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Cutting Corners

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1. INTRODUCTION

The object of this study in this paper is the *corner cut polyhedron*, which we define as follows:

$$P_n^d := \operatorname{conv}\{\lambda^1 + \cdots + \lambda^n : \lambda^1, \dots, \lambda^n \text{ are } n \text{ distinct vectors in } \mathbf{N}^d\} \subset \mathbf{R}^d.$$

The following result demonstrates its significance in computational commutative algebra.

THEOREM 5.0. *The normal fan of the corner cut polyhedron P_n^d equals the Gröbner fan of the vanishing ideal of the generic configuration of n points in affine d-space. Therefore, the distinct reduced Gröbner bases of this ideal are in bijection with the vertices of P_n^d.*

A nonempty finite subset λ of the set $\mathbf{N}^d$ of nonnegative integer vectors is a *staircase* if $u \in \lambda$ and $v \le u$ (coordinatewise) implies $v \in \lambda$. Let $\binom{\mathbf{N}^d}{n}$ be the set of n-element subsets of $\mathbf{N}^d$ and let $\binom{\mathbf{N}^d}{n}_{\text{stair}}$ be its finite subset of staircases. Staircases for $d = 2$ are *partitions* (or *Ferrers diagrams*), and staircases of $d = 3$ are *plane partitions* (cf. [Sta]). These play an important

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role in algebraic combinatorics. We also introduce the *staircase polytope*

$$Q_n^d := \operatorname{conv}\left\{ \sum \lambda \colon \lambda \in \binom{\mathbf{N}^d}{n}_{\text{stair}} \right\} \subset \operatorname{conv}\left\{ \sum \lambda \colon \lambda \in \binom{\mathbf{N}^d}{n} \right\} = P_n^d.$$

A staircase λ is called a *corner cut* if it is linearly separable from its complement $\mathbf{N}^d \setminus \lambda$, i.e., for some $w \in \mathbf{R}^d$ we have $w \cdot v < w \cdot u$ for all $v \in \lambda$ and $u \in \mathbf{N}^d \setminus \lambda$. Let $\binom{\mathbf{N}^d}{n}_{\text{cut}}$ be the set of all n-element corner cuts. Planar and three-dimensional corner cuts appear in various contexts, such as combinatorial number theory [BF1] and computer vision [Bru, Ger]. In this article we examine the set $\binom{\mathbf{N}^d}{n}_{\text{cut}}$ of corner cuts and the corner cut polyhedron P_n^d from four points of view: polyhedral geometry (Sect. 2), computational complexity (Sect. 3), enumerative combinatorics (Sect. 4), and commutative algebra (Sect. 5).

Section 2 concerns the facial structure and the normal fans of P_n^d and Q_n^d. We prove

THEOREM 2.0. *The corner cut polyhedron satisfies $P_n^d = Q_n^d + \mathbf{R}^d_{\geq 0}$ and is hence indeed a polyhedron. The staircase polytope Q_n^d has the same vertex set as P_n^d. The map $\lambda \to \Sigma\lambda$ defines a bijection between the corner cuts and the common vertex set of P_n^d and Q_n^d.*

For $\lambda \in \binom{\mathbf{N}^d}{n}_{\text{stair}}$ let M_λ be the ideal in $k[x] = k[x_1, \dots, x_d]$ which is generated by all monomials $x^u = x_1^{u_1} \cdots x_d^{u_d}$ with $u = (u_1, \dots, u_d) \in \mathbf{N}^d \setminus \lambda$. We may represent λ by the set $\min(M_\lambda)$ of minimal generators of M_λ. Dually, λ can also be represented by its subset $\max(\lambda)$ of coordinate-wise maximal elements. They correspond to the socle monomials of $k[x]/M_\lambda$. For both representations the following computational complexity result holds.

THEOREM 3.0. *There is a polynomial time algorithm for recognizing corner cuts.*

Here the point is that the dimension d is not fixed. A key observation is that if λ is a corner cut then M_λ is *Borel fixed* (Lemma 3.3). This ensures that $\max(\lambda)$ and $\min(M_\lambda)$ have roughly the same size (Corollary 3.6). For $d = 2$ our algorithm can be specialized to the algorithm in [BF1] for recognizing nonhomogeneous spectra of numbers.

The number of Borel fixed ideals grows exponentially in n, even in the plane $d = 2$ (Proposition 4.4). However, the number of corner cuts is polynomial in any fixed dimension.

THEOREM 4.0. *For fixed d, we have* $\#\binom{\mathbf{N}^d}{n}_{\text{cut}} = O(n^{2d(d-1)/(d+1)})$.

Staircases in dimensions 2 and 3 are counted by the classical generating functions

$$\sum_{n=0}^{\infty} \#\binom{\mathbf{N}^2}{n}_{\text{stair}} \cdot z^n = \prod_{k=1}^{\infty} \frac{1}{(1-z^k)} \quad \text{and}$$

$$\sum_{n=0}^{\infty} \#\binom{\mathbf{N}^3}{n}_{\text{stair}} \cdot z^n = \prod_{k=1}^{\infty} \frac{1}{(1-z^k)^k}$$

(see [Sta, Corollary 18.2]). No such formulas are known for $d \geq 4$. We raise the related question of determining $\sum_{n=0}^{\infty} \#\binom{\mathbf{N}^d}{n}_{\text{cut}} \cdot z^n$, the generating functions for corner cuts. An answer for the plane ($d = 2$) is given in [CRST]. Section 4 also contains an efficient procedure for enumerating $\binom{\mathbf{N}^d}{n}_{\text{cut}}$ and hence all vertices of the corner cut polyhedron P_n^d.

In Section 5 we apply our combinatorial results to Gröbner bases of point configurations, starting with Theorem 5.0. We explicitly determine the universal Gröbner basis for any n points in d-space, and we show that its cardinality is polynomial in n for fixed d.

The following example illustrates the objects of study. Let $d = 2$, $n = 6$. This is the first instance when the map $\binom{\mathbf{N}^d}{n}_{\text{stair}} \to \mathbf{R}^d\colon\ \lambda \mapsto \Sigma\lambda$ is not injective. The set of staircases $\binom{\mathbf{N}^2}{6}_{\text{stair}}$ has 11 elements, corresponding to the 11 partitions of the integer 6. We list each partition λ together with the monomial ideal M_λ and its image $\Sigma\lambda$ in Q_6^2.

$1+1+1+1+1+1$	$\langle x^6, y\rangle$	$(15, 0)$
$2+1+1+1+1$	$\langle x^5, xy, y^2\rangle$	$(10, 1)$
$2+2+1+1$	$\langle x^4, x^2y, y^2\rangle$	$(7, 2)$
$2+2+2$	$\langle x^3, y^2\rangle$	$(6, 3)$
$3+1+1+1$	$\langle x^4, xy, y^3\rangle$	$(6, 3)$
$3+2+1$	$\langle x^3, x^2y, xy^2, y^3\rangle$	$(4, 4)$
$3+3$	$\langle x^2, y^3\rangle$	$(3, 6)$
$4+1+1$	$\langle x^3, xy, y^4\rangle$	$(3, 6)$
$4+2$	$\langle x^2, xy^2, y^4\rangle$	$(2, 7)$
$5+1$	$\langle x^2, xy, y^5\rangle$	$(1, 10)$
6	$\langle x, y^6\rangle$	$(0, 15)$.

The corner cut polyhedron P_6^2 has six bounded edges and seven vertices, one for each corner cut. Thus the generic configuration of six points in the plane has seven distinct initial monomial ideals in M_λ. The four staircases which are not corner cuts are those mapped to $(3, 6)$ or $(6, 3)$. The staircase polygon Q_6^2 is obtained from P_6^2 by erasing the unbounded edges on the two coordinate axes and drawing the edge between $(0, 15)$ and $(15, 0)$ instead.

