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Bases for projective modules in $A_n(k)$

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Abstract

Let $A_n(k)$ be the Weyl algebra, with k a field of characteristic zero. It is known that every projective finitely generated left module is free or isomorphic to a left ideal. Let M be a left submodule of a free module. In this paper we give an algorithm to compute the projective dimension of M. If M is projective and $\text{rank}(M) \geq 2$ we give a procedure to find a basis.

Keywords: Projective modules; Non-commutative rings; Gröbner bases

Introduction

The study of finitely generated projective modules over a ring is an interesting topic. We know that over polynomial rings they are free, as it was shown by Quillen and Suslin. There are several algorithmic versions of this theorem (Logar and Sturmfels, 1992; Laubenbacher and Woodburn, 1997; Gago-Vargas, 2002) that compute a basis from a system of generators. All of these procedures use Gröbner bases in polynomial rings. It is natural to extend these results to the Weyl Algebra $A_n(k)$, with k a field with characteristic zero. It is known that if a left finitely generated $A_n(k)$-module is projective and has rank greater or equal 2 then is free (Stafford, 1978). Our goal is to give an algorithm to find a basis of these modules.

Projective modules in $A_n(k)$ are stably free (Stafford, 1977), so the first step is to find an isomorphism $P \oplus A_n(k)^s \simeq A_n(k)^t$ for some s, t. We develop this procedure in Section 1, together with an algorithm to compute the projective dimension of a module, that is valid for a broad class of rings. We note by $\text{pdim}(M)$ the projective dimension of a module M. We require the computation of Gröbner bases in the ring and that every module has a finite free resolution. If M is projective we find a matrix that defines an isomorphism

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$M \oplus R^s \simeq R^t$. The starting point is a left R-module M defined by a system of generators in some R^m.

In Section 2 we follow the proof of Stafford (1978) with algorithmic tools to find a basis of a projective module. We develop, for completeness, the reference to Swan (1968) used in Stafford (1978, Theorem 3.6(a)), to clarify where these computations are needed. We follow describing the minor changes to Hillebrand and Schmale (2002) to obtain two special generators of a left ideal, according to Stafford (1978, Theorem 3.1). Finally, we give an example of this procedure to build a basis of a projective module in $A_2(\mathbb{Q})$.

For all the computations we need an effective field k in the sense of Cohen (1999) to apply the Gröbner bases algorithm in $A_n(k)$. We have used in the examples $k = \mathbb{Q}$.

1. Computing projective dimension

Let R be a ring where it is possible to compute a finite free resolution of a left module, and we can determine if a right submodule of R^k is equal to R^k. Such a ring may be $k[x_1, \ldots, x_n]$, $A_n(k)$ or more general rings like PBW algebras (Bueso et al., 1998). We make use of a characterization given in Logar and Sturmfels (1992), based on a finite free resolution of a module. The existence of a finite free resolution for a projective module M is equivalent for M to be stably free (McConnell and Robson, 1987). With the algorithm described in this section we test wether M is projective, and if the answer is yes we compute an isomorphism $M \oplus R^s \simeq R^t$ for some s, t. The procedure is by induction on the length of the resolution. We identify the homomorphisms with their matrices to simplify the notation.

Suppose

$$0 \to F_1 \stackrel{\alpha_1}{\to} F_0 \stackrel{\alpha_0}{\to} M \to 0$$

is a free resolution of M, with $\mathrm{rank}(F_i) = r_i$. If M is a projective module, this sequence splits, so there exists $\beta_1 : F_0 \to F_1$ such that $\beta_1\alpha_1 = I_{r_1}$. We can compute this matrix from the rows of the matrix α_1: if we consider them as vectors of F_1, the right R-module generated must be equal to F_1. We express each vector of the canonical basis of F_1 as a linear combination of the rows of α_1, and with these coefficients we construct the matrix β_1. So we can give the isomorphism $F_1 \oplus \ker(\beta_1) \simeq F_0 \simeq F_1 \oplus M$ and a basis of $F_1 \oplus \ker(\beta_1)$.

