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Multiplicative Bases, Gröbner Bases, and Right Gröbner Bases

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In this paper, we study conditions on algebras with multiplicative bases so that there is a Gröbner basis theory. We introduce right Gröbner bases for a class of modules. We give an elimination theory and intersection theory for right submodules of projective modules in path algebras. Solutions to homogeneous systems of linear equations with coefficients in a quotient of a path algebra are studied via right Gröbner basis theory.

1. Introduction

Before surveying the results of the paper, we introduce path algebras. Path algebras play a central role in the representation theory of finite-dimensional algebras (Gabriel, 1980; Auslander *et al.*, 1995; Bardzell, 1997) and the theory of Gröbner bases (Bergman, 1978; Mora, 1986; Farkas *et al.*, 1993) has been an important tool in some results (Feustel *et al.*, 1993; Green and Huang, 1995; Bardzell, 1997; Green *et al.*, to appear). In this paper we show that in sense, if there is a Gröbner basis theory for an algebra, that algebra is naturally a quotient of a path algebra; see the survey of results below. To understand the results of the paper, the reader needs to know what a path algebra is.

Let Γ be a directed graph with vertex set Γ_0 and arrow set Γ_1. We usually assume that both Γ_0 and Γ_1 are finite sets but Γ_1 need not be finite in what follows except as noted below. Let $\mathcal{B}$ be the set of finite directed paths in Γ, including the vertices viewed as paths of length 0. The *path algebra* $K\Gamma$, has as K-basis $\mathcal{B}$. Multiplication is given by concatenation of paths if they meet or 0. More precisely, if p is a path from vertex v to vertex w and q is a path from vertex x to vertex y, then $p \cdot q$ is the path pq from v to y if $w = x$ or else $p \cdot q = 0$ if $w \neq x$. See Auslander *et al.* (1995) for a fuller description. Note that the free associative algebra on n noncommuting variables is the path algebra with Γ having one vertex and n loops. The loops correspond to the variables and the basis of paths correspond to the words in the variables. Note that $\mathcal{B} \cup \{0\}$ is a monoid with 0. The multiplicative monoid $\mathcal{B} \cup \{0\}$ is finitely generated if and only if Γ_1 is a finite set. The only time we must assume that Γ_1 is finite is in the case where we assume that $\mathcal{B} \cup \{0\}$ is finitely generated or when we study finite-dimensional quotients of $K\Gamma$.

There is a well-established Gröbner basis theory for path algebras; see Farkas *et al.* (1993), Green (1999). In particular, the basis of paths, $\mathcal{B}$, has many admissible orders; see Section 2 for the definition of an admissible order. For example, there is the *length-lexicographic order* where we arbitrarily order the vertices, say $v_1 < v_2 < \cdots < v_r$, and arbitrarily order the arrows all larger than the vertics, say $v_r < a_1 < \cdots < a_s$. If p and q

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are paths, we define $p > q$ if either the length of p is greater than the length of q or if the lengths of p and q are equal and if $p = b_1 \cdots b_m$, $q = c_1, \ldots, c_m$ where b_i, c_i are arrows, then $b_i > c_i$ where i is the smallest integer j less than or equal to m where $b_j \neq c_j$. There are many others and later in the paper we define a noncommutative lex order which is needed in elimination theory.

The paper begins with a study of K-algebras with multiplicative bases. Section 2 investigates quotients of algebras with multiplicative bases and asks when such a basis induces a multiplicative basis on the quotient. It is shown that if R has a multiplicative basis and I is an ideal in R, then the multiplicative basis of R induces a multiplicative basis in R/I if and only if I is a special type of binomial ideal which we call a 2-nomial ideal; that is, I is generated by elements of the form $p - q$ and p, where p and q are in the multiplicative basis of R.

It is known that an algebra with a multiplicative basis will have a Gröbner basis theory if there is an admissible order on the basis. The main result of Section 3 shows that if an algebra R has a multiplicative basis with an admissible order, then there is a unique graph Γ such that R is quotient of path algebra $K\Gamma$ by a 2-nomial ideal and the basis of R comes from the basis of paths of $K\Gamma$. See Theorem 3.8. In this sense, all algebras with Gröbner basis theories are quotients of path algebras by 2-nomial ideals I. For a large portion of the paper, we study the "simplest" case where $I = (0)$.

Section 4 introduces a theory of right Gröbner bases in a class of modules in algebra with a Gröbner basis theory. The modules in question must have ordered bases that satisfy obvious properties with respect to the module structure. Reduced and tip-reduced right Gröbner bases are defined and shown to exist. The section ends by showing that certain projective modules have right Gröbner bases theories.

Right Gröbner basis theory is applied in Section 5 where a generalization of Cohn's theorem that free associative algebras are free ideal rings is given. It is shown that if $R = K\Gamma$ is path algebra and P is a right projective R-module, possibly infinitely generated, then every right submodule of P is a direct sum of is a direct if ideals of the form vR where v is a vertex Γ. If Γ has one vertex and n loops, and if $P = R$, then this says that every right ideal is a direct sum of copies of R which is Cohn's theorem.

Section 6 gives a constructive technique to find a right Gröbner basis of a right submodule of a right projective module over a path algebra. If one is given a finite generating set and there is a finite Gröbner basis, this technique is an algorithm and finds a finite Gröbner basis.

Section 7 shows how to find a right Gröbner basis for a two-sided ideal in a path algebra given a (two-sided) Gröbner basis for the ideal.

Section 8 shows that for path algebras, we have an elimination theory that mirrors elimination theory in commutative polynomial rings. The theory deals with removing arrows from the graph and finding right Gröbner bases for right submodules of projective modules over this new graph from right Gröbner bases of submodules of projective modules over the original graph. As in commutative theory, this is then applied to study the intersection of right submodules of a projective module.

Section 9 applies the results of Sections 5 and 6 to find solutions to homogeneous systems of linear equations with coefficients in a quotient of a path algebra. In this section, we give a proof that if I is a cofinite ideal in an algebra with ordered multiplicative bases then I has a finite reduced Gröbner basis. We show that finding a generating set of solutions to the homogeneous linear system, if the quotient is finite dimensional, is algorithmic.

The final section briefly indicates how the results of the paper can be applied in some earlier work on projective resolution of modules.

Throughout this paper, K will denote a fixed field and we let K^* denote the nonzero elements of K.

2. Multiplicative and Gröbner Bases

Let R be a K-algebra. For R to have a classical (two-sided) Gröbner basis theory, R must have a multiplicative basis with a special type of order on the basis. We say that $\mathcal{B}$ is *multiplicative basis* if $\mathcal{B}$ is a K-basis for R and for all $b_1, b_2 \in \mathcal{B}$, $b_1 \cdot b_2 \in \mathcal{B} \cup \{0\}$. Before addressing the order, we briefly look at algebras with multiplicative bases.

Let R be a K-algebra with multiplicative basis $\mathcal{B}$. We say an ideal I is R is a *2-nomial ideal* if I can be generated by elements of the form $b_1 - b_2$ and b where $b_1, b_2, b \in \mathcal{B}$. The first result classifies 2-nomial ideals in R.

Proposition 2.1. *There is a one-to-one correspondence between the set of equivalence relations on $\mathcal{B} \cup \{0\}$ and 2-nomial ideals.*

Proof. Let $\sim$ be an equivalence relation on $\mathcal{B} \cup \{0\}$. Set $I_\sim$ to be the ideal generated by $b_1 - b_2$ if $b_1 \sim b_2$ and by b if $b \sim 0$ with $b_1, b_2, b \in \mathcal{B}$. On the other hand, if I is a 2-nomial ideal in R, define $\sim_I$ by $b_1 \sim_I b_2$ if $b_1 - b_2 \in I$ and by $b \sim_I 0$ if $b \in I$ for $b_1, b_2, b \in \mathcal{B}$. This is clearly a one-to-one correspondence. □

If I is a 2-nomial ideal, we call the equivalence relation on $\mathcal{B} \cup \{0\}$ corresponding to I the *associated relation (to I)*. We have the following technical result similar to the following well-known statement: a linear combination of monomials is in an ideal generated by monomials if and only if each monomial is in the ideal.

Lemma 2.2. *Let I be a 2-nomial ideal in R with associated relation $\sim_I$. Then $\sum_{i=1}^r \alpha_i b_i \in I$ with $\alpha_i \in K$ and $b_i \in \mathcal{B}$ if and only if for each equivalence class $[b]$ of $\sim_I$, $\sum_{b_i \in [b]} \alpha_i b_i \in I$.*

Proof. Consider an element of R, $\sum_{i=1}^r \alpha_i b_i$. If for each equivalence class $[b]$ of $\sim_I$, $\sum_{b_i \in [b]} \alpha_i b_i \in I$, then clearly $\sum_{i=1}^r \alpha_i b_i \in I$.

Suppose $x = \sum_{i=1}^r \alpha_i b_i \in I$. Then $x = \sum_{j=1}^s \beta_s (b_s - b'_s) + \sum_{l=1}^t \gamma_t b_t$ where $b_s - b'_s, b_t \in I$. Then fixing an equivalence class $[b]$, we see that

$$\sum_{b_i \in [b]} \alpha_i b_i = \sum_{b_s, b'_s \in [b]} \beta_s (b_s - b'_s) + \sum_{b_t \in [b]} \gamma_t b_t.$$

The result follows from this observation. □

If S is a K-algebra with multiplicative basis $\mathcal{C}$, the next result gives necessary and sufficient conditions on an ideal I in S such that S/I has a multiplicative basis induced from $\mathcal{C}$.

Theorem 2.3. *Suppose that S is a K-algebra with multiplicative basis $\mathcal{C}$. Let I be an ideal in S and $\pi : S \to S/I$ be the canonical surjection. Let $\mathcal{B}$ be the nonzero elements of $\pi(\mathcal{C})$. Then $\mathcal{B}$ is a multiplicative basis for S/I if and only if I is 2-nomial ideal.*

Proof. First we show that if I is generated by elements of the form $c_1 - c_2$ and c with $c_1, c_2, c \in \mathcal{C}$, then $\mathcal{B}$ is a multiplicative basis. For, suppose $\pi(c_1), \pi(c_2) \in \pi(\mathcal{C})$. Then $\pi(c_1) \cdot \pi(c_2) = \pi(c_1 c_2) \in \pi(\mathcal{C})$ and $\pi(\mathcal{C}) = \mathcal{B} \cup \{0\}$. We need to show that $\mathcal{B}$ is a K-basis of S/I. Clearly, $\pi(\mathcal{C})$ spans S/I and so it suffices to show that the elements of $\mathcal{B}$ are linearly independent. Suppose $\sum_{i=1}^n \alpha_i \pi(c_i) = 0$, for each i, $\pi(c_i) \neq 0$, and that the $\pi(c_i)$ are distinct. We need to show that each $\alpha_i = 0$. But $\sum_{i=1}^n \alpha_i c_i \in I$. Let $\sim$ be the relation on $\mathcal{C} \cup \{0\}$ associated with I. By Lemma 2.2, for each equivalence class $[c]$, $\sum_{c_i \in [c]} \alpha_i c_i \in I$. If $c_i \in [c]$ then $\pi(c_i) = \pi(c)$. Thus, different c_i's are in different equivalence classes (since the $\pi(c_i)$ are distinct). Hence, $\alpha_i c_i \in I$ and we conclude that either $c_i \in I$ (in which case, $\pi(c_i) = 0$) or $c_i \notin I$, in which case, $\alpha_i = 0$. Since each $\pi(c_i) \neq 0$, we conclude that each $\alpha_i = 0$.