In this article we consider only finite staircases λ; i.e., we assume that M_λ is Artinian. With suitable care, many of our results can be extended to the infinite situation as well.

2. THE CORNER CUT POLYHEDRON AND THE STAIRCASE POLYTOPE

For any subset $\mathscr{F} \subseteq \binom{\mathbb{N}^d}{n}$ we abbreviate $\Sigma\mathscr{F} := \{\Sigma\lambda \in \mathbb{N}^d : \lambda \in \mathscr{F}\}$. In this section we describe the facial structure and the normal fan of the corner cut polyhedron $P_n^d = \operatorname{conv}\Sigma\binom{\mathbb{N}^d}{n}$ and the staircase polytope $Q_n^d = \operatorname{conv}\Sigma\binom{\mathbb{N}^d}{n}_{\text{stair}}$.

We start with a lemma. We denote by $\mu^{(j)}$ the corner cut $\{i \cdot e_j : i = 0, 1, \dots, n-1\}$.

LEMMA 2.1. *For every staircase $\lambda \in \binom{\mathbb{N}^d}{n}_{\text{stair}}$, the sum of the coordinates of the vector $\Sigma\lambda$ is at most $\binom{n}{2}$. Equality holds if and only if $\lambda \in \{\mu^{(1)}, \mu^{(2)}, \dots, \mu^{(d)}\}$.*

Proof. We use induction on n. The case $n \le 2$ is trivial. Choose $u = (u_1, \dots, u_d) \in \lambda$ such that $\lambda \backslash \{u\}$ is a staircase of cardinality $n - 1$. There are $\prod_{i=1}^d (u_i + 1) - 1$ nonnegative vectors strictly below u. Each of them must lie in $\lambda \backslash \{u\}$. Hence

$$n \ge \prod_{i=1}^{d} (u_i + 1) \ge u_1 + u_2 + \cdots + u_d + 1. \tag{2.1}$$

By induction, the coordinate sum of $\Sigma\lambda \backslash \{u\}$ is at most $\binom{n-1}{2}$, and hence the desired inequality follows from (2.1) and $\binom{n-1}{2} + n - 1 = \binom{n}{2}$. Finally, equality holds in (2.1) if and only if all but one coordinate of u is zero. ∎

We now prove Theorem 2.0 and show that P_n^d deserves its name.

Proof of Theorem 2.0. Let $w \in \mathbf{R}^d_{\ge 0}$ be a vector whose coordinates are $\mathbf{Q}$-linearly independent. We sort the nonnegative integer vectors according to their w-value, say, $\mathbb{N}^d = \{u_1 = 0, u_2, u_3, u_4, \dots\}$, so that $w \cdot u_i < w \cdot u_j$ if and only if $i < j$. The unique minimum of the map $\binom{\mathbf{N}^d}{n} \mapsto \mathbf{R}\colon \lambda \mapsto w \cdot \Sigma\lambda$ is attained at the corner cut $\lambda = \{u_1, \ldots, u_n\}$. Hence the point $\Sigma\lambda$ is the common vertex of P_n^d and Q_n^d at which the linear functional $\mathbf{R}^d \mapsto \mathbf{R}$: $u \mapsto w \cdot u$ attains its minimum. Every corner cut $\lambda \in \binom{\mathbf{N}^d}{n}_{\text{cut}}$ arises this way for some $w \in \mathbf{R}^d_{\geq 0}$ and hence defines a common vertex of P_n^d and Q_n^d.

Next consider $w \in \mathbf{R}^d \backslash \mathbf{R}^d_{\geq 0}$. Then w is not bounded below over P_n^d. This shows that the map $\lambda \mapsto \Sigma\lambda$ is a bijection between $\binom{\mathbf{N}^d}{n}_{\text{cut}}$ and the vertex set of P_n^d, and proves that $P_n^d = Q_n^d + \mathbf{R}^d_{\geq 0}$. Suppose now $\alpha := w_j < 0$ is uniquely the smallest coordinate of w. Then

$$w \cdot \sum \lambda = \sum_{i=1}^{d} w_i \Big(\sum \lambda\Big)_i \geq \alpha \cdot \sum_{i=1}^{d} \Big(\sum \lambda\Big)_i \geq \alpha \cdot \binom{n}{2} \tag{2.2}$$

holds for any staircase λ, with equality if and only if $\lambda = \mu^{(j)}$, by Lemma 2.1. Hence the map $u \mapsto w \cdot u$ attains its minimum over Q_n^d at the vertex $\Sigma \mu^{(j)}$ of P_n^d. We conclude that every vertex of Q_n^d is also a vertex of P_n^d, and so Q_n^d and P_n^d have the same vertex set. ∎

Next, we describe the facial structure and the normal fans of P_n^d and Q_n^d. For each $n \in \mathbf{N}$ and $w \in \mathbf{R}^d_{>0}$, we construct a polytope P_n^w as follows. Let w_0 be the smallest real number such that $\#\{v \in \mathbf{N}^d : w \cdot v \leq w_0\} \geq n$. Let $L := \{v \in \mathbf{N}^d : w \cdot v < w_0\}$, let $H := \{v \in N^d : w \cdot v = w_0\}$, let $h := n - |L| \geq 1$, and let $\Lambda := \{L \cup M : M \in \binom{H}{h}\} \subset \binom{\mathbf{N}^d}{n}$. We define the polytope P_n^w to be

$$P_n^w := \operatorname{conv} \sum \Lambda = \sum L + \operatorname{conv} \sum \binom{H}{h} \subset \mathbf{R}^d.$$

The following theorem shows that every bounded face of P_n^d equals P_n^w for some $w \in \mathbf{R}^d_{>0}$.

THEOREM 2.2. *Let $w \in \mathbf{R}^d$ and let F^w be the face of the corner cut polyhedron P_n^d at which the linear functional $x \mapsto w \cdot x$ is minimized. Then,*

(a) *If w is positive then $F^w = P_n^w$. If $h = |H|$ then P_n^w is the point $\Sigma(L \cup H)$, hence a vertex, and $L \cup H$ is a corner cut in $\binom{\mathbf{N}^d}{n}_{\text{cut}}$. If $h < |H|$ then* $\dim(P_n^w) = \dim(H) \geq 1$.

(b) *If w is nonnegative and $I := \{i \in [d] : w_i = 0\}$ is nonempty then F^w is the unbounded $|I|$-dimensional face $P_n^d \cap \mathbf{R}^I$, which is isomorphic to the corner cut polyhedron $P_n^{|I|}$.*

(c) *If w has a negative coordinate then w is unbounded below; hence* $F^w = \varnothing$.

Proof. First, suppose w is positive. Let L, H, h, Λ be determined by w as described above. Then $w \cdot a < w \cdot b < w \cdot c$ for all $a \in L$, $b \in H$, and $c \in \mathbf{N}^d \backslash (L \cup H)$, and w is constant over H. Therefore, w is minimized at $\Sigma\lambda$ if and only if λ contains L and any h elements from H, which holds precisely when $\lambda \in \Lambda$. This shows that $F^w = P_n^w$. Now, P_n^w is a point if and only if $\Sigma\binom{H}{h}$ is, which holds if and only if $h = |H|$. In this case $(L \cup H) \in \binom{\mathbf{N}^d}{n}_{\mathrm{cut}}$ and $P_n^w = \Sigma(L \cup H)$. Suppose next $h < |H|$. Then the affine span of $\binom{H}{h}$ is a translate of the affine span of H; hence $\dim(P_n^w) = \dim(H)$. This proves (a).

