

## PRIMARY DECOMPOSITION OF ZERO-DIMENSIONAL IDEALS OVER FINITE FIELDS

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**ABSTRACT.** A new algorithm is presented for computing primary decomposition of zero-dimensional ideals over finite fields. Like Berlekamp's algorithm for univariate polynomials, the new method is based on the invariant subspace of the Frobenius map acting on the quotient algebra. The dimension of the invariant subspace equals the number of primary components, and a basis of the invariant subspace yields a complete decomposition. Unlike previous approaches for decomposing multivariate polynomial systems, the new method does not need primality testing nor any generic projection, instead it reduces the general decomposition problem directly to root finding of univariate polynomials over the ground field. Also, it is shown how Gröbner basis structure can be used to get partial primary decomposition without any root finding.

### 1. INTRODUCTION

Let  $k$  be any field and  $I$  an ideal in the polynomial ring  $k[x_1, \dots, x_n]$ . We are interested in computing a primary decomposition of  $I$ . Most approaches in the literature follow essentially two stages. In the first stage, the decomposition of  $I$  is reduced to the decomposition of a sequence of ideals that are zero-dimensional over rational function fields containing  $k$ . In the second stage, the decomposition of each zero-dimensional ideal is then reduced to factorization of (multivariate) polynomials. In the current paper, we are interested in the second stage, that is, developing a new algorithm for decomposition of zero-dimensional ideals.

We focus on the case when  $k$  has positive characteristic, say  $k$  contains a finite field  $\mathbb{F}_q$ . For fields with characteristic zero, one usually uses modular approaches that reduce the problem to fields of positive characteristic. In the following,  $k$  may be a finite field  $\mathbb{F}_q$ , or any finitely generated function field over  $\mathbb{F}_q$ , that is,

$$k \cong \mathbb{F}_q(t_1, \dots, t_d)[y_1, \dots, y_m]/J$$

where  $t_i$ 's and  $y_i$ 's are independent variables,  $\mathbb{F}_q(t_1, \dots, t_d)$  the rational function field with  $d$  variables over  $\mathbb{F}_q$ , and  $J \subset \mathbb{F}_q(t_1, \dots, t_d)[y_1, \dots, y_m]$  a maximal ideal. Let  $I \subset k[x_1, \dots, x_n]$  be a zero-dimensional ideal. Then the current approaches use generic projection and reduce the decomposition of  $I$  to factorization of polynomials

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in  $k[t_1, \dots, t_d, z]$ . It is well recognized that generic projections are expensive to compute and the polynomials from projections are usually “large” in certain sense. Hence the current implementations of primary decomposition in major computer algebra systems (including Maple, Magma, Macaulay 2, Singular, etc.) are quite slow even for small systems of polynomials. As remarked by Decker, Greuel and Pfister [5], finding an efficient algorithm for primary decomposition is a difficult task and still one of the big challenges for computational algebra and computational algebraic geometry. For more details of the current approaches in the literature, we refer the reader to the books [1, 14, 25] and the papers [22, 13, 7, 23, 12, 5, 21, 24, 27, 28, 29].

It is somewhat surprising to us that all the existing techniques for decomposing zero-dimensional ideals over finite fields so far do not generalize any of the known methods for factoring univariate polynomials over finite fields. It was pointed out in [26] that any formula for computing zeta functions over finite fields should be useful in decomposing the ideal. This naturally explains the three algorithms [26] for factoring univariate polynomials over finite fields: Berlekamp’s algorithm [2] using the fixed points of the Frobenius map, Niederreiter’s algorithm [18, 19] using solutions of certain differential equations (fixed points of Cartier’s operator) and Wan’s algorithm [26] using fixed points of Dwork’s operator. It is thus very natural to try to extend each of the above three algorithms to the general multivariate case. In this paper, we extend Berlekamp’s approach to zero-dimensional ideals over finite fields and explore some novel features resulting from the use of Gröbner basis. For higher dimensional ideals over finite fields, Berlekamp’s approach does not seem to generalize, as there is no corresponding known formula for computing the zeta function of higher dimensional ideal using the Frobenius map (which only works for zero-dimensional ideals). However, we expect that the other two approaches (Niederreiter’s approach and Wan’s approach) should generalize to higher dimensional cases, since there are corresponding formulas for zeta functions in the higher dimensional cases.

Thus, our new method in this paper does not require any generic projection. Like Berlekamp’s algorithm, the new method is based on the invariant subspace of the Frobenius map acting on the quotient algebra. The dimension of the invariant subspace equals the number of primary components, and a basis of the invariant subspace yields a complete decomposition. The method tells automatically whether a computed component is primary, hence no need for a separate procedure to test primality. Since our method needs no generic projection, it may take advantage of possible sparsity of polynomial systems especially when the systems have sparse Gröbner bases. We shall also show how Gröbner basis structure can be used to get Gröbner bases for primary components.

The paper is organized as follows. In Section 2, we present the basic theory that works for any field  $k$  containing a finite field. In Section 3, we show how to turn the theory into an algorithm when  $k$  is a finite field. A detailed example is presented to demonstrate the ideas. Finally, we make a few comments, especially on what needs to be done when  $k$  is a finitely generated rational function field.

2. THEORY

In this section, we assume that  $k$  is any field containing  $\mathbb{F}_q$ . For basic notions in commutative algebra, especially on primary decomposition, we refer the reader to the books [1, 4, 6, 14].