Let

$$\mathcal{F} : 0 \to F_t \xrightarrow{\alpha_t} F_{t-1} \xrightarrow{\alpha_{t-1}} F_{t-2} \xrightarrow{\alpha_{t-2}} F_{t-3} \xrightarrow{\alpha_{t-3}} \cdots \xrightarrow{\alpha_1} F_0 \xrightarrow{\alpha_0} M \to 0$$

be a finite free resolution of M with $\mathrm{rank}(F_i) = r_i$ and $t \geq 2$ (we take α_{-1} the null homomorphism). Again, if M is a projective module, then the short exact sequence

$$0 \to \ker(\alpha_0) \to F_0 \to M \to 0$$

splits, so $\ker(\alpha_0) = \mathrm{im}(\alpha_1)$ is projective. By induction, the modules $\mathrm{im}(\alpha_i)$, $i = 1, \ldots, t$ are projective. In particular, $\mathrm{im}(\alpha_{t-1})$ is projective and the exact sequence

$$0 \to F_t \xrightarrow{\alpha_t} F_{t-1} \xrightarrow{\alpha_{t-1}} \mathrm{im}(\alpha_{t-1}) \to 0$$

splits. Then there exists $\beta_t : F_{t-1} \to F_t$ such that $I_{r_t} = \beta_t \alpha_t$. The module $\ker(\beta_t)$ is projective, isomorphic to $\mathrm{im}(\alpha_{t-1})$ and we can compute the isomorphism $\ker(\beta_t) \oplus F_t \simeq F_{t-1}$. We consider the following sequence:

$$0 \to F_t \xrightarrow{\widetilde{\alpha}_t} F_{t-1} \oplus F_t \xrightarrow{\widetilde{\alpha}_{t-1}} F_{t-2} \oplus F_t \xrightarrow{\widetilde{\alpha}_{t-2}} F_{t-3} \xrightarrow{\alpha_{t-3}} \cdots \xrightarrow{\alpha_1} F_0 \xrightarrow{\alpha_0} M \to 0$$

where

$$\widetilde{\alpha}_t(\mathbf{v}_t) = (\alpha_t(\mathbf{v}_t), \mathbf{0}), \qquad \widetilde{\alpha}_{t-1}(\mathbf{v}_{t-1}, \mathbf{v}_t) = (\alpha_{t-1}(\mathbf{v}_{t-1}), \mathbf{v}_t),$$
$$\widetilde{\alpha}_{t-2}(\mathbf{v}_{t-2}, \mathbf{v}_t) = \alpha_{t-2}(\mathbf{v}_{t-2}).$$

Then it is an exact sequence and again the module $\mathrm{im}(\widetilde{\alpha}_{t-1})$ is projective. As before, the sequence

$$0 \to F_t \xrightarrow{\widetilde{\alpha}_t} F_{t-1} \oplus F_t \xrightarrow{\widetilde{\alpha}_{t-1}} \mathrm{im}(\widetilde{\alpha}_{t-1}) \to 0 \tag{1}$$

splits and there exists $\widetilde{\beta}_t : F_{t-1} \oplus F_t \to F_t$ such that $I_{r_t} = \widetilde{\beta}_t \widetilde{\alpha}_t$. In this case,

$$\widetilde{\beta}_t = (\, \beta_t \quad \theta \,)$$

where θ is the null matrix with order $r_t \times r_t$. Then $\widetilde{\beta}(\mathbf{v}_{t-1}, \mathbf{v}_t) = \beta_t(\mathbf{v}_{t-1})$, so $\ker(\widetilde{\beta}_t) = \ker(\beta_t) \oplus F_t \simeq F_{t-1}$. We can compute the isomorphism

$$\widetilde{\nu}_{t-1} : F_{t-1} \to \ker(\widetilde{\beta}_t).$$

Let

$$\widetilde{\gamma}_{t-1} = \widetilde{\alpha}_{t-1} \widetilde{\nu}_{t-1} : F_{t-1} \to F_{t-2} \oplus F_t. \tag{2}$$

Then the sequence

$$0 \to F_{t-1} \xrightarrow{\widetilde{\gamma}_{t-1}} F_{t-2} \oplus F_t \xrightarrow{\widetilde{\alpha}_{t-2}} F_{t-3} \xrightarrow{\alpha_{t-3}} \cdots \xrightarrow{\alpha_1} F_0 \xrightarrow{\alpha_0} M \to 0$$

is exact. Because the sequence (1) splits, the homomorphism $\widetilde{\alpha}_{t-1}$ is an isomorphism between $\ker(\widetilde{\beta}_t)$ and $\mathrm{im}(\widetilde{\alpha}_{t-1})$, so $\widetilde{\gamma}_{t-1}$ is an isomorphism between F_{t-1} and $\mathrm{im}(\widetilde{\alpha}_{t-1}) = \ker(\widetilde{\alpha}_{t-2})$, and we have the exactness of the sequence (2). We apply again the process to $\widetilde{\gamma}_{t-1}$ to check the projectiveness of the module M.