Next, suppose w is nonnegative and $I := \{i \in [d] : w_i = 0\} \neq \emptyset$. Clearly, $P_n^d \cap \mathbf{R}^I$ is a face of P_n^d isomorphic to the unbounded $|I|$-dimensional corner cut polyhedron $P_n^{|I|}$. Now, consider any $\lambda \in \binom{\mathbf{N}^d}{n}$. Then $w \cdot \Sigma\lambda = 0$ if $\Sigma\lambda \in \mathbf{R}^I$, whereas $w \cdot \Sigma\lambda > 0$ otherwise. This shows that w is minimized at $\Sigma\lambda$ precisely when $\Sigma\lambda \in P_n^d \cap \mathbf{R}^I$. Hence (b) follows.

Finally, (c) holds since if w has a negative coordinate then it is unbounded over P_n^d. ∎

We similarly describe the facial structure and normal fan of the staircase polytope.

THEOREM 2.3. *Let $w \in \mathbf{R}^d$ and let F^w be the face of the staircase polytope Q_n^d at which the linear functional $x \to w \cdot x$ is minimized. Then*

(a) *If w is positive then F^w is the polytope P_n^w, as in Theorem* 2.2(a).

(b) *If w is nonnegative and the set $I := \{i \in [d] : w_i = 0\}$ is nonempty then F^w is the $|I|$-dimensional face $Q_n^d \cap \mathbf{R}^I$, which is isomorphic to the staircase polytope $Q_n^{|I|}$.*

(c) *If $\alpha := \min\{w_1, \dots, w_d\} < 0$ and $I := \{i \in [d] : w_i = \alpha\}$ then the face F^w is the $(|I| - 1)$-simplex* $\mathrm{conv}\{\binom{n}{2} \cdot e_i : i \in I\}$.

Proof. Part (a) follows from the observation that $P_n^w \subseteq Q_n^d$ for every positive w. Part (b) is analogous to part (b) of Theorem 2.2. It remains to prove part (c). Let α and I be as above. Then the inequality (2.2) holds for every staircase $\lambda \in \binom{\mathbf{N}^d}{n}_{\mathrm{stair}}$. By Lemma 2.1, the last inequality in (2.2) is strict unless λ is some $\mu^{(j)}$. By definition of I, the middle inequality in (2.2) is strict unless $\lambda = \mu^{(j)}$ for some $j \in I$. This shows that w attains its minimum over Q_n^d precisely at the $(|I| - 1)$-simplex $\mathrm{conv}\{\binom{n}{2} \cdot e_j : j \in I\}$, as claimed. ∎

We summarize the results of this section in the following theorem.

THEOREM 2.4. *The corner cut polyhedron P_n^d and the staircase polytope Q_n^d satisfy*

(a) $P_n^d = Q_n^d + \mathbf{R}^d_{\geq 0}$ *and* $Q_n^d = P_n^d \cap \{x \in \mathbf{R}^d : \sum_i x_i \leq \binom{n}{2}\}$.

(b) *The set of vertices of P_n^d and the set of vertices of Q_n^d equal* $\Sigma\binom{\mathbf{N}^d}{n}_{\mathrm{cut}}$.

(c) *The face poset of Q_n^d is obtained from the face poset of P_n^d as follow:*

1. *Each bounded face of P_n^d is included.*

2. *For $I \subset [d]$ with $1 < |I| < d$, the face $P_n^d \cap \mathbf{R}^I$ is replaced by the face $Q_n^d \cap \mathbf{R}^I$.*

3. *The face $\binom{n}{2} \cdot e_i + \mathbf{R}^{\{i\}}$ is removed for each $i \in [d]$.*

4. *The simplex* $\mathrm{conv}\{\binom{n}{2} \cdot e_1, \ldots, \binom{n}{2} \cdot e_d\}$ *and its faces of dimension ≥ 1 are added.*

3. RECOGNIZING CORNER CUTS

In this section a polynomial time algorithm is given for deciding whether a staircase λ is a corner cut. Here the staircase λ is represented either by its subset $\max(\lambda)$ of maximal elements, or by the set $\min(\mathbf{N}^d \setminus \lambda)$ of minimal elements in its complement. We identity $\min(\mathbf{N}^d \setminus \lambda)$ with the set $\min(M_\lambda)$ of minimal generators of the monomial ideal M_λ. For instance, in the plane ($d = 2$) every staircase is represented by two integer sequences $0 = a_1 < a_2 < \cdots < a_m$ and $b_1 > b_2 > \cdots > b_m = 0$, which are interpreted as follows:

$$\min(\mathbf{N}^d \setminus \lambda) = \{(a_1, b_1), \ldots, (a_m, b_m)\},$$

$$M_\lambda := \langle y^{b_1}, x^{a_2} y^{b_2}, \ldots, x^{a_{m-1}} y^{b_{m-1}}, x^{a_m} \rangle,$$

$$\max(\lambda) = \{(a_2 - 1, b_1 - 1), \ldots, (a_m - 1, b_{m-1} - 1)\}.$$

Since a staircase is, by definition, nonempty and finite, the set $\min(\mathbf{N}^d \setminus \lambda)$ contains a positive multiple of each unit vector. This gives the following "corner cut criterion."

LEMMA 3.1. *A staircase λ is a corner cut if and only if the system of linear equalities*

$$(\mathrm{LP}): \quad \forall v \in \max(\lambda) \quad \forall u \in \min(\mathbf{N}^d \setminus \lambda) : (u - v) \cdot w \geq 1$$

has a solution $w \in \mathbf{Q}^d$. *In other words*, λ *is a corner cut if and only if* (LP) *is feasible*. *Moreover*, *any solution* w *to* (LP) *is necessarily coordinatewise positive*.

We call a solution w to (LP) a *separator* for λ. Recall that our input is either the set $\max(\lambda)$ or the set $\min(\mathbb{N}^d \setminus \lambda)$ but not both. Thus in order to write down the linear program (LP) we must first compute $\min(\mathbb{N}^d \setminus \lambda)$ from $\max(\lambda)$ or vice versa. This is a nontrivial task. Agnarsson [Agn, Theorem 19] showed that for every $m \geq d \geq 1$, there exists a staircase λ with $\#\min(\mathbb{N}^d \setminus \lambda) = m$ and $\#\max(\lambda) \geq c(d) \cdot m^{\lfloor d/2 \rfloor}$. The same example can be dualized to show that for each $m' \geq d \geq 1$ there exists a staircase λ' with $\#\max(\lambda') = m'$ and $\#\min(\lambda') \geq c'(d) \cdot \#(m')^{\lfloor d/2 \rfloor}$. Hence the size of $\max(\lambda)$ can be exponential in d if $\min(\mathbb{N}^d \setminus \lambda)$ is given, and vice versa. This implies:

PROPOSITION 3.2. *For varying dimension* d, *there is no polynomial time algorithm for computing* $\min(\mathbb{N}^d \setminus \lambda)$ *from* $\max(\lambda)$, *or for computing* $\max(\lambda)$ *from* $\min(\mathbb{N}^d \setminus \lambda)$.

We shall overcome this obstacle by restricting to a special class of staircases. A staircase λ in $\mathbb{N}^d$ is called *Borel fixed* if $v + (e_j - e_i) \notin \mathbb{N}^d \setminus \lambda$ for all $i < j$ and $v \in \lambda$. This is equivalent to saying that the monomial ideal M_λ is Borel fixed. Borel-fixed monomial ideals play an important role in computational algebraic geometry (see [BaS or Eis]).

LEMMA 3.3. *Up to a permutation of coordinates*, *every corner cut is Borel-fixed*.

Proof. Let $\lambda \subseteq \mathbb{N}^d$ be a corner cut with separator $w = (w_1, \ldots, w_d)$. Permuting coordinates if necessary we may assume $w_1 \geq \cdots \geq w_d$. Then, if $v \in \lambda$ and $i < j$, we have found that $w \cdot (v + (e_j - e_i)) = w \cdot v + (w_j - w_i) \leq w \cdot v$ hence $v + (e_j - e_i) \notin \mathbb{N}^d \setminus \lambda$. ∎

The bit size of a vector $v \in \mathbb{N}^d$ is the number $d + \sum_{i=1}^{d} \lceil \log_2(v_i + 1) \rceil$ of bits needed to present it. The bit size of an input $V \subset \mathbb{N}^d$ is the sum of the bit sizes of its members.