An ideal  $I \subset k[x_1, \dots, x_n]$  is called *quasi-primary* if  $\sqrt{I}$  is a prime ideal, that is, if  $I$  has only one minimal component and all other components are embedded. For example,  $I = \langle x^2, xy \rangle \subset k[x, y]$  is quasi-primary but not primary. In fact,  $\sqrt{I} = \langle x \rangle$  and  $I$  has the primary decomposition

$$I = \langle x \rangle \cap \langle x^2, y \rangle = \langle x \rangle \cap \langle x^2, xy, y^m \rangle$$

where  $m \geq 1$  and the second component is embedded on the first one. Our method is based on the following key lemma.

**Lemma 2.1.** *Let  $k$  be any field containing  $\mathbb{F}_q$  as a subfield, and  $Q \subset k[x_1, \dots, x_n]$  a quasi-primary ideal. Then, for any  $g \in k[x_1, \dots, x_n]$ ,*

$$(1) \quad g^q \equiv g \pmod{Q}$$

*iff there exists  $\alpha \in \mathbb{F}_q$  such that*

$$(2) \quad g \equiv \alpha \pmod{Q}.$$

*Proof.* Clearly, (2) implies (1). Assume (1) holds. Then

$$g^q - g \equiv \prod_{\alpha \in \mathbb{F}_q} (g - \alpha) \pmod{Q}.$$

That is,  $\prod_{\alpha \in \mathbb{F}_q} (g - \alpha) \in Q \subseteq \sqrt{Q}$ . Since  $\sqrt{Q}$  is a prime ideal, there exists  $\alpha \in \mathbb{F}_q$  such that  $g - \alpha \in \sqrt{Q}$ . Hence  $(g - \alpha)^m \in Q$  for some integer  $m$ . Let  $g_1 = g - \alpha$ . Then  $g_1^q \equiv g_1 \pmod{Q}$  and  $g_1^m \equiv 0 \pmod{Q}$ . If  $m \leq q$ , then  $g_1 \equiv g_1^q \equiv 0 \pmod{Q}$ , so (2) is satisfied. Suppose  $m > q$ . Write  $m = qu + v$  where  $0 \leq v < q$  and  $u \geq 1$ . Then

$$g_1^m \equiv (g_1^q)^u \cdot g_1^v \equiv g_1^u \cdot g_1^v = g_1^{u+v} \pmod{Q}.$$

Hence  $g_1^{u+v} \equiv 0 \pmod{Q}$  and  $u + v < uq + v = m$ . If  $u + v > q$ , we can repeat this process until we find an integer  $m_1 \leq q$  so that  $g_1^{m_1} \equiv 0 \pmod{Q}$ . Therefore we have  $g_1 \equiv 0 \pmod{Q}$  as required.  $\square$

Now consider an arbitrary ideal  $I \subseteq k[x_1, \dots, x_n]$ . Suppose  $I$  has an irredundant primary decomposition

$$(3) \quad I = Q_1 \cap Q_2 \cap \dots \cap Q_r$$

where  $Q_i \in k[x_1, \dots, x_n]$  are primary ideals. In general, the components  $Q_i$  are not unique, but the number  $r$  depends only on  $I$ , not on the specific decomposition. Our goal is to determine  $r$  and find an irredundant primary decomposition as in (3).

For  $g \in k[x_1, \dots, x_n]$ , consider the equation

$$(4) \quad g^q \equiv g \pmod{I}.$$

Let  $R = k[x_1, \dots, x_n]/I$  and define

$$(5) \quad G = \{g \in R \mid g \text{ satisfies (4)}\}.$$

Then  $G$  is an  $\mathbb{F}_q$ -linear subspace of  $R$ .

We hope that the dimension of  $G$  over  $\mathbb{F}_q$  is equal to  $r$ , the number of primary components of  $I$ . Unfortunately, this is not true in general. For example, let  $I = \langle xy \rangle \subset k[x, y]$ , generated by one polynomial. Then  $I = \langle x \rangle \cap \langle y \rangle$  has two primary components. But (4) has only the trivial solutions  $g = \alpha$ ,  $\alpha \in \mathbb{F}_q$ , modulo  $I$ .

In the following, we assume that  $I$  is a zero-dimensional ideal with an irredundant primary decomposition as in (3). We note that the  $Q_i$ 's are zero-dimensional and are uniquely defined by  $I$ . Most importantly, the  $Q_i$ 's are pairwise coprime, that is, for each pair  $i \neq j$ ,

$$Q_i + Q_j = \langle 1 \rangle = k[x_1, \dots, x_n].$$

By the Chinese remainder theorem, there exist  $E_1, \dots, E_r \in k[x_1, \dots, x_n]$  such that

$$(6) \quad E_i \equiv \begin{cases} 1 & \text{mod } Q_i, \\ 0 & \text{mod } Q_j, \quad j \neq i. \end{cases}$$

These  $E_i$ 's are unique mod  $I$  and are linearly independent over  $\mathbb{F}_q$ . For  $\lambda_i \in \mathbb{F}_q$ ,  $1 \leq i \leq r$ , let

$$g = \sum_{i=1}^r \lambda_i E_i \in k[x_1, \dots, x_n].$$

Then  $g^q \equiv \lambda_i^q \equiv \lambda_i \equiv g \pmod{Q_i}$ , that is,  $g^q - g \in Q_i$  for  $1 \leq i \leq r$ . Hence  $g^q - g \in I$  and  $g$  is a solution to (4).