We need the following result:

Theorem 1.1. *Let R be a ring and*

$$\mathcal{F} : \cdots \to F_d \to F_{d-1} \to \cdots \to F_1 \to F_0 \to M \to 0$$

a projective resolution. Let d be the smallest number such that $\{\mathrm{im} F_d \to F_{d-1}\}$ is projective. Then d does not depend on the resolution and $\mathrm{pdim}(M) = d$.

Proof. Eisenbud (1995, Exercise A.3.13). □

Theorem 1.2. *The previous algorithm allows us to compute the projective dimension of a module.*

Proof. Let

$$0 \to F_n \xrightarrow{\alpha_n} F_{n-1} \xrightarrow{\alpha_{n-1}} \cdots \to F_1 \xrightarrow{\alpha_1} F_0 \xrightarrow{\alpha_0} M \to 0$$

be a finite free resolution given by the procedure. Then $\mathrm{im}(\alpha_{n-1})$ is not projective, because the matrix α_n has no left inverse. We can suppose that M is not projective, otherwise we have shortened the resolution. Then the sequence

$$0 \to \ker(\alpha_0) \to F_0 \to M \to 0$$

does not split, so $\mathrm{im}(\alpha_1) = \ker(\alpha_0)$ is not projective. In the same way, the short exact sequence

$$0 \to \ker(\alpha_1) \to F_1 \to \mathrm{im}(\alpha_1) \to 0$$

does not split and $\mathrm{im}(\alpha_2) = \ker(\alpha_1)$ is not projective. Then the modules

$$\mathrm{im}(\alpha_1), \mathrm{im}(\alpha_2), \ldots, \mathrm{im}(\alpha_{n-1})$$

are not projective and the module $\mathrm{im}(\alpha_n)$ is projective. Then the projective dimension of M is equal to n. $\square$

Algorithm. Projective dimension.

Input: a left R-module M defined by its generators in R^r.

Output: a projective dimension of M and a minimal length free resolution. If $\mathrm{pdim}(M) = 0$, i.e. M is projective, the algorithm returns an isomorphism $M \oplus R^s \simeq R^t$.

Let $\mathcal{F}$ be a finite free resolution of M:

$$0 \to F_t \xrightarrow{\alpha_t} F_{t-1} \xrightarrow{\alpha_{t-1}} F_{t-2} \xrightarrow{\alpha_{t-2}} F_{t-3} \xrightarrow{\alpha_{t-3}} \cdots \xrightarrow{\alpha_1} F_0 \xrightarrow{\alpha_0} M \to 0$$

START:

if α_t has no left inverse **then**
  $\mathrm{pdim}(M) = t$. STOP.
else
  let β_t be a left inverse of α_t.
end if
if $t = 1$ **then**
  $\mathrm{pdim}(M) = 0$ and $M \oplus F_1 \simeq \ker(\beta_1) \oplus F_1 \simeq F_0$. STOP.
else
  compute the exact sequence

$$0 \to F_t \xrightarrow{\widetilde{\alpha}_t} F_{t-1} \oplus F_t \xrightarrow{\widetilde{\alpha}_{t-1}} F_{t-2} \oplus F_t \xrightarrow{\widetilde{\alpha}_{t-2}} F_{t-3} \xrightarrow{\alpha_{t-3}} \cdots \xrightarrow{\alpha_1} F_0 \xrightarrow{\alpha_0} M \to 0$$

and the matrix $\widetilde{\nu}_{t-1}$ that gives the isomorphism $\ker(\beta_t) \oplus F_t \simeq F_{t-1}$.
end if
Let $\widetilde{\gamma}_{t-1} = \widetilde{\alpha}_{t-1}\widetilde{\nu}_{t-1}$.
Let $\mathcal{F}$ be the finite free resolution

$$0 \to F_{t-1} \xrightarrow{\widetilde{\gamma}_{t-1}} F_{t-2} \oplus F_t \xrightarrow{\widetilde{\alpha}_{t-2}} F_{t-3} \xrightarrow{\alpha_{t-3}} \cdots \xrightarrow{\alpha_1} F_0 \xrightarrow{\alpha_0} M \to 0.$$

go to START.