LEMMA 3.4. *Let* λ *be a staircase which is represented by either* $\min(\mathbb{N}^d \setminus \lambda)$ *or by* $\max(\lambda)$. *There exists a polynomial time algorithm for deciding whether* λ *is Borel fixed*.

Proof. A staircase λ is Borel fixed if and only if the following equivalent conditions hold:

$$\forall v \in \max(\lambda) : v_i > 0 \text{ and } i < j \Rightarrow \exists v' \in \max(\lambda) : v + e_j - e_i \le v', \tag{3.1}$$

$$\forall u \in \min(\mathbb{N}^d \setminus \lambda) : u_j > 0 \quad \text{and} \quad i < j$$
$$\Rightarrow \exists u' \in \min(\mathbb{N}^d \setminus \lambda) : u - e_j + e_i \ge u'. \tag{3.2}$$

Either (3.1) or (3.2) can be tested in time polynomial in the size of the input. ∎

The two dual representations of a Borel fixed staircase λ can be transformed into each other by the following explicit rules. For $i \in \{1, \dots, d\}$ let $\max(\lambda)^{[i]}$ denote the subset of maximal elements in the set $\{(v_1, \dots, v_i, 0, \dots, 0) \in \mathbb{N}^d : v = (v_1, \dots, v_d) \in \max(\lambda)\}$.

PROPOSITION 3.5. *Let λ be a Borel fixed staircase. Then*

(a) $\max(\lambda) = \{u - e_d : u \in \min(\mathbb{N}^d \setminus \lambda),\ u_d > 0\}$, *and*

(b) $\min(\mathbb{N}^d \setminus \lambda) = \{v + e_i : v \in \max(\lambda)^{[i]},\ i = 1, \dots, d\}$.

Proof. We claim that, if $v \in \lambda$, $u \in \mathbb{N}^d \setminus \lambda$, and $u \le v + e_d$, then $u = v + e_d$. For such u, v clearly $u_d = v_d + 1$. If $u_i < v_i$ for some i, then $v + (e_d - e_i) \ge u$ and $v + (e_d - e_i) \in \lambda$, which is impossible. Therefore, $u_i = v_i$ for all $i < d$ hence $u = v + e_d$ as claimed. It follows that if $v \in \max(\lambda)$ then $u := v + e_d \in \min(\mathbb{N}^d \setminus \lambda)$, and if $u \in \min(\mathbb{N}^d \setminus \lambda)$ with $u_d > 0$ then $v := u - e_d \in \max(\lambda)$ which proves part (a). For part (b) note first that the set on the right-hand side is an antichain in $\mathbb{N}^d \setminus \lambda$. It thus remains to show that it contains $\min(\mathbb{N}^d \setminus \lambda)$. Consider any $u \in \min(\mathbb{N}^d \setminus \lambda)$ and assume u_i is its last positive coordinate. Let $\lambda^{(i)} = \{(v_1, \dots, v_i) : v = (v_1, \dots, v_d) \in \lambda\}$ be the projection of λ to $\mathbb{N}^i$. Then $\lambda^{(i)}$ is Borel fixed and $\max(\lambda)^{[i]} = \{(v_1, \dots, v_i, 0, \dots, 0) : (v_1, \dots, v_i) \in \max(\lambda^{(i)})\}$. Further, $(u_1, \dots, u_i) \in \min(\lambda^{(i)})$. Part (a) applied to $\lambda^{(i)}$ in $\mathbb{N}^i$ shows $(u_1, \dots, u_i) = (v_1, \dots, v_i) + e_i$ for some $(v_1, \dots, v_i) \in \max(\lambda^{(i)})$ hence $u = v + e_i$ for some $v \in \max(\lambda)^{[i]}$. ∎

COROLLARY 3.6. *Let λ be a Borel fixed staircase. Then*

$$\#\max(\lambda) + d - 1 \le \#\min(\mathbb{N}^d \setminus \lambda) \le d \cdot \#\max(\lambda).$$

Proof. The second inequality is clear from part (b) of Proposition 3.5. The first inequality follows from part (a) of Proposition 3.5 and the fact that, λ being finite, the set $\min(\mathbb{N}^d \setminus \lambda)$ contains at least $d - 1$ vectors with zero last coordinate. ∎

Corollary 3.6 stands in contrast to Proposition 3.2 and the results in [Agn] for general staircases. It shows that Borel fixed staircases are much more nicely behaved than general staircases. We are now prepared to prove the complexity result stated in the Introduction.

Proof of Theorem 3.0. We describe an algorithm for deciding whether a given staircase λ is a corner cut, and, as we go along, we shall argue that all steps can be done in polynomial time in the bit size of the input. We only explain how this is done for the case when λ is represented by $\max(\lambda)$, where we make use of condition (3.1) of Lemma 3.4 and part (b) of Proposition 3.5. The case when λ is represented by $\min(\mathbb{N}^d \setminus \lambda)$ is analogous and makes use of condition (3.2) of Lemma 3.4 and part (a) of Proposition 3.5 instead.

The first step is to decide whether λ is Borel fixed after some permutation of the variables, and in the affirmative case, apply such a permutation. We define a directed graph G on the set $[d] = \{1, 2, \dots, d\}$ as follows. We include the arc (i, j) in G if and only if, for each $v \in \max(\lambda)$ with $v_i > 0$ we have $v + e_j - e_i \le v'$ for some $v' \in \max(\lambda)$. The number of operations needed to construct G is quadratic in d and quadratic in $\#\max(\lambda)$ and hence is polynomial in the input size. We now try to construct a permutation π on $[d]$ by the following procedure, which is easily carried out using quadratically many operations. For $i = 1, 2, \dots, \dots$, we define $\pi(i)$ to be any *source* in the digraph $G - \{\pi(j)\colon j < i\}$, where a source is defined to be a vertex having outgoing arcs to all other vertices. If this procedure successfully completes a permutation $\pi = (\pi(1), \dots, \pi(d))$ then condition (3.1) of Lemma 3.4 holds with the coordinate order $\pi(1), \dots, \pi(d)$, so π makes λ Borel fixed. We claim that, if this procedure fails at some i to find a source, then no permutation makes λ Borel fixed. To see this, suppose that $\pi(j)$ had been determined for all $j < i$ but $G - \{\pi(j)\colon j < i\}$ contains no source. Assume indirectly that λ is Borel fixed under some permutation τ. Let $r \in [d]$ be smallest with $\tau(r) \in S := [d] \setminus \{\pi(j)\colon j < i\}$. Since $\tau(r)$ is not a source in $G[S]$, there exists $s > r$ with $\tau(s) \in S$ and $(\tau(r), \tau(s))$ not an arc in G. By the construction of G, this implies that there exists $v \in \max(\lambda)$ with $v_{\tau(r)} > 0$ such that $v + e_{\tau(s)} - e_{\tau(r)} \le v'$ fails for all $v' \in \max(\lambda)$. This shows that condition (3.1) fails for the coordinate order specified by τ, contradicting the choice of τ.