Conversely, if  $g \in k[x_1, \dots, x_n]$  satisfies (4), then  $g^q \equiv g \pmod{Q_i}$ ,  $1 \leq i \leq r$ . By the lemma above,  $g \equiv \alpha_i \pmod{Q_i}$  for some  $\alpha_i \in \mathbb{F}_q$ . This implies that  $g \equiv \sum_{j=1}^r \alpha_j E_j \pmod{Q_i}$  for  $1 \leq i \leq r$ , hence  $g \equiv \sum_{j=1}^r \alpha_j E_j \pmod{I}$ . We have proved the following theorem.

**Theorem 2.2.** *Let  $k$  be any field containing  $\mathbb{F}_q$ . Suppose  $I \subset k[x_1, \dots, x_n]$  is a zero-dimensional ideal with  $r$  primary components. Then the space  $G$  in (5) has dimension  $r$  over  $\mathbb{F}_q$ . Furthermore, suppose  $I$  is decomposed as in (3), and  $E_i \in k[x_1, \dots, x_n]$  satisfies (6). Then every  $g \in G$  is of the form*

$$(7) \quad g \equiv \sum_{i=1}^r \lambda_i E_i \pmod{I}$$

where  $\lambda_i \in \mathbb{F}_q$  for  $1 \leq i \leq r$ .

A solution  $g$  is called trivial if  $g \equiv \lambda \pmod{I}$  for some  $\lambda \in \mathbb{F}_q$ . Next, we show how to get a proper decomposition of  $I$  from any nontrivial solution  $g$ .

**Theorem 2.3.** *Let  $g \in k[x_1, \dots, x_n]$  be any solution of (4). Then*

$$(8) \quad I = \bigcap_{\lambda \in \mathbb{F}_q} \langle I, g - \lambda \rangle.$$

Furthermore, assume  $g$  is of the form (7) and  $I$  has a primary decomposition as in (3). Then

$$\langle I, g - \lambda \rangle = \bigcap_{1 \leq i \leq r, \lambda_i = \lambda} Q_i.$$

Hence  $\langle I, g - \lambda \rangle \neq \langle 1 \rangle$  or  $I$  iff  $\lambda = \lambda_i$  for at least one but not all  $i$ .

*Proof.* Note that, for  $1 \leq i \leq r$ ,  $g \equiv \lambda_i E_i \equiv \lambda_i \pmod{Q_i}$ . We have

$$\langle Q_i, g - \lambda \rangle = \langle Q_i, \lambda_i - \lambda \rangle = \begin{cases} \langle 1 \rangle, & \text{if } \lambda_i \neq \lambda, \\ Q_i, & \text{if } \lambda_i = \lambda. \end{cases}$$

Since the  $Q_i$ 's are pairwise coprime, we have

$$\langle I, g - \lambda \rangle = \bigcap_{i=1}^r \langle Q_i, g - \lambda \rangle = \bigcap_{1 \leq i \leq r, \lambda_i = \lambda} \langle Q_i, g - \lambda \rangle = \bigcap_{1 \leq i \leq r, \lambda_i = \lambda} Q_i.$$

This completes the proof. □

*Remark.* We should mention that the following equation, which we used above,

$$\langle Q_1 \cap Q_2 \cap \dots \cap Q_r, J \rangle = \langle Q_1, J \rangle \cap \langle Q_2, J \rangle \cap \dots \cap \langle Q_r, J \rangle$$

does not hold for general ideals  $Q_1, \dots, Q_r$  and  $J$ , but is true whenever the  $Q_i$ 's are pairwise coprime.

When  $q$  is small, (8) can be computed by plugging in each value  $\lambda \in \mathbb{F}_q$ . When  $q$  is large, we need a better method to find all the  $\lambda_i$  that appear in  $g$ . Traditionally, this is handled by computing the characteristic polynomial  $h(z)$  of  $g$  as a linear operator on the quotient algebra  $R$ , or by computing the minimal polynomial  $h(z)$  of  $g \pmod I$ , i.e., the polynomial  $h(z) \in \mathbb{F}_q[x]$  of smallest degree so that  $h(g) \equiv 0 \pmod I$ . Then all the  $\lambda_i$  are the roots of  $h(z)$ . We shall see in the next section that  $h(z)$  can be computed via Gröbner basis technique that simultaneously gives Gröbner bases for all the components  $\langle I, g - \lambda \rangle$ .

Next we show how to tell if  $\langle I, g - \lambda \rangle$  is primary. If it is not primary, then we want to know how many components it has.

**Theorem 2.4.** *Let  $B$  be any linear basis of  $G$  over  $\mathbb{F}_q$ . For  $g \in G$  and  $\lambda \in \mathbb{F}_q$ , set*

$$I_\lambda = \langle I, g - \lambda \rangle$$

and

$$B_\lambda = B \pmod{I_\lambda},$$

which means that reducing every element of  $B$  by  $I_\lambda$ , and getting rid of linearly dependent elements. Then  $B_\lambda$  is a linear basis over  $\mathbb{F}_q$  for the solution space of (5) with  $I_\lambda$  in place of  $I$ . Particularly,  $|B_\lambda|$  is equal to the number of primary components of  $I_\lambda$ .