Next, suppose that $\mathcal{B}$ is a multiplicative basis of S/I. Define the relation $\sim$ on $\mathcal{C} \cup \{0\}$ by $c \sim c'$ if $\pi(c) = \pi(c')$. It is easy to see that the 2-nomial ideal corresponding to $\sim$ (by Propositon 2.1) is I. This completes the proof. □

We now introduce orders on the multiplicative basis $\mathcal{B}$ of R. We say that $>$ is an *admissible order on* $\mathcal{B}$ if the following properties hold:

A0. $>$ is well-order on $\mathcal{B}$.
A1. For all $b_1, b_2, b_3 \in \mathcal{B}$, if $b_1 > b_2$ then $b_1 b_3 > b_2 b_3$ if both $b_1 b_3$ and $b_2 b_3$ are nonzero.
A2. For all $b_1, b_2, b_3 \in \mathcal{B}$, if $b_1 > b_2$ then $b_3 b_1 > b_3 b_2$ if both $b_3 b_1$ and $b_3 b_2$ are nonzero.
A3. For all $b_1, b_2, b_3, b_4 \in \mathcal{B}$, if $b_1 = b_2 b_3 b_4$ then $b_1 \geq b_3$.

We use the terminology *R has an ordered multiplicative basis* $(\mathcal{B}, >)$ if $\mathcal{B}$ is a multiplicative basis for R and $>$ is an admissible order on $\mathcal{B}$. For the remainder of this section, let R be a K-algebra with ordered multiplicative basis $(\mathcal{B}, >)$.

The K-algebra R has a Gröbner basis theory associated to $(\mathcal{B}, >)$. We refer the reader to Green (1999) for more details. We just summarize the main notions. Let $x = \sum_{i=1}^n \alpha_i b_i$ be a nonzero element of R with $\alpha_i \in K^*$ and the b_i are distinct elements of $\mathcal{B}$. The *tip of* x, denoted $\mathrm{Tip}(x)$, is the largest basis element b_i occurring in x. That is, $\mathrm{Tip}(x) = b_i$ where $b_i \geq b_j$ for $j = 1, \ldots, n$. If I is a subset R, we define $\mathrm{Tip}(I) = \{b \mid b = \mathrm{Tip}(x)$ for some $x \in I \setminus \{0\}\}$. We let $\mathrm{NonTip}(I) = \mathcal{B} \setminus \mathrm{Tip}(I)$.

We say a subset $\mathcal{G}$ of I is a *Gröbner basis of I with respect to* $>$ if the ideal generated by $\mathrm{Tip}(\mathcal{G})$ equals the ideal generated by $\mathrm{Tip}(I)$. Later in this paper, we will introduce and study right Gröbner bases in some detail. In the next section, we show that a K-algebra must be of a very special form to have a Gröbner basis theory.

3. The Ubiquity of Path Algebras

In this section, we show that every K-algebra with a ordered multiplicative basis is a quotient of a path algebra. Throughout this section, R will be K-algebra with ordered multiplicative basis $(\mathcal{B}, >)$. We assume R has a multiplicative identity, 1, but do not assume that $1 \in \mathcal{B}$. Let $1 = \sum_{i=1}^n \alpha_i b_i$ where each $\alpha_i \in K^*$ and the b_i are distinct elements of $\mathcal{B}$. The next result shows that the b_i are special.

Lemma 3.1. *The set $\{b_1, \ldots, b_n\}$ is a set of orthogonal idempotents and each $\alpha_i = 1$.*

PROOF. Suppose $i \neq j$ and that $b_ib_j \neq 0$. Then $b_ib_j = b \in \mathcal{B}$. We assume that $b \neq b_i$ (with the case $b \neq b_j$ handled in a similar fashion). Then

$$b_i = b_i \cdot 1 = \sum_{j=1}^{n} \alpha_j b_i b_j. \tag{$*$}$$

Hence, there is some $l \neq j$ such that $b_ib_l = b$ since $b = b_ib_j$ must be cancelled by some b_ib_l. Now either $b_l > b_j$ or $b_j > b_l$ by A0. In either case, we cannot have $b_ib_l = b_ib_j = b$ by A2. Hence, we conclude that if $i \neq j$ then $b_ib_j = 0$.

Next, by $(*)$, $b_i = \alpha_i b_i b_i$ and the result follows. □

The argument in the above proof can be generalized to give a cancellation result.

PROPOSITION 3.2. *If b_1 and b_2 are distinct elements of $\mathcal{B}$ then for all $b \in \mathcal{B}$, $b_1b = b_2b$ implies $b_1b = 0$ and $bb_1 = bb_2$ implies $bb_1 = 0$.*

PROOF. Without loss of generality, we may assume that $b_1 < b_2$. The result follows from A1 and A2. □

As we are beginning to see, the idempotents $\{b_1, \ldots, b_n\}$ such that $1 = \sum_{i=1}^n b_i$ are very special elements of $\mathcal{B}$. To distinguish them, we will write $1 = \sum_{i=1}^n v_i$ with b_i replaced by v_i. The next result shows that multiplying basis elements by the v_i is very restrictive.

PROPOSITION 3.3. *If $b \in \mathcal{B}$ then there exist unique i, j such that $v_ib = b$ and $bv_j = b$. If $k \neq i$ then $v_kb = 0$. If $k \neq j$ then $bv_k = 0$.*

PROOF. Since $b = b \cdot 1 = \sum_{i=1}^n bv_i$, we conclude that $bv_j = b$ for some j. But if $k \neq j$, since $v_jv_k = 0$, $bv_k = 0$. By a similar argument multiplying $1 \cdot b$ gives the remaining part of the result. □

If $b \in \mathcal{B}$, we let $o(b) = v_i$ if $v_ib = b$. Similarly, we let $t(b) = v_j$ if $bv_j = b$. The next result shows that the v_i's have a minimality property with respect to the order $>$.

LEMMA 3.4. *If $b \in \mathcal{B}$ such that either $o(b) = v_i$ or $t(b) = v_i$ then $b \geq v_i$.*

PROOF. Suppose $b = bv_i$. Then $b = bv_iv_i$ and by A3, $b \geq v_i$. □

COROLLARY 3.5. *If $b \in \mathcal{B} \setminus \{v_1, \ldots, v_n\}$ then $b^2 \neq b$.*

PROOF. If $b^2 = 0$ we are done. By Lemma 3.4, there is some i, such that $b > v_i$ and $bv_i = b$. Then by A2, $b^2 > b$. Hence $b^2 \neq b$. □

The next result continues to indicate the importance of the v_i's. Let $\Gamma_0 = \{v_1, \ldots, v_n\}$. The above results show that Γ_0 can be described as the set of the idempotent elements of $\mathcal{B}$. Recall that an idempotent is called *primitive* if cannot be written as a sum of two orthogonal nonzero idempotents.

LEMMA 3.6. *The set $\Gamma_0 = \{v_1, \ldots, v_n\}$ is a full set of primitive orthogonal idempotents for R.*

Proof. Since $1 = \sum_{i=1}^{n} v_i$, Γ_0 is full. By Lemma 3.1 it suffices to show that each v_i is primitive. Suppose not. Let $v_i = x + y$ for some i, where x and y are nonzero orthogonal idempotents. Now $x = \sum_j \alpha_j v_j + \sum_l \beta_l b_l$ where $\alpha_j, \beta_l \in K$ and $b_l \in \mathcal{B}$. Since $v_i = v_i x v_i + v_i y v_i$ we see that $\alpha_j = 0$ if $j \neq i$ and that if $\beta_l \neq 0$ then $o(b_l) = v_i = t(b_l)$. Thus,

$$x = \alpha_i v_i + \sum_l \beta_l b_l,$$

and

$$y = (1 - \alpha_i)v_i - \sum_l \beta_l b_l.$$

However $xy = 0$ and $b_l b_j \neq v_i$ so we conclude that $\alpha_i(1 - \alpha_i) = 0$. Hence we may assume that $\alpha_i = 1$. Now let b denote the smallest b_l occurring in $\sum_l \beta_l b_l$. Again using that $xy = 0$, $v_i b = b$, and that b is smaller than all the nonzero products $b_l b_j$, we see that in xy we cannot cancel $v_i b$. Thus all the β_l's must be 0. This completes the proof. □

Next we define $\Gamma_1 = \{b \in \mathcal{B} \mid b \notin \Gamma_0$ and b cannot be written as a product $b_1 b_2$ with $b_1, b_2 \in \mathcal{B} \setminus \Gamma_0\}$. That is, Γ_1 is the *product indecomposable* elements in $\mathcal{B} \setminus \Gamma_0$. Note that Γ_1 is a unique set, as is Γ_0.

Proposition 3.7. *Let R be a K-algebra with ordered multiplicative basis $(\mathcal{B}, >)$. If Γ_0 and Γ_1 are the unique subsets of $\mathcal{B}$ defined by $1 = \sum_{v \in \Gamma_0} v$ and Γ_1 are the product indecomposable of $\mathcal{B} \setminus \Gamma_0$, then every $b \in \mathcal{B} \setminus \Gamma_0$ is product $b_1 \cdots b_r$ where $b_i \in \Gamma_1$. In particular, $\Gamma_0 \cup \Gamma_1$ generate the multiplicative basis $\mathcal{B}$.*

Proof. Let $b \in \mathcal{B}$. If $b \in \Gamma_0$ or if b is a product indecomposable element then $b \in \Gamma_0 \cup \Gamma_1$ and we are done. If $b \notin \Gamma_0 \cup \Gamma_1$ then $b = b_1 b_2$ for some $b_1, b_2 \in \mathcal{B} \setminus \Gamma_0$. Since $b = o(b_1) b_1 b_2 = b_1 b_2 t(b_2)$, we see that $b \geq b_1$ and $b \geq b_2$ by A3. We claim that $b \neq b_1$ and and $b \neq b_2$. If, say, $b = b_1$ then $bt(b) = b_1 b_2 = b b_2$. By Proposition 3.2 $t(b) = b_2$, a contradiction since $t(b)$ is a vertex and b_2 is not. Hence, $b > b_1$ and $b > b_2$. If both b_1 and b_2 are in Γ_1 then we are done. Continuing in this fashion, we get $b = b_{i_1} b_{i_2} \dots b_{i_r}$ with $b_{i_j} \in \mathcal{B} \setminus \Gamma_0$. Since $>$ is a total order, we have a proper descending chain $b > b_{i_{s_1}} > \cdots > b_{i_{s_u}}$ where the chain is $r + 1$ elements long. But $>$ is a well-order and this process must stop. That is, each b_{i_j} must be product indecomposable and we are done. □

Now let Γ be the directed graph with vertex set Γ_0 and arrow set Γ_1; that is, if $b \in \Gamma_1$, we view it as an arrow from $o(b)$ to $t(b)$. We call Γ the *graph associated to $\mathcal{B}$*. We now state the main result of this section.

Theorem 3.8. *Let R be a K-algebra with an ordered multiplicative basis $(\mathcal{B}, >)$. Let Γ be graph associated to $\mathcal{B}$. Then there is a surjective K-algebra homomorphism $\phi : K\Gamma \to R$ such that:*

(1) if p is a directed path in Γ, then $\phi(p) \in \mathcal{B} \cup \{0\}$,
(2) if $b \in \mathcal{B}$ then there is a path $p \in \Gamma$ such that $\phi(p) = b$,
(3) the kernel of ϕ is a 2-nomial ideal; that is, it is generated by elements of form p or $p - q$ where p and q paths in Γ.