So if a permutation was not found then λ is not a corner cut by Lemma 3.3 and we are done. Assume now that a permutation had been found and applied to the coordinates, so that λ is Borel fixed. We can then determine $\min(\mathbb{N}^d \setminus \lambda)$ by Proposition 3.5(b), in polynomial time (cf. Corollary 3.6). Having at hand now both $\max(\lambda)$ and $\min(\mathbb{N}^d \setminus \lambda)$, we can write down the linear program (LP) in Lemma 3.1. It is well known by the work of Khachiyan and Karmarkar [Sch, Sects. 13–15] that the feasibility of a system of linear inequalities can be decided in polynomial time. This completes the proof. ∎

In any fixed dimension d, the feasibility of the linear program (LP) can be checked in *strongly* polynomial time, say, by Fourier–Motzkin elimination (cf. [Sch]). In particular, in small dimensions $d = 2, 3$ it is possible to write down the Fourier–Motzkin eliminated system of inequalities explicitly in terms of $\min(\mathbb{N}^d \setminus \lambda)$ and $\max(\lambda)$. This gives an analytical criterion for λ to be a corner cut. Let us demonstrate this for the plane $d = 2$. We may assume $w_2 = 1$ and ask for $w_1 \geq 0$. With $m := \#\min(\mathbb{N}^d \setminus \lambda)$, we obtain a system of $m^2 - m$ inequalities $(u_1 - v_1) \cdot w_1 + (u_2 - v_2) > 0$ where $v = (v_1, v_2)$ runs through $\max(\lambda)$ and $u = (u_1, u_2)$ runs through $\min(\mathbb{N}^2 \setminus \lambda)$. Each such inequality can be rewritten as $w_1 > (v_2 - u_2)/(u_1 - v_1)$ if $u_1 > v_1$ and as $w_1 < (v_2 - u_2)/(u_1 - v_1)$ if $u_1 < v_1$, and can be omitted if $u_1 = v_1$. Let

$$L_\lambda := \max\left\{ \frac{v_2 - u_2}{u_1 - v_1} : v \in \max(\lambda),\, u \in \min(\mathbb{N}^2 \setminus \lambda),\, u_1 > v_1 \right\},$$

$$U_\lambda := \min\left\{ \frac{v_2 - u_2}{u_1 - v_1} : v \in \max(\lambda),\, u \in \min(\mathbb{N}^2 \setminus \lambda),\, u_1 < v_1 \right\}.$$

Then we obtain the following criterion for a staircase $\lambda \subset \mathbb{N}^2$ to be a corner cut, which is equivalent to the result of Boshernitzan and Fraenkel [BF1] on spectra of numbers.

COROLLARY 3.7. *A staircase $\lambda \subseteq \mathbb{N}^2$ is a corner cut if and only if $L_\lambda < U_\lambda$.*

Remark 3.8. Based on this criterion, Boshernitzan and Fraenkel gave a quadratic algorithm for recognizing nonhomogeneous spectra of numbers, which is basically our algorithm for $d = 2$. Later, in [BF2], they refined it to a linear time algorithm. A natural question is whether a linear time recognition algorithm for corner cuts exists in any dimension.

4. COUNTING AND ENUMERATING CORNER CUTS

In this section we discuss the number of corner cuts $\mathbb{N}\binom{\mathbb{N}^d}{n}_{\text{cut}}$. This number grows polynomially with n for fixed d, while the number of Borel fixed staircases is exponential even in the plane. We also show that in fixed dimension all n-element corner cuts can be efficiently enumerated. For the upper bound we shall make use of the following classical result.

PROPOSITION 4.1 (Andrews [And]). *For every fixed* d, *the number of vertices of any lattice polytope* P *in* $\mathbf{R}^d$ *satisfies* $\#\mathrm{vert}(P) = O(\mathrm{vol}(P)^{(d-1)/(d+1)})$.

See [BV] for recent developments in discrete geometry related to Andrews' theorem.

Proof of Theorem 4.0. Fix the dimension d. By Theorem 2.0, the corner cuts are in bijection with the vertices of the corner cut polytope Q_n^d. By Lemma 2.1, Q_n^d is contained in the d-simplex $\mathrm{conv}\{0, \binom{n}{2} \cdot e_1, \ldots, \binom{n}{2} \cdot e_d\}$; hence its volume satisfies $\mathrm{vol}(Q_n^d) \le (\frac{1}{d!})\binom{n}{2}^d = O(n^{2d})$. Since Q_n^d is a lattice polytope, Proposition 4.1 and Theorem 2.0 imply

$$\#\binom{\mathbf{N}^d}{n}_{\mathrm{cut}} = \#\mathrm{vert}\big(Q_n^d\big) = O\Big(\big(n^{2d}\big)^{(d-1)/(d+1)}\Big).$$

This completes the proof of Theorem 4.0. ∎

The bound just proved, which relies on Theorem 2.0, is much better than the bound of $O(n^{d^2})$ which one can derive from results on separable partitions (se [AO]).

Remark 4.2. The number of vertices of any subpolytope of Q_n^d satisfies the same bound.

Next, we show that, in contrast with Theorem 4.0, the number of Borel fixed staircases grows exponentially with n, even in the plane $d = 2$. We use a bijection between finite plane staircases and *RD-sequences*—finite sequences over the alphabet $\{R, D\}$ starting with R and terminating with D. Under this bijection, the RD-sequence

$$R^{r_1}D^{d_1}R^{r_2}D^{d_2} \cdots R^{r_m}D^{d_m}, \qquad m, r_1, d_1, \ldots, r_m, d_m \ge 1$$

corresponds to the staircase λ given by

$$\min\big(\mathbf{N}^d \setminus \lambda\big) = \left\{\left(0, \sum_{i=1}^{m} d_i\right), \left(r_1, \sum_{i=1}^{m-1} d_i\right), \right.$$
$$\left. \left(r_1 + r_2, \sum_{i=1}^{m-2} d_i\right), \ldots, \left(\sum_{i=1}^{m} r_i, 0\right)\right\}.$$

The sequence describes the directions "Right" and "Down" while walking on the boundary of $\mathbf{N}^d \setminus \lambda$. The following characterization of planar Borel fixed staircases is straightforward.

LEMMA 4.3. *The staircase corresponding to an RD-sequence as above is Borel fixed if and only if* $r_1 = r_2 = \cdots = r_m = 1$.

PROPOSITION 4.4. *The number of Borel fixed staircases in* $\binom{\mathbb{N}^2}{n}_{\text{stair}}$ *is* $2^{\Omega(\sqrt{n})}$.

Proof. Given $n \geq 15$, let k be the largest integer such that $n \geq 12k^2 + 3k$, and let $m := 4k$. For each k-subset $I \subset [2k] = \{1, \ldots, 2k\}$ we define an RD-sequence $RD^{d_1} \ldots RD^{d_m}$ by setting $d_i := d_{m+1-i} := 1$ if $i \in I$ and $d_i := d_{m+1-i} := 2$ if $i \notin I$. The number of elements of the corresponding Borel fixed staircase λ is $\#\lambda = \sum_{i=1}^{m} i \cdot d_i = 3k \cdot (m+1) = 12k^2 + 3k$. So the number of n-element planar Borel fixed staircases, which is no smaller than the number of planar Borel fixed staircases with $12k^2 + 3k$ elements, is at least the number $\binom{2k}{k} \geq 2^k$ of k-subsets $I \subset [2k]$. Since $k > \sqrt{\frac{n}{13}}$ for all large n, this number is $2^{\Omega(\sqrt{n})}$. ∎

Remark 4.5. While RD-sequences of planar corner cuts have been studied in various contexts under different names (e.g., in computer vision under the term "chain codes of digitized lines"), no simple characterization of such sequences (say, as the one in Lemma 4.3 for Borel fixed staircases) seems to be known. See [Bru] for a recursive characterization.