*Proof.* Suppose  $B = \{g_1, \dots, g_r\}$ . Express  $g_j$  in the form (7):

$$g_j \equiv \sum_{i=1}^r \lambda_{ij} E_i \pmod I$$

where  $\lambda_{ij} \in \mathbb{F}_q, 1 \leq i \leq r$ . In other words,

$$(g_1, \dots, g_r) = (E_1, \dots, E_r)\Lambda,$$

where  $\Lambda = (\lambda_{ij})$  is an  $r \times r$  matrix. Since both  $g_i$ 's and  $E_i$ 's are bases for  $G$ ,  $\Lambda$  has rank  $r$ .

For  $g \in G$ , write  $g$  as in (7). For  $\lambda \in \mathbb{F}_q$ , let  $i_1, \dots, i_t$  be all the  $i, 1 \leq i \leq r$ , such that  $\lambda_i = \lambda$ . By the previous theorem,  $I_\lambda = \bigcap_{j=1}^t Q_{i_j}$  has  $t$  primary components. Let

$$G_\lambda := \{g \in R_1 \mid g^q \equiv g \pmod{I_\lambda}\},$$

where  $R_1 = k[x_1, \dots, x_n]/I_\lambda$ . By Theorem 2.2,  $\dim_{\mathbb{F}_q} G_\lambda = t$ . In fact, let  $\Lambda_{i_1 \dots i_t}$  be the submatrix of  $\Lambda$  consisting of the rows indexed by  $i_1, \dots, i_t$ . Then

$$(g_1, \dots, g_r) \equiv (E_{i_1}, \dots, E_{i_t})\Lambda_{i_1 \dots i_t} \pmod{I_\lambda}.$$

Note that  $\Lambda_{i_1, \dots, i_t}$  has rank  $t$ , and by Theorem 2.2  $E_{i_1}, \dots, E_{i_t}$  form a basis for  $G_\lambda$  over  $\mathbb{F}_q$ . Hence we have that  $g_1, \dots, g_r \pmod{I_\lambda}$  contains a basis for  $I_\lambda$ . As  $B_\lambda$  is obtained from them by deleting the dependent ones, we see that  $|B_\lambda| = t$ , the number of primary components of  $I_\lambda$ .  $\square$

Berlekamp’s algorithm [2] is the special case when  $I$  is generated by one polynomial in the univariate polynomial ring  $\mathbb{F}_q[x]$ . Suppose  $f \in \mathbb{F}_q[x]$  with

$$f = f_1^{e_1} f_2^{e_2} \cdots f_r^{e_r}$$

where  $f_1, f_2, \dots, f_r \in \mathbb{F}_q[x]$  are distinct irreducible. Then the factorization of  $f$  above is equivalent to the primary decomposition

$$\langle f \rangle = \langle f_1^{e_1} \rangle \cap \langle f_2^{e_2} \rangle \cap \cdots \cap \langle f_r^{e_r} \rangle.$$

The two theorems above immediately yield Berlekamp’s algorithm (note that  $f$  does not need to be squarefree).

### 3. IMPLEMENTATION AND GRÖBNER BASES

In this section we first assume that  $k$  is a finite field. At the end of the section, we make some comments about the case when  $k$  is infinite. To convert the theory of the previous section into an algorithm, we need an explicit representation of ideals and linear bases of quotient algebras. We do this via Gröbner basis theory. For an ideal  $I \subseteq \mathbb{F}_q[x_1, \dots, x_n]$  given by an arbitrary set of generators, it is NP-hard to compute a Gröbner basis for  $I$  (for any term order). For large systems of polynomials, computing Gröbner bases is most likely the bottleneck of our approach (which is true for almost all the algorithms in the literature). In the following, we assume that a Gröbner basis for  $I$  is already known or computed for some term order. This assumption is reasonable for several applications. For example, in a Pham system,  $I$  is generated by polynomials of the form  $x_i^{d_i} + f_i$  where  $f_i \in k[x_1, \dots, x_n]$  has total degree  $< d_i$  for  $1 \leq i \leq n$ , so they form a Gröbner basis under a graded order. Our aim is to compute a primary decomposition of  $I$ .

Assume  $I$  is a zero-dimensional ideal in  $\mathbb{F}_q[x_1, \dots, x_n]$ . Then  $R = \mathbb{F}_q[x_1, \dots, x_n]/I$  is a finite dimensional vector space over  $\mathbb{F}_q$ . Suppose  $I = \langle f_1, \dots, f_m \rangle$  where  $f_1, \dots, f_m$  form a Gröbner basis for  $I$  (for some term order). By Gröbner basis theory, the following set of monomials form a linear basis for  $R$  over  $\mathbb{F}_q$ :

$$B = (X^\alpha | \alpha \in \mathbb{N}^n, \text{ and } X^\alpha \text{ is not divisible by any } \text{LT}(f_i), 1 \leq i \leq m),$$

where  $\text{LT}(f)$  denotes the leading term of  $f$  (we emphasize that  $\text{LT}(f)$  is a monomial without the corresponding coefficient of  $f$ ), and for  $\alpha = (i_1, \dots, i_n) \in \mathbb{N}^n$ ,

$$X^\alpha = x_1^{i_1} \cdots x_n^{i_n}.$$

Here and hereafter  $X = x_1, \dots, x_n$  represents the list of variables. We order  $B$  as a row vector

$$B = (X^{\alpha_1}, \dots, X^{\alpha_d}),$$

where  $d$  is the dimension of  $R = \mathbb{F}_q[x_1, \dots, x_n]/I$  over  $\mathbb{F}_q$ . Note that with any linear basis for  $R$ , one can perform addition and multiplication in  $R$ .