PROOF. Since R has a multiplicative basis $\mathcal{B}$ with admissible order $>$, we have unique subsets Γ_0 and Γ_1 of $\mathcal{B}$. Furthermore, by Lemma 3.6, Γ_0 is a full set of orthogonal idempotents. Letting Γ be the graph associated with $\mathcal{B}$, we note that the path algebra, as a tensor algebra (see Green, 1975), has a universal mapping property determined. In particular, sending the vertices v in Γ to the corresponding elements of Γ_0 in $\mathcal{B}$ and sending the arrows of Γ to the corresponding elements of Γ_1 in $\mathcal{B}$, we obtain a K-algebra homomorphism $\phi : K\Gamma \to R$. By construction, paths in Γ map to elements of $\mathcal{B}$ or 0. By Proposition 3.7, ϕ is surjective.

Finally, we note that the multiplicative basis of paths in $K\Gamma$ maps onto $\mathcal{B} \cup \{0\}$. Hence, by Theorem 2.3, we conclude that the kernel of ϕ is generated by elements of the form $p - q$ and p where p and q are paths in $K\Gamma$. This completes the proof. □

The above theorem states that if a K-algebra R is to have a Gröbner basis theory then R is of the form $K\Gamma/I$ where I is a 2-nomial ideal and the multiplicative basis with an admissible order is the image of the paths in $K\Gamma$. As mentioned earlier, if $I = (0)$ the set of all paths admits an admissible order and hence $K\Gamma$ has a Gröbner basis theory. The following question is open and an answer would be of interest:

QUESTION 3.9. *Given a graph Γ, what are necessary and sufficient conditions on a 2-nomial ideal I such that $K\Gamma/I$ has a Gröbner basis theory in the sense that the image of the paths admit an admissible order?*

4. The Theory of Right Gröbner Bases

We begin by sketching the theory of right Gröbner bases for modules over a K-algebra with an ordered multiplicative basis. We know of no reference in the literature for such a theory and hence we include this summary here.

Throughout this section, R will be K-algebra with ordered multiplicative basis $(\mathcal{B}, >)$. Let M be a right (unital) R-module. As with algebras, we need a K-basis of M and an admissible order on the basis. Let $\mathcal{M}$ be a K-basis of M. We say that $\mathcal{M}$ is a *coherent* basis if for all $m \in \mathcal{M}$ and all $b \in \mathcal{B}$, $mb = 0$ or $mb \in \mathcal{M}$.

LEMMA 4.1. *If $\mathcal{M}$ is a coherent K-basis of M then for all $m \in \mathcal{M}$ there is $v \in \Gamma_0$ such that $mv = m$.*

PROOF. Let $m \in \mathcal{M}$. Then $m = m \cdot 1 = \sum_{v \in \Gamma_0} mv$. But each $mv \in \mathcal{M} \cup \{0\}$ and $\mathcal{M}$ is a K-basis of M. The result now follows. □

We say that a well-order $\succ$ on $\mathcal{M}$ is a *right admissible order on* $\mathcal{M}$ if the following properties hold:

M1. For all $m_1, m_2 \in \mathcal{M}$ and all $b \in \mathcal{B}$, if $m_1 \succ m_2$ then $m_1 b \succ m_2 b$ if both $m_1 b$ and $m_2 b$ are nonzero.

M2. For all $m \in \mathcal{M}$ and all $b_1, b_2 \in \mathcal{B}$, if $b_1 > b_2$ then $mb_1 \succ mb_2$ if both mb_1 and mb_2 are nonzero.

If M is a right R-module, we say that $(\mathcal{M}, \succ)$ is an *ordered basis of* M if $\mathcal{M}$ is coherent K-basis and $\succ$ is a right admissible order on $\mathcal{M}$. If M has an ordered basis we say that

M has a right Gröbner basis theory with respect to $\succ$. Note that if $(\mathcal{M}, \succ)$ is an ordered basis of M with $m \in \mathcal{M}$ and $b \in \mathcal{B} \setminus \Gamma_0$, then $mb \succ m$ if both are not zero. For, there is some $v \in \Gamma_0$ such that $mv = m$. Thus, if $mb \neq 0$ then $vb \neq 0$. By Lemma 3.4, $b > v$. By M2, $mb \succ mv = m$.

For the remainder of this section, let M be a right R-module with ordered basis $(\mathcal{M}, \succ)$. If $x \in M \setminus \{0\}$, then $x = \sum_{i=1}^{r} \alpha_i m_i$ where each $\alpha_i \in K^*$ and the m_i are distinct elements of $\mathcal{M}$. We let $\mathrm{Tip}(x) = m_i$ where $m_i \succeq m_j$ for all $j = 1, \ldots, r$. If X is a subset of M, we let $\mathrm{Tip}(X) = \{m \in \mathcal{M} \mid m = \mathrm{Tip}(x) \text{ for some } x \in M \setminus \{0\}\}$. Similarly, we have $\mathrm{NonTip}(X) = \mathcal{M} \setminus \mathrm{Tip}(X)$.

If N is a right submodule of M, we say $\mathcal{G}$ is a *right Gröbner basis of N with respect to $\succ$* if $\mathcal{G} \subset N$ and the right submodule of M generated by $\mathrm{Tip}(\mathcal{G})$ equals the right submodule of M generated by $\mathrm{Tip}(N)$.

The following basic properties have proofs analogous to the usual ideal theoretic proofs, and we only sketch the proof.

Proposition 4.2. *Let N be a submodule of M. Then:*

(1) There is a right Gröbner basis for N with respect to $\succ$.
(2) If $\mathcal{G}$ is a right Gröbner basis for N with respect to $\succ$, then $\mathcal{G}$ generates N as a submodule.
(3) As vector spaces, $M = N \oplus \mathrm{Span}(\mathrm{NonTip}(N))$.

Proof. Clearly, right Gröbner bases exist. We sketch the (standard) proof that a right Gröbner basis for a right submodule generates the submodule. Assume that the Gröbner basis, $\mathcal{G}$, does not generate the submodule N. Let $z \in N$ such that $\mathrm{Tip}(z)$ is minimal such that z is not in the submodule generated by $\mathcal{G}$. By definition, there is some $g \in \mathcal{G}$ such that $\mathrm{Tip}(g)$ left divides $\mathrm{Tip}(z)$. Let $b \in \mathcal{B}$ be such that $\mathrm{Tip}(g)b = \mathrm{Tip}(z)$. Let α be the coefficient of $\mathrm{Tip}(g)$ in g and β be the coefficient of $\mathrm{Tip}(z)$ in z. Then the tip of $z - (\beta/\alpha)gb$ is less than the tip of z. It follows that $z - (\beta/\alpha)gb$ is in the submodule generated by $\mathcal{G}$. But then so is z. This is a contradiction.

Part 3 is a linear Gröbner basis property and a proof can be found in Green (1999). □

If N is a right submodule of M and $m \in M$, we define the *normal form of m with respect to $\succ$* to be $\mathrm{Norm}(m)$ where $m = n_m + \mathrm{Norm}(m)$ with $n_m \in N$ and $\mathrm{Norm}(m) \in \mathrm{Span}(\mathrm{NonTip}(N))$. Thus, as vector spaces, M/N is isomorphic to $\mathrm{Span}(\mathrm{NonTip}(N))$. As with ideal-theorectic Gröbner basis theory, right Gröbner basis theory allows one to work with factor modules via normal forms. Furthermore, viewing this as an identification, the K-basis $\mathrm{NonTip}(N)$ of M/N is particularly well-suited to work with the Gröbner basis of R once we are given a right Gröbner basis of N with respect to $\succ$.

We should remark that most modules do NOT have a right Gröbner basis theory. Clearly every one-dimensional right R-modules has an ordered basis. The following example shows that even two-dimensional modules need not have such a basis.

Example 4.3. Let R be the free associative algebra on two noncommuting variables x and y. Then, as noted earlier, the monomials in R with (total degree)-left lexicographic order with $x < y$ is an ordered multiplicative basis for R. Thus R has a Gröbner basis theory. Now let M be the two-dimensional module where x acts as $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and y acts as $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. It is easy to show that there is no possible ordered basis for M. In fact, it can be shown that there is no admissible order on the monomials in R such that there is an ordered basis of M.

Before giving a class of right modules that has a right Gröbner basis theory, we define reduced bases. Recall that we are assuming that M has an ordered basis $(\mathcal{M}, \succ)$. If $m, m' \in \mathcal{M}$, we say m *left divides* m' if there is some $b \in \mathcal{B}$ such that $m' = mb$.

Let N be a right submodule of M. We say a right Gröbner basis $\mathcal{G}$ of N with respect to $\succ$ is *reduced* if the following holds. Let $g \in \mathcal{G}$ and $g = \sum_{i=1}^{r} \alpha_i m_i$ with $\alpha_i \in K^*$ and the m_i are distinct elements of $\mathcal{M}$. Then, for each $g' \in \mathcal{G} \setminus \{g\}$ and each $i = 1, \ldots, r$, we have that $\operatorname{Tip}(g')$ does not left divide m_i and the coefficient α_i of $\operatorname{Tip}(g)$ is 1. Note that the definition that $\mathcal{G}$ is a right Gröbner basis of N is reduced is equivalent to the statement that for all $g \in \mathcal{G}$ then $g - \operatorname{Tip}(g) \in \operatorname{Span}(\operatorname{NonTip}(I))$

We say $\mathcal{G}$ is *tip-reduced* if $g, g' \in \mathcal{G}$ and $\operatorname{Tip}(g)$ left divides $\operatorname{Tip}(g')$ then $g = g'$. From the definitions, it is immediate that a reduced right Gröbner basis is a tip-reduced right Gröbner basis. The next result proves the existence of reduced and tip-reduced right Gröbner bases for N.

PROPOSITION 4.4. *Let R be a K-algebra with ordered multiplicative basis $(\mathcal{B}, >)$ and let M be a right R-module with ordered basis $(\mathcal{M}, \succ)$. Let N be a right submodule of M. Then there is a tip-reduced right Gröbner basis for N with respect to $\succ$ and there is a unique reduced right Gröbner basis for N.*

PROOF. We just show existence of a reduced right Gröbner basis for N. Consider the set $\operatorname{Tip}(N)$. Let $\mathcal{T} = \{t \in \operatorname{Tip}(N) \mid \text{no tip } t' \in \operatorname{Tip}(N) \text{ properly left divides } t\}$. That is, $\mathcal{T}$ is the set of tips such that if $t' \in \operatorname{Tip}(N)$ left divides t then $t = t'$. This is a unique set in $\operatorname{Tip}(N)$. Let $\mathcal{G} = \{t - \operatorname{Norm}(t) \mid t \in \mathcal{T}\}$. It is easy to verify that $\mathcal{G}$ is a reduced Gröbner basis for N.

For uniqueness, consider $h \in N$. Since $\mathcal{G}$ is a Gröbner basis of N, there is some $g \in \mathcal{G}$ such that $\operatorname{Tip}(g)$ left divides $\operatorname{Tip}(h)$. Now suppose that $\mathcal{H}$ is another reduced right Gröbner basis of N and let $h \in \mathcal{H}$. Then for some $g \in \mathcal{G}$, $\operatorname{Tip}(g)$ left divides $\operatorname{Tip}(h)$. Since $\mathcal{H}$ is a right Gröbner basis, there is some $h' \in \mathcal{H}$ such that $\operatorname{Tip}(h')$ left divides $\operatorname{Tip}(g)$. It follows that $\operatorname{Tip}(h')$ left divides $\operatorname{Tip}(h)$. But $\mathcal{H}$ is tip-reduced. Thus, $\operatorname{Tip}(h) = \operatorname{Tip}(h') = \operatorname{Tip}(g)$.

