recursion on s, we gradually fill T (as a union) by s-cones with $0 \leq s \leq m$. We start with $s = m$. Assume that a current union $T_0 \subset T$ of m-cones is already constructed (at the very beginning we put $T_0 = \varnothing$) and that an m-cone of the form (61) with $s = m$ is the smallest one (with respect to the fixed linear order on m-cones) that is contained in T and not contained in T_0. Observe that each predecessor of this m-cone was added to T_0 at earlier steps of its construction. Since the total number of m-cones added to T_0 does not exceed $c_m m! < c$, we see that the degree of every such m-cone is less than $c_m m!$ (we use the fact that the first m-cone added to T_0 has degree 0).

For the recursive step, assume that the current T_0 is a union of all possible m-cones, $(m-1)$-cones, …, $(s+1)$-cones and perhaps, some s-cones. This can be expressed as $\deg(H_T - H_{T_0}) \leq s$. Again, as in the base, we take the smallest s-cone of the form (61) that is contained in T and not contained in T_0. Observe that each predecessor of the type (62) of this s-cone is contained in an appropriate r-cone Q, $r \geq s$, such that Q was added to T_0 at earlier steps of its construction and $Q \subset \{X_{j_p} = i_p - 1\}$. Hence,

$$|Q| \geq i_p - 1. \tag{63}$$

This construction terminates when $T_0 = T$. We denote by t_s the number of s-cones added to T_0 and by k_s the maximum of their degrees. We have already seen that $t_m, k_m < c$.

Now, we use inverse induction on s to prove that $t_s, k_s \leq (cn)^{2^{O(m-s)}}$. For this, we introduce a special semilattice on the set of cones. Let $\mathcal{C} = \{C_{\alpha,\beta}\}_{\alpha,\beta}$, $\quad 0 \leq \beta \leq \gamma_\alpha$, be a family of cones of the form (61), where $\dim C_{\alpha,\beta} = \alpha$. By an α-piece we call an α-cone that is the intersection of some cones in $\mathcal{C}$. All the pieces constitute a semilattice $\mathcal{L}$ with respect to intersection with the maximal elements in $\mathcal{C}$. We treat $\mathcal{L}$ also as a partially ordered set with respect to inclusion. Clearly, the depth of $\mathcal{L}$ is at most $n+1$. Our nearest purpose is to estimate the size of $\mathcal{L}$ from above. To simplify the bound, we assume (and this will suffice for our goal in the sequel) that $\gamma_\alpha \le (cn)^{2^{O(m-\alpha)}}$ for $s \le \alpha \le m$ and $\gamma_\alpha = 0$ when $\alpha < s$, although in the general case the required bound can be obtained in the same way. Moreover, we assume that the constant in $O(\dots)$ is sufficiently large. In what follows all the constants in $O(\dots)$ coincide.

Lemma 14. *Suppose that $\gamma_\alpha \le (cn)^{2^{O(m-\alpha)}}$ for all $s \le \alpha \le m$; see above. Then the number of α-pieces in $\mathcal{L}$ does not exceed $(cn)^{2^{O(m-\alpha)}+1}$ for $s \le \alpha \le m$, or $(cn)^{2^{O(m-s)}(s-\alpha+1)+1}$ for $\alpha < s$.*

Proof. For each α-piece, we choose its arbitrary irredundant representation as the intersection of cones in $\mathcal{C}$. Let δ be the minimal dimension of those cones. Then this intersection contains at most $\delta - \alpha + 1$ cones. Therefore, the number of possible α-pieces does not exceed

$$\sum_{\max\{\alpha,s\}\le\delta\le m} (cn)^{2^{O(m-\delta)}(\delta-\alpha+1)},$$

which proves the lemma. □

Now we return to estimating t_s, k_s by inverse induction on s. In the construction described above, let the current T_0 be the union of all added m-cones, $(m-1)$-cones, ..., s-cones. We denote this family of cones by $\mathcal{C}$ and consider the corresponding semilattice $\mathcal{L}$ (see above). Our next purpose is to represent T_0 as a $\mathbb{Z}$-linear combination of pieces in $\mathcal{L}$ via a kind of the inclusion-exclusion formula. We assign the coefficients of this combination by recursion in $\mathcal{L}$. As a base, we assign 1 to each maximal piece, i.e., the elements of $\mathcal{C}$. At a recursive step, if for some piece $P \in \mathcal{L}$ the coefficients are already assigned to all the pieces greater than P, then we assign to P the coefficient ϵ_P in such a way that the sum of the coefficients assigned to P and to all greater pieces equals 1. Therefore,

$$T_0 = \sum_{P\in\mathcal{L}} \epsilon_P P,$$

where the sum is understood in the sense of multisets. Consequently,

$$H_{T_0}(z) = \sum_{P\in\mathcal{L}} \epsilon_P \binom{z - |P| + \dim P}{\dim P} \tag{64}$$

for sufficiently large z. We recall that $\deg(H_T - H_{T_0}) \le s - 1$.

Now we estimate the coefficients $|\epsilon_P|$ with the help of induction in the semilattice $\mathcal{L}$. The inductive hypothesis on $t_\alpha \le (cn)^{2^{O(m-\alpha)}}, s \le \alpha \le m$, and Lemma 14 imply that

$$\sum_{\dim P=\lambda} |\epsilon_P| \le (cn)^{2^{O(m-\lambda)}}, \quad s-1 \le \lambda \le m,$$

in accordance with inverse induction on λ and the definition of ϵ_P. In fact, one could estimate also $\sum_{\dim P=\lambda} |\epsilon_P|$ in a similar way when $\lambda < s-1$, but we do not need this. The inductive hypothesis on $k_\alpha \le (cn)^{2^{O(m-\alpha)}}$, $s \le \alpha \le m$, and (64) imply that the coefficient in $H_{T_0}(z)$ of the power z^α does not exceed $(cn)^{2^{O(m-\alpha)}}$, $s-1 \le \alpha \le m$ (actually, by the inequality $\deg(H_T - H_{T_0}) \le s-1$, the coefficients of the powers z^α for $s \le \alpha \le m$ are less than c). In particular, the coefficient of the power z^{s-1} does not exceed $(cn)^{2^{O(m-s+1)}}$. Denote $H_T - H_{T_0} = \eta z^{s-1} + \cdots$. When constructing T_0, we add $(s-1)$-cones to it $t_{s-1} = \eta(s-1)!$ times. Hence, $t_{s-1} \le (cn)^{2^{O(m-s+1)}}$. This justifies the inductive step for t_{s-1}.

We prove that $k_{s-1} \le (cn)^{2^{O(m-s+1)}}$. We observe that, for each $(s-1)$-cone P added to T_0, either every one of its predecessors is contained in a cone of dimension at least s, or some predecessor is an $(s-1)$-cone. In the former case, $|P| \le (\max_{s\le\alpha\le m} k_\alpha + 1)(n-s+1)$

(by (63)), while in the latter case, $|P|$ is greater by 1 than the degree of that predecessor. Thus, $k_{s-1} \leq (\max_{s\leq\alpha\leq m} k_\alpha+1)(n-s+1)+t_{s-1}$. Finally, we use the inductive hypothesis for $k_m, \dots, k_s$ and the inequality on t_{s-1} obtained above.

To complete the proof of the proposition, it suffices to observe that for any vector in the basis V treated as a 0-cone, each of its predecessors of the form (62) for $s = 0$ is included in an appropriate r-cone occurring in the above construction, whence the degree of V does not exceed $(\max_{0\leq\alpha\leq m} k_\alpha + 1)n$, again by (63) (cf. above). □

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