Now we turn the congruence equation (4) into an explicit linear system over  $\mathbb{F}_q$ . For  $1 \leq j \leq n$ , compute

$$(X^{\alpha_j})^q \equiv \sum_{i=1}^d c_{ij} X^{\alpha_i} \pmod{I},$$

where  $c_{ij} \in \mathbb{F}_q$ . For large  $q$ ,  $X^{\alpha_j q} \pmod{I}$  can be computed by the square and multiply method. Hence each  $X^{\alpha_j q} \pmod{I}$  can be computed using  $\mathcal{O}(n \log q)$  multiplications in  $R$ . Let  $C = (c_{ij})$  be a  $d \times d$  matrix over  $\mathbb{F}_q$ . Then the above equation can be written as

$$(9) \quad B^q = B \cdot C.$$

An element  $g \in R$  can be represented as

$$g = B(a_1, \dots, a_d)^T$$

where  $a_i \in \mathbb{F}_q$  are unknowns. Then

$$g^q = B^q(a_1, \dots, a_d)^T \equiv B \cdot C(a_1, \dots, a_d)^T \pmod{I}.$$

Hence  $g^q \equiv g \pmod{I}$  if and only if

$$(10) \quad (C - I)(a_1, \dots, a_d)^T = 0,$$

where  $I$  is the  $d \times d$  identity matrix. The equation (10) is equivalent to (4). One can compute a linear basis for  $G$  using any fast algorithm for solving linear systems.

Next let  $g \in G$  be any non-trivial element and suppose  $g = \sum_{i=1}^r \lambda_i E_i$  as in (7). We need to show how to find these  $\lambda_i$ 's. Define

$$h(z) = \prod(z - \lambda_i),$$

where the product runs over all distinct values of  $\lambda_i$ 's, so  $h(z)$  is squarefree. The following theorem is a special case of the structure theorem of Gröbner bases in [11]. For completeness, we give a self-contained proof here. Note that a Gröbner basis is called minimal if the leading term of each polynomial in the basis is not divisible by the leading term of any other polynomial in the basis.

**Theorem 3.1.** *Let  $g = \sum_{i=1}^r \lambda_i E_i$  as in (7) and let  $h(z)$  be as defined above. Let*

$$J_z = \langle I, z - g \rangle \subseteq \mathbb{F}_q[x_1, \dots, x_n, z].$$

*Then, under any elimination ordering with  $x_1, \dots, x_n > z$ , any minimal Gröbner basis  $G_z$  of  $J_z$  has the following properties:*

- (a) *It contains the polynomial  $h(z)$ .*
- (b) *For any polynomial  $f$  in  $G_z$ , viewed as a polynomial in  $\mathbb{F}_q[z][x_1, \dots, x_n]$ , say*

$$f = u(z) \cdot X^\alpha + \dots + (\text{lower order terms}),$$

*where  $X^\alpha$  is the leading monomial and  $u(z) \in \mathbb{F}_q[z]$  is the leading coefficient. Then  $u(z) \mid h(z)$ .*

- (c) *For any root  $\lambda$  of  $h(z)$ , the specialization  $G_\lambda$  of  $G_z$  is a Gröbner basis for  $\langle I, \lambda - g \rangle$ .*

*Proof.* For part (a), as  $g \equiv \lambda_i \pmod{Q_i}$ , we have

$$\langle Q_i, z - g \rangle = \langle Q_i, z - \lambda_i \rangle.$$

Since the  $Q_i$ 's are pairwise coprime, we have

$$(11) \quad J_z = \bigcap_{i=1}^r \langle Q_i, z - g \rangle = \bigcap_{i=1}^r \langle Q_i, z - \lambda_i \rangle.$$

Hence  $h(z) \in J_z$  and must be one of the polynomials in  $G_z$ .

For part (b), let  $w(z) = \gcd(u(z), h(z))$ . Suppose  $u(z) \nmid h(z)$ . Then  $\ell_1 = \deg(w(z)) < \deg(u(z)) = \ell$ . Let  $s(z), t(z) \in \mathbb{F}_q[z]$  be such that  $w(z) = s(z)u(z) + t(z)h(z)$ . Define

$$w_1 := s(z) \cdot f + t(z)X^\alpha \cdot h(z) = w(z)X^\alpha + \dots \in J_z.$$

Then  $\text{LT}(w_1) = X^\alpha z^{\ell_1} < \text{LT}(f) = X^\alpha z^\ell$ , and  $\text{LT}(w_1) \mid \text{LT}(f)$ . Since  $G_z$  is a Gröbner basis, there is a polynomial  $g \in G_z$  such that  $\text{LT}(g) \mid \text{LT}(w)$ , hence  $\text{LT}(g) \mid \text{LT}(f)$ , contradictory to the assumption that  $G_z$  is minimal. Therefore, we must have  $u(z) \mid h(z)$ .