It follows that $h - g$ is a K-linear combination of elements in $\operatorname{Span}(\operatorname{NonTip}(N))$. But, $h - g \in N$ which implies $h - g = 0$ (or else it would have a tip in $\operatorname{Tip}(N)$). Thus, $\mathcal{H} \subseteq \mathcal{G}$. Interchanging the roles of $\mathcal{G}$ and $\mathcal{H}$, we see $\mathcal{G} = \mathcal{H}$. This proves uniqueness of the reduced Gröbner basis. □

COROLLARY 4.5. *Keeping the hypothesis of Proposition 4.4, let $\mathcal{T} = \{t \in \operatorname{Tip}(N) \mid$ no tip $t' \in \operatorname{Tip}(N)$ properly left divides $t\}$. If $\mathcal{G}$ is a subset of N such that $\mathcal{T} \subseteq \operatorname{Tip}(\mathcal{G})$ then $\mathcal{G}$ is a right Gröbner basis for N with respect to $\succ$.*

PROOF. Suppose that $\operatorname{Tip}(\mathcal{G})$ contains $\mathcal{T}$. Since $\mathcal{T}$ generates the right submodule generated by $\operatorname{Tip}(N)$, so does $\operatorname{Tip}(\mathcal{G})$. □

We need one more concept before continuing. If $x = \sum_{i=1}^{r} \alpha_i p_i \in M$ where $\alpha_i \in K^*$

and $p_i \in \mathcal{M}$, we say that x is *(left) uniform* if there is some $v \in \Gamma_0$ such that $xv = x$. Note that every element of M is a finite sum of uniform elements since $1 = \sum_{v\in\Gamma_0} v$. In fact, we have the following consequence of Proposition 4.4

Corollary 4.6. *Let R be a K-algebra with ordered multiplicative basis $(\mathcal{B}, >)$ and let M be a right R-module with ordered basis $(\mathcal{M}, \succ)$. Let N be a right submodule of M. Then there is a tip-reduced uniform right Gröbner basis for N with respect to $\succ$.*

Proof. By Proposition 4.4, there is a tip-reduced right Gröbner basis $\mathcal{G}'$ of N. Since $\mathrm{Tip}(g) \in \mathcal{M}$, by Lemma 4.1, we see that for each $g \in \mathcal{G}'$ there is a unique $v \in \Gamma_0$ such that $\mathrm{Tip}(g)v = \mathrm{Tip}(g)$. Let $\mathcal{G} = \{gv \mid g \in \mathcal{G}', v \in \Gamma_0, \text{ and } \mathrm{Tip}(gv) = \mathrm{Tip}(g)\}$. It follows that $\mathcal{G}$ is a uniform tip-reduced right Gröbner basis. □

We conclude this section by describing a class of modules that always have ordered bases. For each $v \in \Gamma_0$, let $v\mathcal{B} = \{b \in \mathcal{B} \mid b = vb\}$. The right R-module vR is a right projective R-module since $\oplus_{v\in\Gamma_0} vR = R$. Now restricting the admissible order $>$ on $\mathcal{B}$ to $v\mathcal{B}$, it is easy to see that $(v\mathcal{B}, >)$ is an ordered basis for vR. Next, we show that arbitrary direct sums of these types of right projective modules admits an ordered basis.

Let $\mathcal{I}$ be a set, $V : \mathcal{I} \to \Gamma_0$, and $P = \coprod_{i\in\mathcal{I}} V(i)R$. Then P is right projective R-module. (In this paper, $\coprod$ denotes the direct sum.) We now construct an ordered basis for P. For each $i \in \mathcal{I}$, let $\mathcal{P}_i = \{x \in P \mid x_j = 0 \text{ if } j \in \mathcal{I} \text{ and } j \neq i, \text{ and } x_i \in V(i)\mathcal{B}\}$. The basis for P is $\mathcal{P} = \cup_{i\in\mathcal{I}} \mathcal{P}_i$. Thus, if $x \in P$ has only one component with a nonzero entry and that entry is in the i'th-component, then the entry is in $V(i)\mathcal{B}$. We wish to find a right admissible order $\succ$ on $\mathcal{P}$. First, choose some well-order $>_{\mathcal{I}}$ on $\mathcal{I}$. If $x_1, x_2 \in \mathcal{P}$, we define $x_1 \succ x_2$ if the nonzero entry of x_1 is greater than the nonzero entry of x_2 (viewed as elements of $\mathcal{B}$) or if the nonzero entries are equal, then the nonzero entry of x_1 occurs in the ith-component, the nonzero entry of x_2 occurs in the i'th-component, and $i >_{\mathcal{I}} i'$. The reader may check that $(\mathcal{P}, \succ)$ is an ordered basis for P. We summarize these remarks in the following result.

Theorem 4.7. *Let R be a K-algebra with ordered multiplicative basis $(\mathcal{B}, >)$. Let $\mathcal{I}$ be an index set and $V : \mathcal{I} \to \Gamma_0$. Then, keeping the notation and definitions above, the right projective module $\coprod_{i\in\mathcal{I}} V(i)R$ has an ordered basis $(\mathcal{P}, \succ)$.*

In Section 6, we give a constructive procedure for finding a tip-reduced right Gröbner basis for a submodule of a right projective module of the form $\coprod_{i\in\mathcal{I}} V(i)R$.

In the next section, we use this result to generalize Cohn's theorem on firs to path algebras.

5. A Generalization of a Theorem of P. M. Cohn

In this section R is a fixed path algebra $K\Gamma$ and $\mathcal{B}$ is the set of finite directed paths in $K\Gamma$. We also fix an admissible order $>$ on $\mathcal{B}$; for example, $>$ might be the length-lexicographic order described in Section 1.

Let $\mathcal{I}$ be an index set and $V : \mathcal{I} \to \Gamma_0$ be a set map. Let $P = \coprod_{i\in\mathcal{I}} V(i)R$. Recall that P is a right projective R-module. In this section we will show that all right projective R-modules are of this form. We apply the results of the previous section and let $(\mathcal{P}, \succ)$ be the ordered basis of P where the elements of $\mathcal{P}$ are tuples having one nonzero component and if the nonzero component is the ith component, then the entry is in $V(i)\mathcal{B}$. The order $\succ$ is defined in the previous section and depends on the order $>$ on $\mathcal{B}$ and on a choice of well-order $>_{\mathcal{I}}$ on $\mathcal{I}$.

The next result holds for path algebras but not general K-algebras. Recall that an element $x \in R \setminus \{0\}$ is uniform if there is a vertex v such that $xv = x$.

Lemma 5.1. *If x is a uniform element with $xv = x$ and $v \in \Gamma_0$, then, as right R-modules, $xR = vR$ and hence xR is a right projective R-module. Furthermore, if $x = xv$ and $r \in R \setminus \{0\}$ such that $vr = r$, then* $\mathrm{Tip}(xr) = \mathrm{Tip}(x)\mathrm{Tip}(r)$.

Proof. First, assume $r \in R$ such that $vr = r$ and x is a uniform element with $xv = x$ for some vertex $v \in \Gamma$. Then $\mathrm{Tip}(x) = \mathrm{Tip}(x)v$ and $\mathrm{Tip}(r) = v\mathrm{Tip}(r)$. Hence these paths concatenate and we have shown the last part of the result by the multiplicative properties of an admissible order.

Define $\phi : vR \to xR$ by $\phi(vr) = xr$. Clearly, ϕ is onto. Suppose that $\phi(vr) = 0$. Then $xr = 0$. But by the first part of the proof, $\mathrm{Tip}(x)v \neq 0$ and $v\mathrm{Tip}(r) \neq 0$ if $vr \neq 0$. However, then $\mathrm{Tip}(x)\mathrm{Tip}(vr) \neq 0$ and we conclude that $\mathrm{Tip}(xr) \neq 0$. This contradicts $xr = 0$. Hence, $vr = 0$ and the proof is complete. □

The next result is fundamental and used frequently in what follows.

Theorem 5.2. *If $\mathcal{G}$ is a uniform tip-reduced subset of P, then the right submodule generated by $\mathcal{G}$ is the right projective module $\coprod_{g\in\mathcal{G}} gR$.*

Proof. Let $\mathcal{G}$ be a uniform tip-reduced subset of P. Let Q be the right submodule generated by $\mathcal{G}$. We need to show that if $x = \sum_{g\in\mathcal{G}} gr_g = 0$ in Q then each $r_g = 0$. Suppose that at least one $r_g \neq 0$. Then $r_g v \neq 0$ for some $v \in \Gamma_0$ and some $g \in \mathcal{G}$. Hence, replacing x by $\sum_{g\mathcal{G}} gr_g v$, we may assume that x is uniform. Furthermore, since every element of $\mathcal{G}$ is uniform, we may assume that for each g, there is a vertex v_g such that $g = gv_g$. It follows that $gr_g = gv_g r_g$. Hence we may assume that for each g, $r_g = v_g r_g$. From these assumptions on uniformity, we conclude, using the definition of $\mathcal{P}$ and Lemma 5.1 that for each $g \in \mathcal{G}$, $\mathrm{Tip}(gr_g) = \mathrm{Tip}(g)\mathrm{Tip}(r_g)$.

Since $r_g = 0$ for all but a finite number of r_g, we consider all $\mathrm{Tip}(gr_g)$ for nonzero gr_g. Let $\mathrm{Tip}(g_0 r_{g_0})$ be maximal in this set. By the order $\succ$ on $\mathcal{P}$, $\mathrm{Tip}(g_0 r_{g_0})$ is 0 in all components but one, say i_0, and in that component it is a path, say p_0.

Since $x = 0$, there must be some other $g \in \mathcal{G}$ such that in the i_0th-component the path p_0 must occur (to get cancellation). By the order $\succ$, and by maximality of $\mathrm{Tip}(g_0 r_{g_0})$, we conclude that $\mathrm{Tip}(gr_g) = \mathrm{Tip}(g_0 r_{g_0})$. Thus, we have that $\mathrm{Tip}(g)\mathrm{Tip}(r_g) = \mathrm{Tip}(g_0)\mathrm{Tip}(r_{g_0})$ with $g \neq g_0$. Let p be the path in the i_0th-component of $\mathrm{Tip}(g)$. We conclude that $p\mathrm{Tip}(r_g) = p_0\mathrm{Tip}(r_{g_0})$. Hence either $p = p_0 q$ or $pq = p_0$ for some path q. It follows that $\mathrm{Tip}(g) = \mathrm{Tip}(g_0)q$ or $\mathrm{Tip}(g)q = \mathrm{Tip}(g_0)q$ for some path q. Since $g \neq g_0$, this contradicts the assumption that $\mathcal{G}$ is tip-reduced. This concludes the proof. □

We have an important consequence of the above theorem.

Corollary 5.3. *Keeping the above notation, let Q be a right submodule of P. Suppose that Q has a finite generating set. Then every uniform tip-reduced right Gröbner basis is finite.*

Proof. Let $\mathcal{G}$ be a uniform tip-reduced right Gröbner basis. By Theorem 5.2, $Q = \coprod_{g\in\mathcal{G}} gR$. If $\mathcal{G}$ is an infinite set, $\coprod_{g\in\mathcal{G}} gR$ cannot be finitely generated. □

In the next section we stengthen the above corollary. We now present a simple proof of a generalization of a result of Cohn for free algebras. For path algebras, this result is folklore.