The set $\binom{\mathbb{N}^2}{n}_{\text{stair}}$ of all planar staircases (or partitions) has the generating function

$$\sum_{n=0}^{\infty} \#\binom{\mathbb{N}^2}{n}_{\text{stair}} \cdot z^n = \prod_{k=1}^{\infty} \frac{1}{(1-z^k)}$$
$$= 1 + z + 2z^2 + 3z^3 + 5z^4 + 7z^5 + 11z^6 + 13z^7 + \ldots .$$

Staircases in 3-space are called *plane partitions* in combinatorics. The generating function for counting $\binom{\mathbb{N}^3}{n}_{\text{stair}}$ is derived in [Sta, Theorem 18.2]. It is MacMahon's classical formula:

$$\sum_{n=0}^{\infty} \#\binom{\mathbb{N}^3}{n}_{\text{stair}} \cdot z^n = \prod_{k=1}^{\infty} \frac{1}{(1-z^k)^k}$$
$$= 1 + z + 3z^2 + 6z^3 + 13z^4 + 24z^5 + 48z^6 + \ldots .$$

To the best of our knowledge no such formulas are known for $d \geq 4$.

Is it possible to find an explicit formula for the generating function $\sum_{n=0}^{\infty} \#\binom{\mathbb{N}^d}{n}_{\text{cut}} \cdot z^n$ which enumerates the subset of corner cuts among all staircases? Of special interest is the number of planar corner cuts (cf. Remark 4.5). The following table is for small values of n:

n	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
$\#\binom{\mathbb{N}^2}{n}_{\text{cut}}$	6	7	8	10	12	13	16	16	18	20	23	24	26	26	30	32	35	34

In an earlier version of this paper we raised the problem to determine this sequence. This problem was solved by Corteel *et al.* [CRST].

We finish this section with an algorithm for enumerating all corner cuts and all vertices of P_n^d. It builds on results in [HOR2] and runs in strongly polynomial time for fixed d.

PROPOSITION 4.6. *There is an algorithm that, given any d and n, produces the set $\binom{\mathbb{N}^d}{n}_{\text{cut}}$ of corner cuts and the set of vertices P_n^d using $n^{O(d^2)}$ arithmetic operations.*

Proof. Put $N := \{0, 1, \dots, n-1\}$. Call a subset $\lambda \subseteq N^d \subset \mathbb{N}^d$ *separable* if λ is strictly separable by a hyperplane from $N^d \setminus \lambda$. Clearly, any n-element corner cut in $\mathbb{N}^d$ is a separable subset of N^d. The collection $\mathscr{S}$ of all separable subsets of N^d is determined by the collection of $\#\binom{N^d}{d+1} \le n^{d(d+1)}$ orientations of all $(d+1)$-simplices spanned by points of N^d, and can be produced using $n^{O(d^2)}$ arithmetic operations. The exact details involve symbolic perturbation of the points in N^d to general position and suitable determinant computations and can be found in [HOR2]. Let $\mathscr{T}$ be the subcollection of $\mathscr{S}$ of all n-element λ which satisfy $\sum_{i=1}^d (\Sigma\lambda)_i \le \binom{n}{2}$, and let $V := \{\Sigma\lambda \colon \lambda \in \mathscr{T}\}$. From Theorem 2.0 and Lemma 2.1, it follows that $Q_n^d = \operatorname{conv}(V)$ and $\lambda \in \mathscr{T}$ is a corner cut if and only if $\Sigma\lambda$ is a vertex of $\operatorname{conv}(V)$. So $\lambda \in \mathscr{T}$ is a corner cut if and only if $\Sigma\lambda \notin \operatorname{conv}(U)$ for every $(d+1)$-subset $U \subseteq V \setminus \{\Sigma\lambda\}$. Now V is contained in $\{v \in \mathbb{N}^d : \sum_{i=1}^d v_i \le \binom{n}{2}\}$, hence $\#V \le \binom{\binom{n}{2}+d}{d} \le n^{2d}$, and there are $\binom{\#V}{d+1} = n^{O(d^2)}$ such subsets U of V. Therefore, the set of corner cuts $\binom{\mathbb{N}^d}{n}_{\text{cut}} \subseteq \mathscr{T}$ and the corresponding set $\{\Sigma\lambda \colon \lambda \in \binom{\mathbb{N}^d}{n}_{\text{cut}}\} \subseteq V$ of vertices of P_n^d can be computed in $n^{O(d^2)}$ arithmetic operations as claimed. ∎

The procedure described above gives, for every fixed d, a polynomial time algorithm that, given n and $v \in \mathbb{N}^d$, decides if v is a vertex of P_n^d, and if it is, finds the (unique) corner cut $\lambda \in \binom{\mathbb{N}^d}{n}_{\text{cut}}$ with $\Sigma\lambda = v$. It would be interesting to know if this task can be done in polynomial time even in *varying* dimension d, perhaps using the methods of [HOR1].

In this section we have seen that, for fixed d and varying n, the map

$$\binom{\mathbf{N}^d}{n}_{\text{stair}} \to Q_n^d \cap \mathbf{Z}^d \colon \lambda \mapsto \Sigma\lambda \tag{4.1}$$

compresses a set of exponential size to a set of polynomial size. On the boundary it restricts to the bijection between $\binom{\mathbf{N}^d}{n}_{\text{cut}}$ and the vertices of Q_n^d. The typical fiber over an interior lattice point of Q_n^d is expected to have exponential size. It would be interesting to study the fibers of this map in detail. Is there an interesting *fiber polytope*, in the sense of [BiS]?

5. THE GRÖBNER BASES OF A POINT CONFIGURATION

Let k be an infinite field and let $\mathscr{P} = \{p_1, \dots, p_n\}$ be a configuration of n distinct points in affine d-space k^d. Each point $p_i = (p_{i1}, \dots, p_{id})$ corresponds to a maximal ideal $M(p_i) = \langle x_1 - p_{i1}, \dots, x_d - p_{id}\rangle$ in the polynomial ring $k[x] = k[x_1, \dots, x_d]$. The configuration $\mathscr{P}$ is an algebraic variety whose vanishing ideal is the intersection of these n maximal ideals

$$I_{\mathscr{P}} = M(p_1) \cap M(p_2) \cap \cdots \cap M(p_n) \subset k[x].$$

Thus $I_{\mathscr{P}}$ is the radical ideal consisting of those polynomials $f \in k[x]$ which vanish on $\mathscr{P}$.

For any nonnegative vector w in $\mathbf{R}^d_{\geq 0}$, the *initial ideal* $in_w(I_{\mathscr{P}})$ is the ideal of w-leading forms $in_w(f)$ where f runs over $I_{\mathscr{P}}$. We call two nonnegative vectors w and w' *equivalent* if $in_w(I_{\mathscr{P}}) = in_{w'}(I_{\mathscr{P}})$. The equivalence classes are the relatively open cones in a subdivsion of $\mathbf{R}^d_{\geq 0}$ which is called the *Gröbner fan* of $I_{\mathscr{P}}$. A vector w lies in an open cell of the Gröbner fan if and only if $in_w(I_{\mathscr{P}})$ is a monomial ideal; see [BM, MR, Stu].

In this section we construct a convex polyhedron $state(\mathscr{P})$ in $\mathbf{R}^n$ whose normal fan equals the Gröbner fan of $I_{\mathscr{P}}$. Following [BM] we call $state(\mathscr{P})$ the *state polyhedron* of $\mathscr{P}$. We thus obtain a one-to-one-to-one-to-one correspondence between the following objects:

(a) the distinct reduced Gröbner bases of the ideal $I_{\mathscr{P}}$;
(b) the distinct initial monomial ideals of the ideal $I_{\mathscr{P}}$;
(c) the open cones in the Gröbner fan of $I_{\mathscr{P}}$;
(d) the vertices of the state polyhedron $state(\mathscr{P})$.