Finally, the proof of (c) is a little longer. Let  $\{g_1, \dots, g_t\}$  be any minimal Gröbner basis for the ideal  $\langle I, \lambda - g \rangle \subseteq \mathbb{F}_q[X]$ . Suppose

$$\text{LT}(g_i) = X^{\alpha_i}, \quad 1 \leq i \leq t.$$

We show below that  $G_z$  contains polynomials  $f_i \in G_z$ ,  $1 \leq i \leq t$ , of the form

$$(12) \quad f_i(X, z) = u_i(z)X^{\alpha_i} + \dots$$

with  $u_i(\lambda) \neq 0$ , where  $X^{\alpha_i} = \text{LT}(f_i)$  and  $u_i(z) \in \mathbb{F}_q[z]$  is the coefficient of  $X^{\alpha_i}$  in  $f_i(X, z)$ . Hence  $\text{LT}(f_i(X, \lambda)) = X^{\alpha_i} = \text{LT}(g_i)$ . As  $f_i(X, \lambda) \in \langle I, \lambda - g \rangle$  and  $\{g_1, \dots, g_t\}$  form a Gröbner basis, we conclude that  $\{f_1(X, \lambda), \dots, f_t(X, \lambda)\} \subseteq G_\lambda$  form a Gröbner basis for  $\langle I, \lambda - g \rangle$  as claimed.

It remains to prove that  $f_i$  in (12) exists in  $G_z$  for each  $1 \leq i \leq t$ . Without loss of generality, we assume that  $\lambda = \lambda_1$  and that  $\lambda_1, \dots, \lambda_v$  are the distinct values of  $\lambda_1, \dots, \lambda_r$ . Let

$$w(z) = \prod_{j=2}^v (z - \lambda_j).$$

Note that

$$J_z = \langle I, z - g \rangle = \bigcap_{i=1}^v \langle I, \lambda_i - g, z - \lambda_i \rangle.$$

Since  $g_i \in \langle I, \lambda - g, z - \lambda \rangle = \langle I, \lambda_1 - g, z - \lambda_1 \rangle$ , we have  $w(z)g_i \in J_z$ . By the standard division algorithm, there are polynomials  $f_j \in G_z$  and  $u_j \in \mathbb{F}_q[X, z]$ ,  $1 \leq j \leq s$ , such that

$$(13) \quad w(z)g_i(X) = u_1(X, z)f_1(X, z) + \dots + u_s(X, z)f_s(X, z)$$

with  $\text{LT}(u_j f_j) \leq \text{LT}(w(z)g_i)$ ,  $1 \leq j \leq s$ . Since  $G_z$  is assumed to be a Gröbner basis under an elimination order with  $X > z$ , we also have that

$$(14) \quad \text{LT}_X(u_j(X, z)f_j(X, z)) \leq \text{LT}_X(w(z)g_i(X)) = \text{LT}_X(g_i(X)) = X^{\alpha_i},$$

where  $\text{LT}_X(f)$  means the leading term of  $f$  as a polynomial in  $X$ . Now plugging  $z = \lambda$  into (13), we get

$$w(\lambda)g_i(X) = u_1(X, \lambda)f_1(X, \lambda) + \dots + u_s(X, \lambda)f_s(X, \lambda).$$

Note that (14) implies that

$$\text{LT}(u_j(X, \lambda)f_j(X, \lambda)) \leq \text{LT}(w(\lambda)g_i) = \text{LT}(g_i) = X^{\alpha_i}, \quad 1 \leq j \leq s.$$



There is at least one  $j$  such that  $\text{LT}(u_j(X, \lambda)f_j(X, \lambda)) = X^{\alpha_i}$ , hence  $\text{LT}(f_j(X, \lambda))$  divides  $X^{\alpha_i}$ .

By renaming  $f_1, \dots, f_s$  if needed, we may assume that  $\text{LT}(u_i(X, \lambda)f_i(X, \lambda)) = X^{\alpha_i}$ , thus  $\text{LT}(f_i(X, \lambda))$  divides  $X^{\alpha_i}$ . We claim that  $\text{LT}(f_i(X, \lambda)) = X^{\alpha_i}$ . Otherwise, since  $f_i(X, \lambda) \in \langle I, \lambda - g \rangle$  and  $\{g_1, \dots, g_t\}$  is a Gröbner basis for the latter,  $\text{LT}(f_i(X, \lambda))$  would be divisible by  $\text{LT}(g_\ell)$  for some  $\ell \neq i$ , hence  $\text{LT}(g_\ell)$  would divide  $\text{LT}(g_i)$ , contradicting to the assumption that  $\{g_1, \dots, g_t\}$  is minimal. It follows that  $\text{LT}_X(f_i(X, z)) = X^{\alpha_i}$  and the coefficient of  $X^{\alpha_i}$  in  $f_i(X, z)$  is a polynomial  $u_i(z) \in \mathbb{F}_q[z]$  such that  $u_i(\lambda) \neq 0$ . This proves that  $G_z$  has a polynomial of the form (12) for each  $1 \leq i \leq t$ . The proof is complete.  $\square$

Note that if  $G$  is a Gröbner basis for  $I$  under a term order  $\sigma$ , then  $G \cup \{z - g\}$  is a Gröbner basis for  $J_z$  under the elimination order  $z > x_1, \dots, x_n$  with the  $x_i$ 's ordered by  $\sigma$ . Hence the Gröbner basis  $G_z$  for  $J_z$  under the elimination order  $x_1, \dots, x_n > z$  can be computed by a change of term order using the algorithm in [8]. After  $G_z$  is computed, the Gröbner bases for the components  $\langle I, \lambda - g \rangle$  are obtained from  $G_z$  by specializing  $z$  to the roots of the polynomial  $h(z)$ . Also, the leading coefficients  $u(z)$  automatically give a partial factorization of  $h(z)$ .