Theorem 5.4. *Let R be a path algebra $K\Gamma$ and let $P = \coprod_{i\in\mathcal{I}} V(i)R$ for some set function $V : \mathcal{I} \to \Gamma_0$. Let Q be a right submodule of P. Then there is a tip-reduced, uniform Gröbner basis of Q. Moreover, for every tip-reduced uniform right Gröbner basis, $f_j \in P$, $j \in \mathcal{J}$, of Q,*

$$Q = \coprod_{j\in\mathcal{J}} f_jR.$$

Proof. By Theorem 4.7, P has an ordered basis $(\mathcal{P}, \succ)$. Hence, by Corollary 4.6, every right submodule of P has a uniform tip-reduced right Gröbner basis. The last part follows from Theorem 5.2. □

We now give some consequences of the above theorem. Some are folklore with no proof in the literature.

Corollary 5.5. *Let R be a path algebra $K\Gamma$. Then the following hold.*

(1) The (right) global dimension of R is ≤ 1 and $= 1$ if and only if Γ has at least one arrow.
(2) Every projective right R-module is of the form $\coprod_{i\in\mathcal{I}} V(i)R$ where $V : \mathcal{I} \to \Gamma_0$.
(3) A right projective R-module is indecomposable, if and only if $Q = vR$ for some vertex v.

Proof. If Γ has no arrows, then $K\Gamma$ is a semisimple ring and hence has global dimension 0. Next, suppose that Γ has at least one arrow. Let J denote the ideal in $K\Gamma$ generated by all the arrows of Γ. Then $J^2 \neq J$ and hence $K\Gamma$ is not a semisimple ring. That the global dimension is bounded by 1 follows from Theorem 5.4 above. Hence part 1 is proved.

To prove part 2, let P be a projective right R-module. Then there is a projective right R-module P' such that $P \oplus P'$ is a free R-module. That is there is some index set $\mathcal{J}$ such that $P \oplus P' = \coprod_{j\in\mathcal{J}} R$. But $R = \coprod_{v\in\Gamma_0} R$. Thus P is a submodule of a module of the form $\coprod_{i\in\mathcal{I}} V(i)R$ where $\mathcal{I}$ is an index set and where $V : \mathcal{I} \to \Gamma_0$. Part 2 now follows from Theorem 5.4. Finally part 3 follows from part 2. □

6. Construction of Uniform Tip-reduced Right Gröbner Bases

In this section R is a fixed path algebra $K\Gamma$ and $\mathcal{B}$ is the set of finite directed paths in $K\Gamma$. We also fix an admissible order $>$ on $\mathcal{B}$. Let $\mathcal{I}$ be an index set and $V : \mathcal{I} \to \Gamma_0$ be a set map and $P = \coprod_{i\in\mathcal{I}} V(i)R$. We let $(\mathcal{P}, \succ)$ be the ordered basis discussed in the last two sections.

Let Q be a submodule of P and suppose that $\mathcal{F}'$ is a generating set for the right module Q. Let $\mathcal{F} = \{fv \mid f \in \mathcal{F}' \text{ and } v \in \Gamma_0\}$. Then $\mathcal{F}$ is a set of uniform elements that generates Q. Thus, if we have a generating set for Q, we assume that it is a set of uniform elements. We begin with a general result.

Theorem 6.1. *Keeping the above notations, Q has a uniform tip-reduced generating set. Every uniform tip-reduced generating set of Q is a right Gröbner basis with respect to $\succ$.*

Proof. We have seen that Q has a uniform tip-reduced Gröbner basis and that every right Gröbner basis generates Q. We need to show that if $\mathcal{G}$ is a uniform tip-reduced subset of Q that generates Q then $\mathcal{G}$ is a right Gröbner basis for Q.

By Theorem 5.2, the submodule generated by $\mathcal{G}$ is $\coprod_{g\in\mathcal{G}} gR$. Thus $Q = \coprod_{g\in\mathcal{G}} gR$. Let $x \in Q \setminus \{0\}$. Then $x = \sum_{g\in\mathcal{G}} gr_g$ for some $r_g \in R$ such that all but a finite number of $r_g = 0$. Consider $\mathrm{Tip}(x)$ and the largest element of the form $\mathrm{Tip}(gr_g)$, say $\mathrm{Tip}(g_0 r_{g_0})$. If $\mathrm{Tip}(x) \neq \mathrm{Tip}(g_0 r_{g_0})$ there must cancellation of $\mathrm{Tip}(g_0 r_{g_0})$ by some other $\mathrm{Tip}(gr_g)$. As in the proof of Theorem 5.2, using Lemma 5.1, $\mathrm{Tip}(g_0 r_{g_0}) = \mathrm{Tip}(g_0)\mathrm{Tip}(r_{g_0})$ and $\mathrm{Tip}(gr_g) = \mathrm{Tip}(g)\mathrm{Tip}(r_g)$. Thus, either $\mathrm{Tip}(g_0)$ left divides $\mathrm{Tip}(g)$ or $\mathrm{Tip}(g)$ left divides $\mathrm{Tip}(g_0)$. But this contradicts the assumption that $\mathcal{G}$ is a tip-reduced set. Hence $\mathrm{Tip}(x) = \mathrm{Tip}(g_0)\mathrm{Tip}(r_{g_0})$ and we conclude that $\mathcal{G}$ is a right Gröbner basis for Q. □

6.1. Algorithm for constructing uniform tip-reduced right Gröbner bases for finitely generated submodules Q of P

(1) Given: a finite uniform set $\mathcal{H} = \{h_1, \ldots, h_r\}$ of elements of P.
(2) Remove from 0 from $\mathcal{H}$ if it occurs.
(3) Let $\mathcal{T}_{\mathcal{H}} = \{\mathrm{Tip}(h) \mid \text{ if } h' \in \mathcal{H} \setminus \{h\} \text{ then } \mathrm{Tip}(h') \text{ does not left divide } \mathrm{Tip}(h)\}$.
(4) For each $t \in \mathcal{T}$, choose some $h \in \mathcal{H}$ such that $\mathrm{Tip}(h) = t$, and, renumber so that these element are $h_1, \ldots, h_s$. If $s = r$, we are done. Otherwise let Q^* be the right submodule generated by $\{h_1, \ldots, h_s\}$.
(5) If $s < r$, for each i, $i = s+1, \ldots, r$, write $h_i = h_i^* + \mathrm{Norm}(h_i)$ using $P = Q^* \oplus \mathrm{Span}(\mathrm{NonTip}(Q^*))$. (Note, this is a finite algorithm since by Theorem 6.1, $h_1, \ldots, h_s$ is a uniform tip-reduced right Gröbner basis of Q^*. Reducing tips using $\{h_1, \ldots, h_s\}$ and that $\succ$ is a well-order, after a finite number of reductions, we find $\mathrm{Norm}(h_i)$. Finally, it is easy to see that in this reduction process, each term remains uniform.)
(6) Let $\mathcal{H} = \{h_1, \ldots, h_s, \mathrm{Norm}(h_{s+1}), \ldots, \mathrm{Norm}(h_r)\}$. Go back to step 2.

Not only is the above a finite algorithm by $\succ$ being a well-order on $\mathcal{P}$, but we also note that the produced uniform tip-reduced right Gröbner basis as no more than r elements where r is the number of original generating elements.

7. Right Generators of a Two-sided Ideal

For this section, $R = K\Gamma$ is a path algebra and $(\mathcal{B}, >)$ is an order multiplicative basis of R with $\mathcal{B}$ the directed paths in Γ. We say an element $x \in R \setminus \{0\}$ is *strongly uniform* if there exist vertices v and w in Γ_0 such that $x = vxw$. Let $\mathbf{a}$ be a two-sided ideal and assume $\mathcal{G}$ is the (two-sided) reduced Gröbner basis of strongly uniform elements for $\mathbf{a}$. If $p = p_1p_2$ with $p, p_1, p_2 \in \mathcal{B}$ then we say p_1 is a *prefix* if p. If $p_2 \notin \Gamma_0$ then we say p_1 is a *proper prefix* of p.

Proposition 7.1. *Let* $\mathbf{X} = \{pg \,|\, p \in \mathrm{NonTip}(\mathbf{a}), g \in \mathcal{G}$ *and no proper prefix of* $\mathrm{Tip}(pg)$ *is in the tip ideal of* $\mathbf{a}\}$. *Then* $\mathbf{X}$ *is a tip-reduced uniform right Gröbner basis of* $\mathbf{a}$.

Proof. By the definition of $\mathbf{X}$, it is clearly a tip-reduced set. Since every element of $\mathbf{X}$ is in $\mathbf{a}$ and is uniform, it suffices to show that every element of $\mathbf{a}$ is in the right ideal generated by $\mathbf{X}$. Assume not and let $r \in \mathbf{a}$ such that $\mathrm{Tip}(r)$ is minimal with respect to the property that r is usually not in the right ideal generated $\mathbf{X}$.

Since $\mathcal{G}$ is a Gröbner basis for $\mathbf{a}$, there is some $g \in \mathcal{G}$ such that $\mathrm{Tip}(r) = p\mathrm{Tip}(g)q$ for some paths p and q. Choose this equality so that p is of minimal length. Then no proper prefix of $p\mathrm{Tip}(g)$ is in the tip ideal of $\mathbf{a}$. (This follows since $\mathcal{G}$ is the reduced Gröbner basis for $\mathbf{a}$.)

Thus, $pg \in \mathbf{X}$. Hence, αpgq is in the right ideal generated by $\mathbf{X}$ for any $\alpha \in K$. Let β be the coefficient of $\mathrm{Tip}(r)$ and γ be the coefficient of $\mathrm{Tip}(g)$. Then $r - (\beta/\gamma)pgq$ has smaller tip than r and hence, by the minimality condition on r, $r - (\beta/\gamma)pgq$ is in the right ideal generated by $\mathbf{X}$. But this contradicts the assumption that r is not in the right ideal generated by $\mathbf{X}$. This completes the proof. □

Corollary 7.2. *Let* $\mathbf{a}$ *be an ideal in* $K\Gamma$ *and suppose that* $\mathcal{G}$ *is a reduced Gröbner basis of strongly uniform elements for* $\mathbf{a}$ *with respect to some admissible order* $>$. *Let* $\mathbf{X} = \{pg \,|\, p \in \mathrm{NonTip}(\mathbf{a}), g \in \mathcal{G}$ *and no proper prefix of* $\mathrm{Tip}(pg)$ *is in the tip ideal of* $\mathbf{a}\}$. *Then, as right ideals,*

$$\mathbf{a} = \coprod_{x \in \mathbf{X}} xR.$$

Proof. The result follows from Theorem 5.2. □

We will use this result in the next section.

8. Elimination Theory and the Intersection of Right Submodules

Throughout this section, $R = K\Gamma$ is a path algebra and $\mathcal{B}$ is the basis of paths. Let $P = \coprod_{i \in \mathcal{I}} V(i)R$ where $\mathcal{I}$ is an index set and $V : \mathcal{I} \to \Gamma_0$ is a set map. For each $i \in \mathcal{I}$, let $\mathcal{P}_i = \{x \in P \mid x_j = 0 \text{ if } j \neq i, \text{ and } x_i \in V(i)\mathcal{B}\}$. As before, $\mathcal{P} = \cup_{i \in \mathcal{I}} \mathcal{P}_i$ is a K-basis for P.