This algorithm has been programmed with *Macaulay 2* (Grayson and Stillman, 2000), using the routines for D-modules developed by Leykin and Tsai (2002).

Example. Let $W = A_2(\mathbb{Q})$ and $I = W\langle x\partial_x - 1, x\partial_y, \partial_x^2, \partial_y^2\rangle$. We found a resolution of I of the form

$$0 \leftarrow I \overset{\widetilde{\alpha}_0}{\leftarrow} W^4 \overset{\widetilde{\gamma}_1}{\leftarrow} W^3 \leftarrow 0$$

where

$$\widetilde{\gamma}_1 = \begin{pmatrix} -\partial_x^2 & -x\partial_x + 1 & 0 \\ \partial_y & 0 & -x \\ 0 & \partial_y & \partial_x \\ -\partial_x & -x & 0 \end{pmatrix}.$$

The rows of the matrix $\widetilde{\gamma}_1$ do not generate W^3, because a Gröbner basis is given by the columns of the matrix

$$\begin{pmatrix} 0 & 0 & \partial_y & \partial_x \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Then the ideal I is not projective, and its projective dimension is 1.

2. Computing a basis

Let k be a field of characteristic zero. Given a projective module over $A_n(k)$ with rank greater than 1, we are going to describe a procedure to compute a basis. We will need the standard Gröbner basis theory on $A_n(k)$ to perform the computations. See, for example, Castro (1987) for a description of this algorithm. In Hillebrand and Schmale (2002) we found the following theorem.

Theorem 2.1. *Let $\mathcal{R} = k(x_1, \ldots, x_n)[\partial_1, \ldots, \partial_n]$ and $I = \mathcal{R}\langle a, b, c\rangle$. Then we can compute $\tilde{a}, \tilde{b} \in \mathcal{R}$ such that $I = \mathcal{R}\langle a + \tilde{a}c, b + \tilde{b}c\rangle$.*

As pointed out in Hillebrand and Schmale (2002, Remark 3.15), the algorithm can be extended to $W = A_n(k) = k[x_1, \ldots, x_n][\partial_1, \ldots, \partial_n]$. We need the following stronger result (Stafford, 1978, Theorem 3.1):

Theorem 2.2. *Let $I = W\langle a, b, c\rangle$ be a left W-ideal, and let $d_1, d_2 \in W - \{0\}$ be arbitrary elements. Then we can find $f_1, f_2 \in W$ such that*

$$I = W\langle a + d_1 f_1 c, b + d_2 f_2 c\rangle.$$

This can be accomplished with some minor changes to the proof of Hillebrand and Schmale (2002, Lemma 3.10). Following their notation, it is enough to take $g_1, g_2 \in W$ such that $h_1 d_1 g_1 + h_2 d_2 g_2 = 0$, and to apply (Hillebrand and Schmale, 2002, Lemma 3.9) to $v = t d_2 g_2$. These changes appear in the proof of Stafford (1978, Theorem 3.1). The procedure is analogous for right ideals.

Definition. Let M be a left W-module and $\mathbf{v} \in M$. We say that $\mathbf{v}$ is unimodular in M if there exists $\varphi \in \mathrm{Hom}_W(M, W)$ such that $\varphi(\mathbf{v}) = 1$.

Remark. If $\mathbf{v}$ is a column vector in some W^m then $\mathbf{v}$ is unimodular if and only if the right ideal generated by its entries is equal to W. Through Gröbner bases, we can give the homomorphism that apply $\mathbf{v}$ in 1.

The following Lemma is a direct consequence of Theorem 2.2, and it will allow a 'cancellation' in some direct sums.

Lemma 2.1 (Stafford, 1978, Lemma 3.5). *Let $M \subset W^m$ be a left W-module with* $\text{rank}(M) \geq 2$ *and* $\mathbf{a} \oplus t \in M \oplus W$ *unimodular. Then there is an algorithm to find* $\Phi \in \text{Hom}_W(W, M)$ *such that* $\mathbf{a} + \Phi(t)$ *is unimodular in* M.