Based on the above results, one can write an algorithm explicitly for computing primary decomposition for zero-dimensional ideals over  $\mathbb{F}_q$ . We shall not present the details of such an algorithm, instead we demonstrate below by giving a concrete example, which we hope is more helpful in understanding the ideas. Consider the ideal

$$I = \langle y^2 - xz, z^2 - x^2y, x + y + z - 1 \rangle \subset \mathbb{F}_5[x, y, z].$$

One can use any term order in the following computation, but we happen to use lex order. Under the lex order with  $x > y > z$ ,  $I$  has a Gröbner basis

$$G = [x + y + z - 1, y^2 + 3y - 2z^4 + z^3 + 2z^2 + z, yz + 2y + 2z^4 - z^3 - z^2 - 2z, z^5 - z^4 + 3z^3 - z^2 + 2z],$$

and the corresponding monomial basis for  $R = \mathbb{F}_5[x, y, z]/I$  is

$$B = (z^4, z^3, z^2, z, y, 1).$$

The matrix  $C$  in (9) of the Frobenius map on  $R$  under this basis is

$$C = \begin{bmatrix} -2 & -1 & 1 & 1 & 1 & 0 \\ -1 & -1 & 2 & 2 & 0 & 0 \\ 2 & -1 & 2 & 1 & 0 & 0 \\ -1 & -2 & 2 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The solution space of the linear equation system (10) has a basis with four elements:

$$\begin{aligned} (0, 0, 0, 0, 0, 1) &\longleftrightarrow g_1 = 1, \\ (0, 0, -1, 1, 0, 0) &\longleftrightarrow g_2 = z - z^2, \\ (0, 1, 1, 0, 0, 0) &\longleftrightarrow g_3 = z^2 + z^3, \\ (-2, 1, 0, 0, 0, 0) &\longleftrightarrow g_4 = z^3 - 2z^4. \end{aligned}$$

Hence  $I$  has four primary components.

Choose  $g = g_2 = z - z^2$  and let  $J = \langle I, w - g \rangle \subseteq \mathbb{F}_5[x, y, z, w]$ . A Gröbner basis  $G_w$  of  $J$  under the lex order with  $x > y > z > w$  follows:

$$\begin{aligned} w^4 + w^3 + w^2 + w, & \quad (w - 2)z + 2w^3 + w^2, \quad z^2 - z + w, \\ (w + 1)y + zw - z - w, & \quad yz - 2yw - 2z^2w - 2z^2 + 2zw + 2z, \\ y^2 + yz + z^2 - z, & \quad x + y + z - 1. \end{aligned}$$

The polynomial  $h = w^4 + w^3 + w^2 + w$  has four different roots:  $w = 0, -1, -2, 2$ . Note that two of the roots can be seen from the leading coefficients  $w - 2$  and  $w + 1$ . As the degree of  $h$  equals 4, the dimension of the solution space, we get a primary component for each of the roots. For example, if  $w = 0$ , then the above Gröbner basis becomes

$$G_0 = \{-2z, z^2 - z, y - z, yz + 3z^2 + 2z, y^2 + yz + z^2 - z, x + y + z - 1\}.$$

Deleting the redundant polynomials (i.e., those whose leading terms are divisible by the leading terms of other polynomials), we have

$$Q_1 = \langle G_0 \rangle = \langle -2z, y - z, x + y + z - 1 \rangle = \langle z, y, x - 1 \rangle,$$

where the basis was reduced in the last equation. Similarly, for the other three components:

$$\begin{aligned} w = -1 : & \quad Q_2 = \langle z + 2, y^2 - 2y + 1, x + y + 2 \rangle, \\ w = -2 : & \quad Q_3 = \langle z - 2, y - 1, x + 2 \rangle, \\ w = 2 : & \quad Q_4 = \langle z^2 - z + 2, y + 2z + 1, x - z + 3 \rangle. \end{aligned}$$

Then  $I = Q_1 \cap Q_2 \cap Q_3 \cap Q_4$  is a primary decomposition.

In the above computation, we obtained all the primary components using only the solution  $g$ . In this case, we say  $g$  is separable for  $I$ . Note that  $g$  is separable iff the  $\lambda_i$ 's in (7) are distinct, i.e., iff  $h(z)$  has degree  $r$ . One can prove that when the field  $\mathbb{F}_q$  is large relative to  $r$ , a random solution  $g$  is separable with high probability. In fact, Theorem 2.10 in [9] applies here directly and one has the following proposition.