We now turn to the problem of generating the intersection of two right submodules of P. First we develop an elimination theory works in the noncommutative setting of path algebras (and hence free algebras). We loosely follow the commutative theory for this (Cox *et al.*, 1992). For this discussion, we will use a special admissible order on the paths in R and extend it to a special right admissible order on P. Recall the lex order is used for commutative elimination theory but the usual lexicographic order in a path algebra is usually not a well order on $\mathcal{B}$ and hence not admissible. We bypass this problem by defining a new order.

Let $>_{nc}$ be a *noncommutative lex order* on $\mathcal{B}$. That is, first fix some order of the vertices and arrows of Γ and also require that the vertices are less than the arrows. Next, identifying all vertices as 1 and pretending the arrows commute, we let $>_c$ be the commutative lex order using the order of the arrows fixed above. In other words, we view paths as commutative monomials in a commutative polynomial ring having the arrows as commutative variables and $>_c$ is just the commutative lex order. If $p, q \in \mathcal{B}$ then $p >_{nc} q$

if either $p >_c q$ or $p =_c q$ and $p >_l q$ in the left lexicographic order using the fixed order on the vertices and arrows. It is easy to see that $>_{nc}$ is an admissible order on $\mathcal{B}$. As in Section 4, we extend $>_{nc}$ to a right admissible order $\succ$ on the basis X of P. That is, choose a well-order $>_{\mathcal{I}}$ on $\mathcal{I}$. If $x_1, x_2 \in \mathcal{P}$, we define $x_1 \succ x_2$ if the nonzero entry of x_1 is greater than the nonzero entry of x_2 (viewed as elements of $\mathcal{B}$) or if the zero entries are equal, and the nonzero entry of x_1 occurs in the ith-component, and the nonzero entry of x_2 occurs in the i'th-component, then $i >_{\mathcal{I}} i'$. With these definitions, $(\mathcal{P}, \succ)$ is an ordered basis for P.

If a is an arrow in a quiver Γ, we let Γ_a denote the quiver with $(\Gamma_a)_0 = \Gamma_0$ and $(\Gamma_a)_1 = \Gamma_1 \setminus \{a\}$. That is, Γ_a is Γ minus the arrow a. We view $K\Gamma_a$ as a subalgebra of $K\Gamma$. If S is a subset of $K\Gamma$, we let $S_a = \{s \in S \mid s \in K\Gamma_a\}$. That is, S_a are the elements s in S such that the arrow a does not occur in any path that occurs in s. Equivalently, $S_a = S \cap K\Gamma_a$.

We let $P_a = \coprod_{i\in\mathcal{I}}(V(i)R)_a$. Note that P_a is a projective right $K\Gamma_a$-module. If S is a subset of P, we let $S_a = S \cap P_a$.

We let $(>_{nc})_a$ be the restriction of $>_{nc}$ to $\mathcal{B}_a$, the paths in Γ_a. It is immediate that $(>_{nc})_a$ is an admissible order; in fact it is also a noncommutatvie lex order on $K\Gamma_a$. We will also denote by $\succ_a$, the extension of $(>_{nc})_a$ to P_a as we extended $>_{nc}$ to $\succ$.

The next result is a noncommutative version of elimination theory.

Theorem 8.1. (The Elimination Theorem) *Let Γ be a quiver, $>_{nc}$ be a noncommutative lex order on the paths of Γ, and assume that $a \in \Gamma_1$ is the maximal arrow in Γ with respect to $>_{nc}$. Let $P = \coprod_{i\in\mathcal{I}} V(i)K\Gamma$ where $V : \mathcal{I} \to \Gamma_0$. Let $(\mathcal{P}, \succ)$ be the ordered basis for P given above. Suppose that $\mathcal{G}$ is a (reduced) uniform right Gröbner basis for P with respect to $>_{nc}$. Then $\mathcal{G}_a$ is a (reduced) uniform right Gröbner basis for P_a with respect to $(\succ)_a$.*

Proof. Let P be a right projective R-module of the appropriate form and $>_{nc}$ and $a \in \Gamma_1$ satisfy the hypothesis of the theorem. We wish to show that

$$\mathcal{G}_a = \mathcal{G} \cap P_a$$

is a uniform right Gröbner basis for P_a with respect to $(\succ)_a$. Uniformity follows if we show it is a right Gröbner basis. If we show that $\mathcal{G}_a$ is a right Gröbner basis, then, if $\mathcal{G}$ is in fact the reduced right Gröbner basis for P, then it is immediate that $\mathcal{G}_a$ is the reduced right Gröbner basis for P_a.

To show that $\mathcal{G}_a$ is a right Gröbner basis of P_a, we let z be an arbitrary element of P_a and show that there is some $g \in \mathcal{G}_a$ such that $\mathrm{Tip}(g)$ is a prefix of $\mathrm{Tip}(z)$. Since $z \in P$ there is some $g \in \mathcal{G}$ such that $\mathrm{Tip}(g)$ is a prefix of $\mathrm{Tip}(z)$. If we show that $g \in P_a$, we will be done. Since $z \in P_a$, it follows that $\mathrm{Tip}(z) \in P_a$, and hence, $\mathrm{Tip}(g) \in P_a$. From the definition of $\mathcal{P}$, there is some $i \in \mathcal{I}$ and path $p \in V(i)\mathcal{B}$ such that $\mathrm{Tip}(g)$ has p in the component corresponding to i, and 0 in all other components. Since $\mathrm{Tip}(g) \in P_a$, the arrow a does not occur in p. Now let q be a path such that for some j, αq occurs as a term in the jth component of g for some $\alpha \in K^*$. By the definition of $\succ$, we have $p \geq_{nc} q$. But, by the definition of $>_{nc}$ and the assumption that a is maximal amongst the arrows, we conclude that $q \in K\Gamma_a$ since $p \in K\Gamma_a$. Hence $g \in P_a$ since every term occurring in every component of g is in $K\Gamma_a$. We are done. □

If $U = \{a_1, \ldots, a_r\}$ be a subset of the arrow set Γ_1 then define Γ_U recursively by

$\Gamma_U = (\Gamma_{U\setminus\{a_r\}})_{a_r}$. Note that this definition is independent of the order of the a_i's. Thus Γ_U is the quiver obtained from Γ by removing the arrows in U. If $>$ is an admissible order then $>_U$ is the restriction of $>$ to $K\Gamma_U$. We define P_U analogously.

Corollary 8.2. *Let Γ be a quiver, $>_{nc}$ be a noncommutative lex order on the paths of Γ with arrows $a_1, \ldots, a_n$. Assume that $a_n >_{nc} a_{n-1} >_{nc} \cdots >_{nc} a_1$. Let P be a right projective $K\Gamma$-module as in Theorem 8.1, and suppose that $\mathcal{G}$ is a (reduced) uniform right Gröbner basis for P with respect to the extended order $\succ$. If $1 \leq k < n$ then $\mathcal{G}_{\{a_{k+1},a_{k+2},\ldots,a_n\}}$ is the (reduced) uniform right Gröbner basis for $P_{\{a_{k+1},\ldots,a_n\}}$ with respect to $(\succ)_{\{a_{k+1},a_{k+2},\ldots,a_n\}}$.*

We now turn our attention to the intersection of right submodules of P. For the next result we need another construction.

Let Γ be a quiver. Let $\Gamma[T]$ be the quiver with $\Gamma[T]_0 = \Gamma_0$ and $\Gamma[T]_1 = \Gamma_1 \cup \{T_v \mid v \in \Gamma_0\}$ where T_v is a loop at vertex v. Thus, $\Gamma[T]$ is the quiver obtained from Γ by adding a loop at each vertex. We view $K\Gamma$ as a subalgebra of $K\Gamma[T]$. Let $R[T] = K\Gamma[T]$. If $P = \coprod_{i\in\mathcal{I}} V(i)R$ where $V : \mathcal{I} \to \Gamma_1$ then we let $P[T] = \coprod_{i\in\mathcal{I}} V(i)R[T]$. Note that $P[T]$ is a projective right $R[T]$-module. Furthermore, P can viewed as an R-sumbodule of $P[T]$. Finally, we let $(\mathcal{P}, \succ)$ be an ordered basis for P obtained from a noncommutative lex order $>_{nc}$ on $\mathcal{B}$. If $\mathcal{B}[T]$ is the basis of paths in $R[T]$, we extend $>_{nc}$ to $\mathcal{B}[T]$ by fixing some order to the T_v's and setting $T_v >_{nc} a$ for each $v \in \Gamma_0$ and $a \in \Gamma_1$.

Let $T = \sum_{v\in\Gamma_0} T_v$. Then if $p \in \mathcal{B}$ note that $Tp = T_{o(p)}p$ and $(1-T)p = T_{o(p)} - T_{o(p)}p$. If $x \in P$, and the ith-component of x is $\sum_{j=1}^{r} \alpha_j p_j$ with $\alpha_j \in K^*$ and $q_j \in V(i)\mathcal{B}$ then we let $Tx \in P[T]$ be the element whose ith component is $\sum_{j=1}^{r} \alpha_j Tp_i$. Since we have defined Tp for basis elements, we linearly extend this definition to define Tx for $x \in P$. For $x \in P$, we define $(1-T)x \in P[T]$ similarly.

Let Q be a right submodule of P. Let TQ denote the right submodule of $P[T]$ consisting of elements $\{Tz \mid z \in Q\}$. Similarly, let $(1-T)Q$ denote the right submodule of $P[T]$ consisting of elements $\{(1-T)z \mid z \in Q\}$.

We prove a result that allows us to algorithmically find a generating set for the intersection of two ideals or two right ideals.

Theorem 8.3. *Let $R = K\Gamma$ be a path algebra and let Q_1 and Q_2 be two right submodules of the right projective R-module $P = \coprod_{i\in\mathcal{I}} V(i)P_i$ for some map $V : \mathcal{I} \to \Gamma_0$. Then*

$$Q_1 \cap Q_2 = (TQ_1 + (1-T)Q_2) \cap P.$$

Proof. Let $h \in Q_1 \cap Q_2$. In particular, $h \in P$. Then, viewing $h \in P[T]$, we have that $h = Th + (1-T)h$. Hence $h \in (TQ_1 + (1-T)Q_2) \cap P$.

Next we let $h \in (TQ_1 + (1-T)Q_2) \cap P$. Then $h = Tf + (1-T)g$ for some $f \in Q_1$ and $g \in Q_2$. Let $\mathbf{D}$ denote the right submodule of $P[T]$ generated by elements with having one nonzero component, say the ith, and, in that component $T_{V(i)}$. Let $\psi : P[T] \to K\Gamma[T]/\mathbf{d}$ be the canonical surjection. Note that P can be considered a right R-submodule of $P[T]/\mathbf{d}$ and that ψ restricted to P is the identity map. It is clear that if $f \in P$ then $\psi(f) = f$. Hence $h = \psi(h) = \psi(Tf + (1-T)g) = g$. Thus $h \in Q_2$. As similar argument shows that $h \in Q_1$ and we are done. □

We now describe how the above results allow us to algorithmically write the intersection of two right submodules of a projective module P as a direct sum of modules of the form fR where f is a uniform element of P. Let Q_1 and Q_2 be right submodules of P. Let $>_{nc}$ be a noncommutative lex order on $\mathcal{B}$. Extend $>_{nc}$ to $\succ$ on $\mathcal{P}$ as described earlier. Assume we have reduced right Gröbner bases of uniform elements for Q_2 and Q_2 As above, construct $P[T]$ and construct right submodules TQ_1 and $(1-T)Q_2$ in $P[T]$ as above.