Proof. Let $\mathbf{a}_1 \in M \subset W^m$ be a non-zero element and consider $\Phi_1 : W^m \to W$ a projection such that $\Phi_1(\mathbf{a}_1) \neq 0$. Let $M_1 = M \cap \ker(\Phi_1)$, that we can compute by Gröbner bases. Then $\text{rank}(M_1) = \text{rank}(M) - 1 \geq 1$, so there exists $\mathbf{a}_2 \in M_1 - \mathbf{0}$. Let $\Phi_2 : W^m \to W$ be a projection such that $\Phi_2(\mathbf{a}_2) \neq 0$. If $\Phi_2(\mathbf{a}_1) \neq 0$ we can compute syzygies to get $r_1, r_2 \in W$ such that $\Phi_1(\mathbf{a}_1)r_1 + \Phi_2(\mathbf{a}_2)r_2 = 0$ and replace Φ_2 by the homomorphism $\Phi_1 r_1 + \Phi_2 r_2$. Then $\Phi_1(\mathbf{a}_2) = \Phi_2(\mathbf{a}_1) = 0$. Let $d_1 = \Phi_1(\mathbf{a}_1)$, $d_2 = \Phi_2(\mathbf{a}_2)$ and consider the right ideal

$$I = \langle \Phi_1(\mathbf{a}), \Phi_2(\mathbf{a}), t \rangle W.$$

Then there exist $f_1, f_2 \in W$ such that

$$I = \langle \Phi_1(\mathbf{a}) + t f_1 d_1, \Phi_2(\mathbf{a}) + t f_2 d_2 \rangle W.$$

Let $\Phi : W \to M$ be the homomorphism defined by $\Phi(1) = f_1 \mathbf{a}_1 + f_2 \mathbf{a}_2$. Then, as shown in Stafford (1978, Lemma 3.5), $\mathbf{a} + \Phi(t)$ is unimodular, and we can compute $j \in \text{Hom}_W(M, W)$ such that $j(\mathbf{a} + \Phi(t)) = 1$. $\square$

Remark. The case $\mathbf{a} \neq \mathbf{0}$ is of special interest. In this case we can take $\mathbf{a}_1 = \mathbf{a}$ and obtain $\Phi_2(\mathbf{a}) = 0$, $d_1 = \Phi_1(\mathbf{a})$. We have to find f_1, f_2 such that

$$I = \langle d_1, 0, t \rangle W = \langle d_1 + t f_1 d_1, t f_2 d_2 \rangle W.$$

Note that the problem is not to find two generators for the ideal I. We are looking for two special generators.

Proposition 2.1 (Swan, 1968, Corollary 12.6). *Let $M \subset W^m$ be a left W-module with* $\text{rank}(M) \geq 2$ *and $h : W \oplus N \to W \oplus M$ be an isomorphism with N a left W-module. Then $M \simeq N$.*

Proof. Let $h(1, \mathbf{0}) = (t_0, \mathbf{a}_0) \in W \oplus M$. The vector $(1, \mathbf{0})$ is unimodular so $(t_0, \mathbf{a}_0)$ too. Then we compute $\Phi : W \to M$ such that $\mathbf{a}'_0 = \mathbf{a}_0 + \Phi(t_0)$ is unimodular in M and we get the homomorphism $j : M \to W$ with $j(\mathbf{a}'_0) = 1$. We consider the following homomorphisms:

$$\begin{array}{ll} g : W \oplus M \to W \oplus M, & g(t, \mathbf{a}) = (t, \mathbf{a} + \Phi(t)) \\ k : W \to W, & k(1) = t_0 \\ l : W \oplus M \to W \oplus M, & l(t, \mathbf{a}) = (t - (k \circ j)(\mathbf{a}), \mathbf{a}), \\ i : W \oplus N \to W \oplus M, & i = l \circ g \circ h. \end{array}$$

Then i is isomorphism and $i(1, \mathbf{0}) = (0, \mathbf{a}'_0)$. We have $M = W\mathbf{a}'_0 \oplus \ker(j)$ and the following chain of isomorphisms:

$$N \simeq (W \oplus N)/W\mathbf{e}_1 \stackrel{i}{\to} (W \oplus M)/W\mathbf{a}'_0 = (W \oplus \ker(j) \oplus W\mathbf{a}'_0)/W\mathbf{a}'_0$$
$$\simeq W \oplus \ker(j) \simeq W\mathbf{a}'_0 \oplus \ker(j) = M.$$