**Proposition 3.2.** *The probability that a random solution  $g \in G$  in (5) is separable, is at least  $1 - \frac{r(r-1)}{2q}$ .*

Next we show how to apply Theorem 2.4 when  $g$  is not separable. Suppose we happen to pick  $g = g_3 = z^3 + z^2$ . Then the Gröbner basis of  $J = \langle I, w - g \rangle$  under the lex order with  $x > y > z > w$  follows:

$$\begin{aligned} w^3 + 2w^2 + 2w, & \quad z^3 + z^2 - w, \quad yz + 2y + 2z^2 + 2w^2, \\ (w - 1)z - w^2 + w, & \quad (w - 1)y + 2w^2 + 3w, \\ y^2 + 3y - z^2 - z + 3w^2, & \quad x + y + z - 1. \end{aligned}$$

The leading coefficient  $w - 1$  gives the root  $w = 1$  for  $h = w^3 + 2w^2 + 2w$ , and the other two roots of  $h$  are  $w = 0, 2$ . Note that  $h(z)$  has degree  $3 < r = 4$ , so  $g$  is not separable for  $I$ .

Specializing  $w$  to 0, 1 and 2 in the above basis gives three components:

$$\begin{aligned} w = 0: \quad & Q_1 = \langle z, y, x - 1 \rangle \\ w = 1: \quad & I_2 = \langle x + y + z - 1, y^2 - 2y - z^2 - z - 2, yz + 2y + 2z^2 + 2, z^3 + z^2 - 1 \rangle \\ w = 2: \quad & Q_3 = \langle z - 2, y - 1, x + 2 \rangle. \end{aligned}$$

For each of the components, we apply Theorem 2.4. We demonstrate this on  $I_2$ . Reduce the basis polynomials  $g_1, g_2, g_3, g_4$  by  $I_2$  to get

$$1, -z^2 + z, 1, 2z^2 + 3z + 3,$$

for which 1 and  $-z^2 + z$  are linearly independent over  $\mathbb{F}_5$  and the other two polynomials are linearly dependent on them. Hence the solution space for  $I_2$  in (5) has dimension  $r = 2$ , or equivalently  $I_2$  has two primary components.

To decompose  $I_2$ , we compute a Gröbner basis for the ideal  $\langle I_2, w - (z - z^2) \rangle$ :

$$\begin{aligned} & w^2 - w + 3, \quad (w + 3)z + 2w + 1, \quad yz + 2y + 2z + 3w + 2, \\ & z^2 - z + w, \quad (w + 1)y + z + 2w + 4, \quad y^2 + 3y + 3z + w + 3, \quad x + y + z - 1. \end{aligned}$$

As the factors  $w + 1$  and  $w + 3$  appear as leading coefficients,  $-1$  and  $-3 = 2$  are roots of  $h = w^2 - w + 3$ . Plugging each of them back into this Gröbner basis gives two primary components of  $I_2$  (after reduction):

$$Q_2 = \langle z + 2, y^2 - 2y + 1, x + y + 2 \rangle, \quad Q_4 = \langle z^2 - z + 2, y + 2z + 1, x - z + 3 \rangle.$$

They give the same decomposition of  $I$  as above.

Note that in the above example,  $Q_1, Q_3$  and  $Q_4$  are prime ideals but not  $Q_2$ . In fact,  $\sqrt{Q_2} = \langle z + 2, y - 1, x - 2 \rangle$ . Also, if we change the polynomial  $x + y + z - 1$  to  $x + y + z$  in  $I$  and apply our algorithm to the modified ideal

$$I_1 = \langle y^2 - xz, z^2 - x^2y, x + y + z \rangle \subset \mathbb{F}_5[x, y, z],$$

then  $I_1 = Q_5 \cap Q_6$  where

$$Q_5 = \langle z^2, y^2 + yz, x + y + z \rangle, \quad Q_6 = \langle z^2 + z + 1, y + z + 1, x - 1 \rangle.$$

Note that  $\sqrt{Q_5} = \langle x, y, z \rangle$  and  $\sqrt{Q_6} = Q_6$ . Hence  $Q_6$  is a prime ideal but not  $Q_5$ . (Note that a primary ideal  $Q$  is a prime ideal iff  $\sqrt{Q} = Q$ .)

Our algorithm computes an irredundant primary decomposition of a zero-dimensional ideal over  $\mathbb{F}_q$ . If the associated primes are needed, one can simply compute the radicals of the primary ideals that appear in the decomposition.

Finally, when  $k$  is not a finite field, but a more general function field over  $\mathbb{F}_q$  (or over  $\mathbb{Q}$ ) as mentioned in the introduction, much more needs to be done. Equation (4) is a quasi-linear system over  $k$ , so not a finite linear system over  $\mathbb{F}_q$  as  $k$  has infinite dimension as a vector space over  $\mathbb{F}_q$ . A practical approach is to use a modular method, that is, reduce by a maximal ideal and do Hensel lifting. This will convert the congruence equation (4) to a finite linear system over  $\mathbb{F}_q$ . The main problem is that, under a reduction at a prime ideal, a primary ideal may split into many primary ideals, so one needs to find the right combinations to get back to the original primary components. For polynomial factorization, which corresponds to the special case when  $I$  is generated by one polynomial, efficient algorithms were recently developed by van Hoeij [15] and Lecerf [16]. It is desirable to develop a similar theory for systems of multivariate polynomials. As mentioned in the introduction, the other two approaches are to use differential equations and Dwork's operators, which behave much better than the Frobenius operator in higher

dimensional cases. For factoring bivariate polynomials, the paper [10] shows how to solve related quasi-linear systems over function fields. We hope to explore these approaches for general ideals in a future paper.

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