Let $>_{nc}^*$ be an extension of $>_{nc}$ to the basis of paths in $R[T]$ and $\succ^*$ be an extension of $>_{nc}^*$ to the basis $\mathcal{P}[T]$ such that:

(1) For all $a \in \Gamma_1$ and $v \in \Gamma_0$, $a <^* T_v$.
(2) The order $\succ$ is the restriction of $\succ^*$ to the basis $\mathcal{P}$ of P.

Since we have generating sets of TQ_1 and $(1-T)Q_1$ obtained from uniform tip-reduced right Gröbner bases by applying T on the left and $1-T$ on the left respectively to the bases, we have a uniform tip-reduced generating set of the right submodule $TQ_1 + (1-T)Q_2$ in $P[T]$. Thus, we may algorithmically find a tip-reduced right Gröbner basis of uniform elements of this ideal. By Corollary 8.2 and Theorem 8.3, we can alorithmically find a reduced right Gröbner basis of uniform elements of $Q_1 \cap Q_2$. Finally, Corollary 7.2 gives an algorithmic way of writing $Q_1 \cap Q_2$ as a direct sum of right submodules of P of the form fR, with f uniform.

9. Solving Systems of Equations Over Algebras

In this section, we let $R = K\Gamma$ be a path algebra with ordered multiplicative basis $(\mathcal{B}, >)$ and $\mathcal{B}$ the set of paths in Γ. We fix an ideal I and let Λ denote R/I. We identify Λ with Span(NonTip(I)). We begin by showing that if Λ is finite dimensional, then there is a finite Gröbner basis for I with respect to $>$. In fact, we have a stronger result.

Theorem 9.1. *Let S be a K-algebra with ordered multiplicative basis $(\mathcal{B}, >)$. Assume that $\mathcal{B} \cup \{0\}$ is a finitely generated monoid with 0. Let I be an ideal in S such that S/I is finite dimensional over K. Then there is a finite Gröbner basis for I with respect to $>$.*

Proof. We note that NonTip(I) has $\dim_K(S/I)$ elements. Consider $\mathcal{T} = \{t \in \text{Tip}(I) \mid t$ cannot be properly factored with elements of $\mathcal{B} \setminus \Gamma_0\}$. Recall that $R = I \oplus$ Span (NonTip(I)) as vector spaces and that if $x \in R \setminus \{0\}$, we may write $x = i_x + \text{Norm}(x)$ for unique elements $i_x \in I$ and $\text{Norm}(x) \in \text{Span}(\text{NonTip}(I))$. We see that $\mathcal{G} = \{t - \text{Norm(t)} \mid t \in \mathcal{T}\}$ is a reduced Gröbner basis for I (Green, 1999).

Thus, if we show that $\mathcal{T}$ is finite, we are done. Suppose that $t \in \mathcal{T}$. Then every proper factor of t must be a nontip. Let B be the finite set of generators of $\mathcal{B}$. We show

$$\mathcal{T} \subseteq \{nb \mid n \in \text{NonTip}(I), b \in B\} \cup B.$$

Suppose that $t \in \mathcal{T} \setminus \Gamma_0$. Then $t = o(t)b_1 \cdots b_r$, with $b_i \in B$. Since we are assuming that $t \notin \Gamma_0$, we may suppose that $b_r \in B \setminus \{0\}$. Hence, $o(t)b_1 \cdots b_{r-1} \in \text{NonTip}(I)$ or is a vertex. If $o(t)b_1 \cdots b_{r-1}$ is a vertex, then $b = b_r \in B$. Thus we have shown that $b \in B$ or $b = nb_r$ for some $n \in \text{NonTip}(I)$ and $b_r \in B$. Thus, since $\{nb \mid n \in \text{NonTip}(I), b \in B\} \cup B$ is a finite set, $\mathcal{T}$ is a finite set and we are done. □

Corollary 9.2. *If $\Lambda = K\Gamma/I$ is finite dimensional, and Γ is a finite graph, then I has a finite reduced Gröbner basis with respect to $>$.*

Proof. Since $\Lambda = K\Gamma/I$ is finite dimensional and path set, $\mathcal{B}$, is finitely generated by the vertices and arrows, the result follows. □

We now turn our attention to homogeneous systems of linear equations with coefficients in Λ. We show how the theory of right Gröbner bases can be used to find a generating set of solutions to such a system. We will use the results of the last two sections.

Consider a homogeneous system of n linear equations in m unknowns with coefficients in Λ

$$\begin{aligned} \lambda_{1,1}x_1 + \cdots + \lambda_{1,m}x_m &= 0 \\ \lambda_{2,1}x_1 + \cdots + \lambda_{2,m}x_m &= 0 \\ &\vdots \\ \lambda_{n,1}x_1 + \cdots + \lambda_{n,m}x_m &= 0, \end{aligned} \tag{$*$}$$

where each $\lambda_{i,j} \in \Lambda$ and the x_i's are unknowns.

Let $\Gamma_0 = \{v_1, \ldots, v_r\}$ be the vertices in Γ. The next result allows us to assume strong uniformity of the $\lambda_{i,j}$'s with respect to Γ_0.

Proposition 9.3. *We may associate a new system of linear equations,*

$$\begin{aligned} \lambda'_{1,1}x_1 + \cdots + \lambda'_{1,m'}x_{m'} &= 0 \\ \lambda'_{2,1}x_1 + \cdots + \lambda'_{2,m'}x_{m'} &= 0 \\ &\vdots \\ \lambda'_{n',1}x_1 + \cdots + \lambda'_{n',m'}x_{m'} &= 0, \end{aligned} \tag{$**$}$$

to $(*)$ such that there are functions $V : \{1, \ldots, n'\} \to \Gamma_0$ and $W : \{1, \ldots, m'\} \to \Gamma_0$ such that for each i, $1 \le i \le n'$ and j, $1 \le j \le m'$, $V(i)\lambda'_{i,j}W(j) = \lambda'_{i,j}$. Furthermore, there is a one-to-one correspondence between the solutions of $(*)$ and $(**)$.

Proof. By replacing each $\lambda_{i,j}x_j$ by $\sum_{l=1}^{r}(\lambda_{i,j}v_l)x_{l,j}$, we note each $\lambda_{i,j}v_l$ is uniform and that, fixing j and l, $t(\lambda_{i,j}v_l) = v_l$ is the same for $i = 1, \ldots, n$. This increases the number of variables to mr. We let $W : \{1, \ldots, m\} \times \{1, \ldots, r\} \to \Gamma_0$ be such that $W((j,l)) = v(l)$ for $j = 1, \ldots, m$ and $l = 1, \ldots, r$.

Thus, after this change, the ith equation is now of the form:

$$\lambda_{i,1}v_1x_{1,1} + \cdots + \lambda_{i,1}v_rx_{r,1} + \cdots + \lambda_{i,m}v_mx_{i,m} = 0.$$

For each i, we replace the ith equation by the r equations obtained by multiply by $v_1, \ldots, v_n$ on the left. That is, replace the ith equation by

$$\begin{aligned} v_1\lambda_{i,1}v_1x_{1,1} + \cdots + v_1\lambda_{i,m}v_rx_{r,m} &= 0 \\ v_2\lambda_{i,1}v_1x_{1,1} + \cdots + v_2\lambda_{i,m}v_rx_{r,m} &= 0 \\ &\vdots \\ v_r\lambda_{i,1}v_1x_{1,1} + \cdots + v_r\lambda_{i,m}v_rx_{r,m} &= 0. \end{aligned}$$

In this way we obtain nr equations in mr unknowns. We take $V : \{1, \ldots, n\} \times \{1, \ldots, r\} \to \Gamma_0$ to be $V((i,l)) = v_l$.

This new system has the appropriate properties and there is clearly a one-to-one correspondence between the solutions of $(*)$ and $(**)$; namely, $(a_1, a_2, \ldots, a_m)^t$ is a solution to $(*)$ if and only if $(v_1a_1, v_2a_1, \ldots, v_ra_1, v_1a_2, \ldots, v_ra_m)^t$ is a solution to the new system of nr equations in mr unknowns. □

Thus, without loss of generality, we assume that $(*)$ has the property that there are functions $V : \{1, \ldots, n\} \to \Gamma_0$ and $W : \{1, \ldots, m\} \to \Gamma_0$ such that for each $i = 1, \ldots, n$, and $j = 1, \ldots, m$, $V(i)\lambda_{i,j}W(j) = \lambda_{i,j}$.

Let A be the $n \times m$-matrix over Λ such that the (i, j)th-entry is $\lambda_{i,j}$.

There are certain "trivial" solutions to $(*)$ that we will ignore. Namely, if $X = (a_1, \ldots, a_m)^t$ is a solution, then both $(W(1)a_1, \ldots, W(m)a_m)^t$ and $((1-W(1))a_1, \ldots, (1-W(m))a_m)^t$ are solutions. However, the latter is trivial in the sense that for each $\lambda_{i,j}$, $\lambda_{i,j}(1 - W(j)) = 0$ and thus $((1 - W(j))b_1, \ldots, (1 - W(m))b_m)^t$ is a solution for all possible choices of b_i's. So these solutions are known and uninteresting. In this sense, we study solutions X to $(*)$ such that $X \in \coprod_{j=1}^m W(j)\Lambda$. We want to find a generating set for $M = \{X \in \coprod_{j=1}^m W(j)\Lambda \mid AX = \mathbf{0}\}$. Note that M is a right Λ-module viewing elements of M as $m \times 1$ matrices and right multiplication by elements of Λ as scalar multiplication.

Consider the following exact commutative diagram.

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \uparrow & & \uparrow & & \uparrow \\
0 & \to & M & \to & \coprod_{j=1}^m W(j)\Lambda & \xrightarrow{f_A} & \coprod_{i=1}^n V(i)\Lambda \\
 & & \uparrow s & & \uparrow \pi & & \uparrow = \\
0 & \to & K & \to & \coprod_{j=1}^m W(j)R & \xrightarrow{h_A} & \coprod_{i=1}^n V(i)\Lambda \\
 & & \uparrow i & & \uparrow i & & \uparrow \\
0 & \to & \coprod_{j=1}^m W(j)I & \xrightarrow{=} & \coprod_{j=1}^m W(j)I & \to & 0 \\
 & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & &
\end{array}$$

The maps f_A and h_A are defined as follows. Identifying Λ with Span(NonTip(I)), the view the entries of A as elements of $K\Gamma$. The map $h_A : \coprod_{j=1}^m W(j)R \to \coprod_{i=1}^n V(i)\Lambda$ is given by

$$h_A((r_1, r_2, \ldots, r_m)^t) = \left(\mathrm{Norm}\left(\sum_{j=1}^m \lambda_{1,j}r_j\right), \mathrm{Norm}\left(\sum_{j=1}^m \lambda_{2,j}r_j\right), \ldots, \mathrm{Norm}\left(\sum_{j=1}^m \lambda_{n,j}r_j\right)\right)^t.$$

Similarly, the map $f_A : \coprod_{j=1}^m W(j)\Lambda \to \coprod_{i=1}^n V(i)\Lambda$ is given by

$$f_A((r_1, r_2, \ldots, r_m)^t) = \left(\mathrm{Norm}\left(\sum_{j=1}^m \lambda_{1,j}r_j\right), \mathrm{Norm}\left(\sum_{j=1}^m \lambda_{2,j}r_j\right), \ldots, \mathrm{Norm}\left(\sum_{j=1}^m \lambda_{n,j}r_j\right)\right)^t,$$

where, in this case, each $r_i \in \mathrm{Span}(\mathrm{NonTip}(I))$. Next, $\pi : \coprod_{j=1}^m W(j)R \to \coprod_{j=1}^m W(j)\Lambda$ is the canonical surjection. Finally, $M = \ker(f_A)$ and $K = \ker(h_A)$. The exactness and commutativity are clear. Furthermore, $M = \{X \in \coprod_{j=1}^m W(j)\Lambda \mid AX = \mathbf{0}\}$. Recall that our goal is to find a generating set for M.