The isomorphism is defined as follows. Take $\mathbf{v}_1, \ldots, \mathbf{v}_r$ a set of generators of N. Let $i(0, \mathbf{v}_i) = (\alpha_i, \mathbf{u}_i)$, where $\alpha_i \in W, \mathbf{u}_i \in M$. The map $(W \oplus M)/W\mathbf{a}'_0 \to W \oplus \ker(j)$ works taking an element of $W \oplus M$, decomposes the component in M as a sum $\mathbf{v} + \mathbf{w}$ with $\mathbf{v} \in W\mathbf{a}'_0, \mathbf{w} \in \ker(j)$ and takes $\mathbf{w}$. For this step note that if $\mathbf{u} \in M$ and $\lambda = j(\mathbf{u})$ then $\mathbf{u} = (\lambda\mathbf{a}'_0) + (\mathbf{u} - \lambda\mathbf{a}'_0)$ is the desired decomposition. □

Remark. When the module N is of the form W^s, then M is isomorphic to a free module, so it has a basis. Such a basis is the image of $\mathbf{e}_i$, $i = 1, \ldots, s$.

Algorithm. Computing a basis.

Input: an isomorphism $W^t \stackrel{h}{\simeq} W^s \oplus M$, with $t - s \geq 2$.

Output: a basis of the module M.

START:

if $s = 0$ **then**
 $\{h(\mathbf{e}_1,), \ldots, h(\mathbf{e}_t)\}$ is a basis.
 STOP.
end if

Let $h(1, \mathbf{0}) = (t_0, \mathbf{a}_0)$, with $t_0 \in W, \mathbf{a}_0 \in W^{s-1} \oplus M$.

Compute $\Phi : W \to W^{s-1} \oplus M$ such that $\mathbf{a}'_0 = \mathbf{a}_0 + \Phi(t_0)$ is unimodular.

Compute $j : W^{s-1} \oplus M \to W$ such that $j(\mathbf{a}'_0) = 1$.

Let $i : W \oplus W^{t-1} \to W \oplus (W^{s-1} \oplus M)$ as defined in Proposition 2.1.

Let $h : W^{t-1} \to W^{s-1} \oplus M$ the isomorphism defined by

$$h(\mathbf{e}_i) = \alpha_i\mathbf{a}'_0 + \mathbf{u}_i - \lambda_i\mathbf{a}'_0$$

where $i(0, \mathbf{e}_i) = (\alpha_i, \mathbf{u}_i), \alpha_i \in W, \mathbf{u}_i \in W^{s-1} \oplus M, \lambda_i = j(\mathbf{u}_i)$.

go to START

As in the previous section, this algorithm has been programmed with *Macaulay 2*.

Example. Let $W = A_2(\mathbb{Q})$, and $\mathbf{f} = (\, x\partial_y \quad xy \quad \partial_x \,)$. Then $P = \ker \mathbf{f}$ is a projective module, because $\mathbf{f}$ is a unimodular row. Let

$$\beta = \begin{pmatrix} -y\partial_x \\ \partial_x\partial_y \\ -x \end{pmatrix}.$$

Then $\mathbf{f} \cdot \beta = 1$, and $\operatorname{im} \beta \oplus P = W^3$. The isomorphism $h : W \oplus W^2 \to W \oplus P$ is given by the matrix

$$h = \begin{pmatrix} x\partial_y & xy & \partial_x \\ xy\partial_x\partial_y + x\partial_x + 1 & xy^2\partial_x & y\partial_x^2 \\ -x\partial_x\partial_y^2 & -xy\partial_x\partial_y + 1 & -\partial_x^2\partial_y \\ x^2\partial_y & x^2y & x\partial_x + 2 \end{pmatrix}.$$

Then

$$t_0 = x\partial_y, \qquad \mathbf{a}_0 = \begin{pmatrix} xy\partial_x\partial_y + x\partial_x + 1 \\ -x\partial_x\partial_y^2 \\ x^2\partial_y \end{pmatrix}.$$