For $\lambda = \{\lambda_1, \lambda_2, \dots, \lambda_n\} \in \binom{\mathbf{N}^d}{n}$ and a point configuration $\mathscr{P}$ as above we define

$$[\lambda](\mathscr{P}) := \det\begin{pmatrix} p_1^{\lambda_1} & p_1^{\lambda_2} & \cdots & p_1^{\lambda_n} \\ p_2^{\lambda_1} & p_2^{\lambda_2} & \cdots & p_2^{\lambda_n} \\ \vdots & \vdots & \ddots & \vdots \\ p_n^{\lambda_1} & p_n^{\lambda_2} & \cdots & p_n^{\lambda_n} \end{pmatrix}, \quad \text{where } p_i^{\lambda_j} = p_{i1}^{\lambda_{j1}} p_{i2}^{\lambda_{j2}} \cdots p_{id}^{\lambda_{jd}}.$$

The expression $[\lambda](\mathscr{P})$ is defined only up to sign, since λ and $\mathscr{P}$ are regarded as unordered sets. This is notationally more convenient. Note that all $n!$ terms in the expansion in the determinant $[\lambda](\mathscr{P})$ are distinct monomials in the p_{ij}. This implies

LEMMA 5.1. *The determinant $[\lambda](\mathscr{P})$ is a nonzero polynomial in the dn variables p_{ij}.*

We call a point configuration $\mathscr{P}$ *generic* if $[\lambda](\mathscr{P}) \neq 0$ for all corner cuts $\lambda \in \binom{\mathbf{N}^d}{n}_{\text{cut}}$. By Lemma 5.1, the set of generic configurations is nonempty and Zariski dense in the space k^{dn} of all point configurations. Thus the statement of Theorem 5.0 makes sense and is consistent with standard usage of "generic point configuration" in algebraic geometry.

We define the *state polyhedron* of a point configuration $\mathscr{P}$ as

$$state(\mathscr{P}) := \mathbf{R}^d_{\geq 0} + \operatorname{conv}\left\{ \sum \lambda : \lambda \in \binom{\mathbf{N}^d}{n}_{\text{stair}} \text{ and } [\lambda](\mathscr{P}) \neq 0 \right\}.$$

In view of Theorem 2.0 this is a subpolyhedron of the corner cut polyhedron P_n^d. The equality $state(\mathscr{P}) = P_n^d$ holds if and only if $\mathscr{P}$ is generic. The result stated in the Introduction (Theorem 5.0) is an immediate corollary to the following more general theorem.

THEOREM 5.2. *The normal fan of $state(\mathscr{P})$ equals the Gröbner fan of $I_{\mathscr{P}}$.*

Proof. Let $\lambda \in \binom{\mathbf{N}^d}{n}_{\text{stair}}$. For each $u \in \mathbf{N}^d \setminus \lambda$ we form the $(n+1) \times (n+1)$-determinant

$$f_u := [\lambda \cup \{u\}](\mathscr{P} \cup \{(x_1, \dots, x_d)\}).$$

This is a polynomial in $k[x]$ which is well defined up to sign. By Laplace expansion,

$$f_u = [\lambda](\mathscr{P}) \cdot x^u + \sum_{i=1}^{n} (-1)^i \cdot [\lambda \setminus \{\lambda_i\} \cup \{u\}](\mathscr{P}) \cdot x^{\lambda_i}.$$

We claim that the following seven statements are equivalent for a vector $w \in \mathbf{R}^d_{\geq 0}$:

(1) $[\lambda](\mathscr{P}) \neq 0$ and the linear functional $v \mapsto w \cdot v$ is minimized over $state(\mathscr{P})$ at $\Sigma\lambda$,

(2) $[\lambda](\mathscr{P}) \neq 0$ and $\forall \mu \in \binom{\mathbf{N}^d}{n}_{\text{stair}}$: $\mu \neq \lambda$ and $[\mu](\mathscr{P}) \neq 0 \Rightarrow w \cdot \Sigma\mu > w \cdot \Sigma\lambda$,

(3) $[\lambda](\mathscr{P}) \neq 0$ and $\forall u \in \mathbf{N}^d \setminus \lambda\ \forall i \in \{1, \ldots, n\}$: $[\lambda \setminus \{\lambda_i\} \cup \{u\}](\mathscr{P}) \neq 0 \Rightarrow w \cdot \lambda_i < w \cdot u$,

(4) $[\lambda](\mathscr{P}) \neq 0$ and $\forall u \in \mathbf{N}^d \setminus \lambda$: $in_w(f_u) = x^u$,

(5) $\forall u \in \mathbf{N}^d \setminus \lambda$: $f_u \neq 0$ and $in_w(f_u) = x^u$,

(6) $M_\lambda \subseteq in_w(I_{\mathscr{P}})$,

(7) $M_\lambda = in_w(I_{\mathscr{P}})$.

The implications (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) are straightforward. The implication (3) $\Rightarrow$ (2) holds by the Basis Exchange Lemma of linear algebra. To see the implication (5) $\Rightarrow$ (6), it suffices to note that f_u vanishes at each point in $\mathscr{P}$ and hence $f_i \in I_{\mathscr{P}}$. The statements (6) and (7) are equivalent because both ideals M_λ and $in_w(I_{\mathscr{P}})$ are Artinian of colength n in $k[x]$. Hence if one of them contains the order, then they are equal.

To complete the proof of our claim, we next show (7) $\Rightarrow$ (4). Suppose that (7) holds. Then the set $\{x^{\lambda_1}, x^{\lambda_2}, \ldots, x^{\lambda_n}\}$ is k-linearly independent modulo $I_{\mathscr{P}}$. This implies that the $n \times n$-matrix $(p_i^{\lambda_j})$ has rank n, and hence its determinant $[\lambda](\mathscr{P})$ is nonzero. Therefore x^u is the unique monomial appearing in the expansion of f_u which lies in $M_\lambda = in_w(I_{\mathscr{P}})$. Since $f_u \in I_{\mathscr{P}}$, we conclude $in_w(f_u) = x^u$, and (4) is proved.

The equivalence of (1) and (7) shows that two nonnegative vectors w and w' give the same initial monomial ideal $in_w(I_{\mathscr{P}}) = in_{w'}(I_{\mathscr{P}})$ if and only if they support the same vertex of $state(\mathscr{P})$. Hence w and w' lie in the same open cone of the Gröbner fan of $I_{\mathscr{P}}$ if and only if they lie in the same open cone of the normal fan of $state(\mathscr{P})$. ∎

COROLLARY 5.3. *If $in_w(I_{\mathscr{P}}) = M_\lambda$, then $\{f_u: u \in \min(\mathbf{N}^d \setminus \lambda)\}$ is the reduced Gröbner basis of $I_{\mathscr{P}}$ with respect to w.*

Proof. The initial terms of the elements $f_u \in I_{\mathscr{P}}$ minimally generate the initial monomial ideal $in_w(I_{\mathscr{P}}) = M_\lambda$, and this ideal contains none of the trailing terms of any f_u. ∎

For fixed number of variables d, the number of monomial ideals of colength n grows exponentially in n. Even the subset of Borel fixed ideals grows exponentially in n, even for $d = 2$ as Proposition 4.4 shows. Thus the following result may be somewhat surprising.

COROLLARY 5.4. *Fix d and let $\mathscr{P}$ be any configuration of n points in the affine d-space k^d.*

(a) *The number of distinct reduced Gröbner bases of $I_{\mathscr{P}}$ is* $O(n^{2d(d-1)/(d+1)})$.

(b) *The ideal $I_{\mathscr{P}}$ possesses a universal Gröbner basis of cardinality* $O(n^{2d-3+(3d-1)/d(d+1)})$.

Recall that a *universal Gröbner basis* of the ideal $I_{\mathscr{P}}$ is a finite subset $\mathscr{U}$ which is a Gröbner basis of $I_{\mathscr{P}}$ simultaneously for all weight vectors $w \in \mathbf{R}^d_{\geq 0}$; cf. [Stu, Sect. 1].