For $i = 1, \ldots, m$, let $f_i = (\lambda_{1,j}, \lambda_{2,j}, \ldots, \lambda_{n,j})^t$ be the jth column of A. Thus $f_i \in \coprod_{i=1}^n V(i)\Lambda$. Identifying Λ with $\mathrm{Span}(\mathrm{NonTip}(I))$, we view $f_i \in \coprod_{i=1}^n V(i)R$, for $i = 1, \ldots, n$. Let $P = \coprod_{i=1}^n V(i)R$. Note that P is a right projective R-module, we have two right submodules, $\coprod_{j=1}^m f_iR$ and $\coprod_{i=1}^n V(i)I$.

Theorem 9.4. *Let $R = K\Gamma$ be a path algebra with ordered multiplicative basis $(\mathcal{B}, >)$ where $\mathcal{B}$ is the basis of paths and $>$ is a noncommutative lex order on $\mathcal{B}$. Let P be the right projective R-module $\coprod_{i=1}^n V(i)R$. Let $(\mathcal{P}, \succ)$ be an ordered basis for P constructed as in the last section. Let $\mathcal{G}$ be a tip-reduced uniform right Gröbner basis for $\coprod_{j=1}^m f_jR \cap \coprod_{i=1}^n V(i)I$ in P. For each $g \in \mathcal{G}$, we have $g = \sum_{j=1}^m f_j a(g)_j$ for some $a(g)_j \in R$. Then*

$$\left\{ \begin{pmatrix} \mathrm{Norm}(a(g)_1) \\ \mathrm{Norm}(a(g)_2) \\ \vdots \\ \mathrm{Norm}(a(g)_m) \end{pmatrix} \in \coprod_{j=1}^m W(j)\Lambda \mid g \in \mathcal{G} \right\}$$

is a generating set for M.

Proof. Suppose that $X \in M$. Thus $AX = \mathbf{0}$. If $X = (x_1, \ldots, x_m)^t$, $AX = (0)$ implies $f_1x_1 + f_2x_2 + \cdots + f_mx_m = \mathbf{0}$ in $\coprod_{i=1}^n V(i)\Lambda$. But then, viewing $f_1x_1 + \cdots + f_mx_m$ in $\coprod_{i=1}^n V(i)R$, we conclude that $\sum_{j=1}^m f_jx_j \in \coprod_{i=1}^n V(i)I$. Thus, $\sum_{j=1}^m f_jx_j \in \coprod_{j=1}^m f_jR \cap \coprod_{i=1}^n V(i)I$. Since the elements of $\mathcal{G}$ generate $\coprod_{j=1}^m f_jR \cap \coprod_{i=1}^n V(i)I$ as a right R-module, we conclude that $\sum_{j=1}^m f_jx_j = \sum_{g \in \mathcal{G}} gs_g$ for some $s_g \in R$. (All but a finite number of $s_g = 0$.) Hence

$$\sum_{j=1}^m f_jx_j = \sum_{g \in \mathcal{G}} \left(\sum_{j=1}^m f_j a(g)_j \right) s_g = \sum_{j=1}^m f_j \left(\sum_{g \in \mathcal{G}} a(g)_j s_g \right).$$

However, the sum $\coprod_{j=1}^m f_jR$ is a direct sum, and we conclude that for each j, $x_j = \sum_{g \in \mathcal{G}} a(g)_j s_g$. Since $x_j \in \Lambda$ it follows that $x_j = \mathrm{Norm}(\sum_{g \in \mathcal{G}} a(g)_j s_g) = \sum_{g \in \mathcal{G}} \mathrm{Norm}(a(g)_j) * \mathrm{Norm}(s_g)$ where the last product is in Λ. Finally, we see that

$$X = \sum_{g \in \mathcal{G}} \begin{pmatrix} \mathrm{Norm}(a(g)_1) \\ \mathrm{Norm}(a(g)_2) \\ \vdots \\ \mathrm{Norm}(a(g)_m) \end{pmatrix} * \mathrm{Norm}(s_g),$$

where the right-hand product is as elements of a right Λ-module. The result now follows.□

In fact, the above proof holds for any generating set of the right module $\coprod_{j=1}^m f_jR \cap \coprod_{i=1}^n V(i)I$. We stated it for a tip-reduced right Gröbner basis since these can be constructed from the f_j's and I using Theorem 8.3 and Corollaries 8.2 and 9.2. However, without further assumptions, the computation of the right Gröbner basis $\mathcal{G}$ above need not be finite. We now show that if Λ is finite dimensional, then every computation is finite and hence algorithmic. The next result contains this and a bit more.

THEOREM 9.5. *Let $R = K\Gamma$ be a path algebra and $(\mathcal{B}, >)$ be the order multiplicative basis of paths with a noncommutative lex order. Let I be an ideal such that $\Lambda = K\Gamma/I$ is finite dimensional over K. Let $V : \{1, \ldots, n\} \to \Gamma_0$ be a set function and $P = \coprod_{i=1}^{n} V(i)R$. Let $f_j \in P$, $j = 1, \ldots, m$ be a tip-reduced set of uniform elements. Let $(\mathcal{P}, \succ)$ be the ordered basis of P as in the last section. Then:*

(1) I has a finite reduced Gröbner basis.
(2) As a right submodule of R, I has a finite uniform tip-reduced right Gröbner basis.
(3) The submodule of generated by $\{f_1, \ldots, f_m\}$ of P is $\coprod_{j=1}^{m} f_jR$.
(4) There is a finite uniform tip-reduced right Gröbner basis of $\coprod_{j=1}^{m} f_jR \cap \coprod_{i=1}^{n} V(i)I$ in P with respect to $\succ$.

PROOF. Part 1 follows from Theorem 9.2. Let $\mathcal{G}^*$ be the finite Gröbner basis of I as an ideal. Since Λ is assumed to be finite dimensional, $\mathrm{NonTip}(I)$ has $\dim_K(\Lambda)$ elements and is a finite set. Hence, by Corollary 7.2, there is a finite tip-reduced uniform right Gröbner basis for I and part 2 follows. Part 3 follows from Theorem 5.2. It remains to show that $Q = \coprod_{j=1}^{m} f_jR \cap \coprod_{i=1}^{n} V(i)I$ has a finite uniform tip-reduced right Gröbner basis with respect to $\succ$.

However Q contains the right submodule $Z = \coprod_{j=1}^{m} f_jI$. Then Q/Z is a right submodule of $P/Z = \big(\coprod_{i=1}^{n} V(i)R\big)/\big(\coprod_{i=1}^{n} V(i)I\big)$ which is isomorphic to $\coprod_{i=1}^{n} V(i)(R/I)$. Since R/I is finite dimensional, we see that Q/Z is finite dimensional and has a finite K-basis, say B. For each $b \in B$, choose $b^* \in Q$ such that $b^* + Z = b$ and b^* is uniform. Let $B^* = \{b^* \mid b \in B\}$. If we show that Z is a finite uniform generating set G as a right R-module, then clearly $G \cup B^*$ is a finite uniform set that generates Q. Tip-reducing $G \cup B^*$ yeilds part 4. So it remains to show that $Z = \coprod_{j=1}^{m} f_jI$ has finite uniform generating set. Let $h_1, \ldots, h_t$ be a finite uniform generating set for I as a right ideal (which we have shown to exist in part 2). Then it is easy to see that $\{f_jh_i \mid j = 1, \ldots, m,$ and $i = 1, \ldots, t\}$ is a uniform right generating set for Z. This completes the proof. □

Although the above argument does not make it clear that finding a uniform tip-reduced right Gröbner basis is algorithmic, the work of the last section does. Thus, we get the last result of the section.

THEOREM 9.6. *Let $K\Gamma$ be a path algebra with order multiplicative basis $(\mathcal{B}, >)$ where $\mathcal{B}$ is the set of paths and $>$ is a noncommutative lex order. Suppose that I is an ideal in $K\Gamma$ such that $\Lambda = K\Gamma/I$ is finite dimensional over K. Consider a homogeneous system of n linear equations in m unknowns with coefficients in Λ*

$$\begin{aligned} \lambda_{1,1}x_1 + \cdots + \lambda_{1,m}x_m &= 0 \\ \lambda_{2,1}x_1 + \cdots + \lambda_{2,m}x_m &= 0 \\ &\vdots \\ \lambda_{n,1}x_1 + \cdots + \lambda_{n,m}x_m &= 0, \end{aligned}$$

where each $\lambda_{i,j} \in \Lambda$ and the x_i's are unknowns and assume there are functions

V:$\{1, \ldots, n\} \to \Gamma_0$ and $W : \{1, \ldots, m\} \to \Gamma_0$ such that for each i and j, $V(i)\lambda_{i,j}W(j) = \lambda_{i,j}$. Then there is a finite algorithm to find a generating set for the solutions of the form $(x_1, \ldots, x_n)$ with $x_j \in W(j)\Lambda$.

We summarize the algorithm.

(1) Find the reduced Gröbner basis, $\mathcal{F}$ for I. Since $K\Gamma/I$ is finite dimensional, there is a finite algorithm to do this.
(2) Find NonTip(I). There is a finite algorithm to find this finite set.
(3) Find the uniform reduced right Gröbner basis for I. There is a finite algorithm using Corollary 7.2.
(4) Set up the matrix A and the f_j's as we did prior to Theorem 9.4.
(5) Find a uniform tip-reduced right Gröbner basis for $\coprod_{j=1}^m f_j K\Gamma \cap \coprod_{i=1}^n V(i)I$. By Theorem 9.5 such a basis is finite and hence using the intersection and elimination techniques of the last section, there is an algorithm to do this.
(6) Obtain the desired generating set of solutions from this right Gröbner basis by Theorem 9.4.

10. Projective Resolutions

We conclude the paper with some brief remarks about constructing projective resolutions of right Λ-modules, where Λ is the quotient of a path algebra. Let $R = K\Gamma$ such that $\Lambda = R/I$ for some ideal I.

In Green *et al.* (to appear), a construction of a projective resolution of a right module M was given. In particular, we begin with a presentation, over the path algebra R,

$$0 \to \coprod_{j=1}^{m} f_j^1 R \to \coprod_{i=1}^{n} f_i^0 R \to M \to 0.$$

Here the f_i^0 are vertices and the $f_j^1 \in \coprod_{i=1}^n f_i^0 R$.

The algorithmic construction of the resolution, was dependent on being able to recusively find a direct sum decomposition of $\coprod_s f^n R \cap \coprod_t f^{n-1} I$, given the f_s^n's and the f_t^{n-1}'s. Finding such a direct sum decomposition allowed the construction of the next f^{n+1}'s. However this construction is exactly what was studied in the last two sections and we showed how right Gröbner basis theory provides the "algorithm" to find this intersection.

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