We must find $\Phi : W \to P$ such that $\mathbf{a}'_0 = \mathbf{a}_0 + \Phi(t_0)$ is unimodular. Let $\Phi_1 : P \to W$ be the projection over the first component and $\mathbf{a}_2 \in P \cap \ker(\Phi_1)$ not null. For example,

$$\mathbf{a}_2 = \begin{pmatrix} 0 \\ \partial_x^2\partial_y \\ -xy\partial_x\partial_y - x\partial_x - 2y\partial_y - 2 \end{pmatrix}$$

and let $\Phi_2 : W \to P$ be the projection over the second component. Because $\Phi_2(\mathbf{a}_0) \neq 0$, we have to compute $r_1, r_2 \in W$ such that $\Phi_1(\mathbf{a}_0)r_1 + \Phi_2(\mathbf{a}_2)r_2 = 0$. In this case, we get

$$r_1 = -\partial_x^2\partial_y, \qquad r_2 = xy\partial_x\partial_y - 2y\partial_y + 1,$$

and following the notation of the proof of Lemma 2.1

$$d_1 = xy\partial_x\partial_y + x\partial_x + 1, \qquad d_2 = xy\partial_x^3\partial_y^2 + x\partial_x^3\partial_y + \partial_x^2 dy.$$

We have to find $f_1, f_2 \in W$ such that $\langle d_1, t_0\rangle W = \langle d_1 + t_0 f_1 d_1, t_0 f_2 d_2\rangle W$. Applying the modified procedure of Hillebrand and Schmale (2002), we find

$$f_1 = 0, \qquad f_2 = x + y.$$

Let $\Phi : W \to P$ be the morphism defined by $\Phi(1) = (x + y)\mathbf{a}_2$. Then $\mathbf{a}'_0 = \mathbf{a}_0 + \Phi(t_0)$ is unimodular and we can compute the morphism $j : P \to W$ such that $j(\mathbf{a}'_0) = 1$. The output is too large to be included here, but has the form

$$j = \Big(-\tfrac{2}{63}x^2y^7\partial_x^4\partial_y^5 - \tfrac{2}{63}xy^8\partial_x^4\partial_y^5 + \tfrac{5}{126}x^3y^6\partial_x^3\partial_y^6 + \cdots - \tfrac{433}{9}x\partial_x + 17x\partial_y + 1,$$
$$\tfrac{2}{63}xy^8\partial_x^3\partial_y^4 - \tfrac{5}{126}x^2y^7\partial_x^2\partial_y^5 + \tfrac{10}{63}xy^8\partial_x^2\partial_y^5 + + \cdots + \tfrac{5}{3}xy - \tfrac{137}{6}y^2, 0\Big).$$

Also we can build the matrices associated to the other morphisms

$$g = \begin{pmatrix} 1 & \mathbf{0} \\ \Phi & I_3 \end{pmatrix}, \qquad k = (x\partial_y), \qquad l = \begin{pmatrix} 1 & -k \cdot j \\ \mathbf{0} & I_3 \end{pmatrix},$$
$$i = l \cdot g \cdot h = \begin{pmatrix} 0 & \alpha_2 & \alpha_3 \\ \mathbf{a}'_0 & \mathbf{u}_2 & \mathbf{u}_3 \end{pmatrix}$$

where

$$\begin{aligned}\mathbf{u}_2 = (&xy^2\partial_x, x^2y\partial_x^2\partial_y + xy^2\partial_x^2\partial_y - xy\partial_x\partial_y + 1,\\ &-x^3y^2\partial_x\partial_y - x^2y^3\partial_x\partial_y - x^3y\partial_x - x^2y^2\partial_x\\ &-2x^2y^2\partial_y - 2xy^3\partial_y - x^2y - 2xy^2)^t,\\ \mathbf{u}_3 = (&y\partial_x^2, x\partial_x^3\partial_y + y\partial_x^3\partial_y,\\ &-x^2y\partial_x^2\partial_y - xy^2\partial_x^2\partial_y - x^2\partial_x^2 - xy\partial_x^2 - 4xy\partial_x\partial_y\\ &-3y^2\partial_x\partial_y - 3x\partial_x - 3y\partial_x - 2y\partial_y)^t.\end{aligned}$$

Then

$$\mathbf{w}_1 = (\alpha_2 - \lambda_2)\mathbf{a}'_0 + \mathbf{u}_2, \qquad \mathbf{w}_2 = (\alpha_3 - \lambda_3)\mathbf{a}'_0 + \mathbf{u}_3$$

is a basis of P, where $\lambda_i = j(\mathbf{u}_i)$, $i = 2, 3$.

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