Proof of Corollary 5.4. Every vertex of *state*$(\mathscr{P})$ is a lattice point in Q^d_n. Hence (a) follows from Theorem 5.2 and Remark 4.2. Next note that the union of all reduced Gröbner bases of $I_{\mathscr{P}}$ is a universal Gröbner basis. By Corollary 5.3, the cardinality of the reduced Gröbner basis corresponding to the staircase λ is $\#\min(\mathbf{N}^d \setminus \lambda)$. Multiplying the bound $\#\min(\mathbf{N}^d \setminus \lambda) = O(n^{(d-1)/d})$ from [Ber, Theorem 3] by the bound in (a) we get (b). ▮

Remark 5.5. Two monomial ideals M_λ and M_μ which satisfy $\Sigma\lambda = \Sigma\mu$ cannot both be initial ideals of some fixed nonmonomial ideal I in $k[x]$, even if I is not radical. This is the content of [Stu, Sect. 2, Exercise (2)]. The example in the Introduction shows that there is no ideal I of colength 6 in $k[x, y]$ with $in_w(I) = \langle x^3, y^2 \rangle$ and $in_{w'}(I) = \langle x^4, xy, y^3 \rangle$. ▮

In Section 4 we studied the cardinality of $\binom{\mathbf{N}^d}{n}_{\mathrm{cut}}$ as a function of n and d. In Corollary 5.4(a) we gave an upper bound for the function, $F(n, d) :=$ the maximum number of vertices of *state*$(\mathscr{P})$, where $\mathscr{P}$ runs over all configurations of n points in k^d, and k runs over all fields. Clearly, $\#\binom{\mathbf{N}^d}{n}_{\mathrm{cut}} \leq F(n, d) = O(n^{2d(d-1)/(d+1)})$, but the inequality is generally strict. Configurations in special position may have more distinct reduced Gröbner bases than the generic configuration with the same number of points. Here is the first instance:

PROPOSITION 5.6. $\#\binom{\mathbf{N}^2}{7}_{\mathrm{cut}} = 8 < F(7, 2) = 10.$

Proof. For $n = 7, d = 2$, the map (4.1) is injective. The 15 partitions of the number 7 are mapped to the following 15 distinct points, the first eight of which are the vertices of Q^2_7:

vertices: $(21, 0)$, $(15, 1)$, $(11, 2)$, $\underline{(7, 4)}$, $\underline{(4, 7)}$, $(2, 11)$, $(1, 15)$, $(0, 21)$

not vertices: $\underline{(10, 3)}$, $(9, 3)$, $(6, 5)$, $\underline{(6, 6)}$, $(5, 6)$, $(3, 9)$, $\underline{(3, 10)}$.

No subset of 11 points among these 15 is in convex position. This shows $F(7,2) \leq 10$.

Consider the 10 points which are not underlined. They are in convex position, and each of them is smaller than the other nine with respect to some positive linear functional. We shall present a configuration $\mathscr{P}$ of 7 points in $\mathbf{R}^2$ such that $state(\mathscr{P})$ has precisely these 10 vertices. This will imply $F(7,2) \geq 10$ and thus complete the proof. Set

$$\mathscr{P} = \{(0,0),(1,1),(2,2),(3,4),(5,7),(11,13),(\alpha,\beta)\},$$

where $(\alpha,\beta) \sim (1.82997, 1.82448)$ is the unique real solution of the two equations

$$\begin{aligned}&1468\alpha - 2\beta^2 + 141\beta - 2937\\&\qquad = 4\beta^3 + 2112\beta^2 + 1578145\beta - 2886359 = 0.\end{aligned}$$

This configuration satisfies $[\lambda](\mathscr{P}) = 0$ when λ is any of the partitions $1+1+2+3$, $1+2+4$, or $1+1+1+4$. The points $\sum\lambda$ representing these three partitions are $(7,4)$, $(4,7)$, and $(6,6)$. The other 12 partitions μ satisfying $[\mu](\mathscr{P}) \neq 0$. Among the 12 points $\sum\mu$ representing these 12 partitions, only the two underlined points $(10,3)$ and $(3,10)$ are not extreme. Therefore the vertices of $state(\mathscr{P})$ are exactly the 10 nonunderlined points. ∎

We point out that the computational results in Sections 3 and 4 can now be translated into algorithms for the Gröbner bases theory. In particular, Theorem 3.0 gives a polynomial time algorithm for deciding whether a given monomial ideal M_λ is the initial ideal $in_w(I_{\mathscr{P}})$ of the generic point configuration $\mathscr{P}$ in affine d-space with respect to some term order w. In the affirmative case it produces a suitable term order w. The point here is that d varies.

If we fix the number of variables d, then Proposition 4.6 together with Corollary 5.4 gives a polynomial time algorithm for computing a universal Gröbner basis of $I_{\mathscr{P}}$.

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REFERENCES

[Agn] G. Agnarsson, The number of outside corners of monomial ideals, *in* "Algorithms for Algebra (Eindhoven, 1996)" *J. Pure Appl. Algebra* **117/118** (1997), 3–21.

[AO] N. Alon and S. Onn, Separable partitions, *Discrete Appl. Math.* **91** (1999), 39–51.

[And] G. Andrews, A lower bound for the volume of strictly convex bodies with many boundary points, *Trans. Amer. Math. Soc.* **106** (1965), 270–273.

[BV] I. Bárány and A. Vershik, On the number of convex lattice polytopes, *Geom. Funct. Anal.* **2** (1992), 381–393.

[BaS] D. Bayer and M. Stillman, A criterion for detecting m-regularity, *Invent. Math.* **87** (1987), 1–11.

[BM] D. Bayer and I. Morrison, Standard bases and geometric invariant theory, *J. Symbolic Comput.* **6** (1988), 209–217.

[Ber] D. Berman, The number of generators of a colength N ideal in a power series ring, *J. Algebra* **73** (1981), 156–166.

[BiS] L. Billera and B. Sturmfels, Fiber polytopes, *Ann. Math.* **135** (1992), 527–549.

[BF1] M. Boshernitzan and A. S. Fraenkel, Nonhomogeneous spectra of numbers, *Discrete Math.* **34** (1981), 325–327.

[BF2] M. Boshernitzan and A. S. Fraenkel, A linear algorithm for nonhomogeneous spectra of numbers, *J. Algorithms* **5** (1984), 187–198.

[Bru] A. M. Bruckstein, Self-similarity properties of digitized straight lines, *in* "Vision Geometry (Hoboken, New Jersey, 1989)" *Contemp. Math.* **119** (1991), 1–20.

[CRST] S. Corteel, G. Rémond, G. Schaeffer, and H. Thomas, The number of plane corner cuts, *Adv. Appl. Math.* **23** (1999), 49–53.

[Eis] D. Eisenbud, "Commutative Algebra with a View toward Algebraic Geometry," Springer-Verlag, New York, 1995.

[Ger] Y. Gérard, Analyse locale des droites discrètes. Généralisation et application à la connexité des plans discrets, *C. R. Acad. Sci. Paris, Sér. I Math.* **324** (1997), 1419–1424.

[HOR1] F. Hwang, S. Onn, and U. Rothblum, Representations and characterizations of vertices of bounded-shape partition polytopes, *Linear Algebra Appl.* **278** (1998), 263–284.

[HOR2] F. Hwang, S. Onn, and U. Rothblum, A polynomial time algorithm for shaped partition problems, *SIAM Journal on Optimization*, to appear.

[MR] T. Mora and L. Robbiano, The Gröbner fan of an ideal, *J. Symbolic Comput.* **6** (1988), 183–208.

[Sch] A. Schrijver, Theory of Linear and Integer Programming, Series in Discrete Mathematics, Wiley-Interscience, Chichester, 1986.

[Sta] R. Stanley, Theory and applications of plane partitions, I, II, *Stud. Appl. Math.* **50** (1971), 167–188 and 259–279.

[Stu] B. Sturmfels, "Gröbner Bases and Convex Polytopes," University Lecture Notes **8**, American Mathematical Society, Providence, RI, 1995.