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Gröbner bases and the Nagata automorphism

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Abstract

We study different properties of the Nagata automorphism of the polynomial algebra in three variables and extend them to other automorphisms of polynomial algebras and algebras close to them. In particular, we propose two approaches to the Nagata conjecture: via the theory of Gröbner bases and trying to lift the Nagata automorphism to an automorphism of the free associative algebra. We show that the reduced Gröbner basis of three face polynomials of the Nagata automorphism obtained by substituting a variable by zero does not produce an automorphism, independently of the "tag" monomial ordering, contrary to the two variable case. We construct examples related to Nagata's automorphism which show different aspects of this problem. We formulate a conjecture which implies Nagata's conjecture. We also construct an explicit lifting of the Nagata automorphism to the free metabelian associative algebra. Finally, we show that the method to determine whether an endomorphism of $K[X]$ is an automorphism is based on a general fact for the ideals of arbitrary free algebras and works also for other algebraic systems such as groups and semigroups, etc.

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1. Introduction

Let K be a field and let $K[X] := K[x_1, \dots, x_n]$ be the polynomial algebra in n variables $x_1, \dots, x_n$ over K, where n is a fixed positive integer. In this paper we study the group of K-algebra automorphisms $\mathrm{Aut}\, K[x_1, \dots, x_n]$, $n \geq 2$. (See Nagata [17] for a survey on facts about $\mathrm{Aut}\, K[X]$.)

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Since every map $X:=\{x_1,\dots,x_n\}\to K[X]$ induces an endomorphism of $K[X]$, we identify the endomorphisms $\phi\in\operatorname{End}K[X]$ with the polynomial maps $F=F(X)=\phi(X)=(f_1,\dots,f_n)$, where $f_i=\phi(x_i)$, $i=1,\dots,n$, and use both ϕ and F to denote the endomorphisms of $K[X]$. Sometimes we prefer to denote the elements of X by $x,y,\dots,z$, etc.

The group $\operatorname{Aut}K[X]$ has two important subgroups. The first one is the group $A(n,K)$ consisting of all affine automorphisms. The second one $J(n,K)$ is generated by all triangular automorphisms defined as follows. Given invertible elements $a_1,\dots,a_n$ in K and polynomials $f_i\in K[x_{i+1},\dots,x_n]$, $i=1,\dots,n$ (f_n is a polynomial in zero variables, i.e. $f_n\in K$) then the map $(a_1x_1+f_1,a_2x_2+f_2,\dots,a_nx_n+f_n)$ is called a triangular automorphism of $K[X]$. The group generated by $A(n,K)$ and $J(n,K)$ is the group of tame automorphisms and is denoted by $T(n,K)$. The automorphisms of $K[X]$ which are not in $T(n,K)$ are called wild. Similarly, one can define tame and wild automorphisms of $K[X]$ when K is any commutative ring with identity. Now, we state the main question.

Question 1.1. Is it true that $\operatorname{Aut}K[x_1,\dots,x_n]=T(n,K)$, i.e., is every automorphism of $K[x_1,\dots,x_n]$ tame?

The case $n=2$ of Question 1.1 has an affirmative answer. There are several known proofs of this result originally obtained by Jung [12] for $K=\mathbb{C}$ and by Van der Kulk [23] for any field K.

Consider the Nagata automorphism $v=N=(n_1,n_2,n_3)$ of $K[x,y,z]$ (see [17]):

$$
\begin{aligned}
n_1&=v(x)=x-2y(y^2+xz)-z(y^2+xz)^2,\\
n_2&=v(y)=y+z(y^2+xz),\\
n_3&=v(z)=z.
\end{aligned}
$$

Conjecture 1.2. *For any field K the Nagata automorphism $N=(n_1,n_2,n_3)$ is not tame.*

Conjecture 1.2 was formulated by Nagata in 1972 [17] and is still open.

Moreover, it is still unknown whether there exist wild (i.e., nontame) automorphisms, when K is an arbitrary field and $n>2$. The starting point of our research was to find some new evidences that the Nagata automorphism has some properties which distinguish it from most of the known tame automorphisms. In particular, we propose two approaches to the Nagata conjecture: via the theory of Gröbner bases and trying to lift the Nagata automorphism to an automorphism of the free associative algebra.

In Section 2 we consider Gröbner basis techniques. There is a well-known Gröbner bases algorithm deciding whether an endomorphism of $K[X]$ is an automorphism and, if the answer is affirmative, finding the inverse. An automorphisms of $K[X]$ can be reconstructed from all n^2 face polynomials of its inverse. In the two variables case we prove that only two face polynomials are enough and this new result is closely related to the fact that every automorphism of $K[x,y]$ is tame. Examples show that an anology does not hold for $n \geq 3$. We show that the reduced Gröbner basis of three face polynomials of the Nagata automorphism obtained by substituting $z=0$ does not produce an automorphism, under any "tag" monomial ordering, contrary to all examples of tame automorphisms of $K[x, y, z]$ we have tested, which is similar to the two variable case. Based on these we formulate a conjecture which implies the Nagata conjecture.

In Section 3 we try to lift the Nagata automorphism to an automorphism of the free associative algebra $K\langle x, y, z\rangle$. Obviously, every tame automorphism of $K[X]$ can be lifted to an automorphism of $K\langle X\rangle$. Therefore, although an automorphism of $K[X]$ may be not tame, if it can be lifted to an automorphism of $K\langle X\rangle$, it may be regarded "not wilder" than the corresponding automorphism of the free algebra. If an automorphism of $K[X]$ cannot be lifted to an automorphism of $K\langle X\rangle$, we may ask the question "how far" can it be lifted. The natural way to measure the "distance" is in the language of T-ideals, i.e. ideals of $K\langle X\rangle$ which are invariant under all endomorphisms of $K\langle X\rangle$. Martha Smith [21] has proved that the Nagata automorphism is stably tame, i.e. it becomes tame if we extend it to the polynomial algebra in four variables $K[x, y, z, v]$ assuming that it fixes v. Unfortunately, the procedure of Martha Smith does not give lifting to $\operatorname{Aut} K\langle x, y, z, v\rangle$ which fixes v. We have not succeeded to lift the Nagata automorphism to an automorphism of $K\langle x, y, z\rangle$, but we give an explicit lifting to the algebra $K\langle x, y, z\rangle/C^k$, where C is the commutator ideal of $K\langle x, y, z\rangle$ and k is any positive integer. By a result of Umirbaev [22], such liftings exist for any automorphism of $K[X]$ but our lifting has the advantage that it fixes z as the Nagata automorphism itself.

Finally, in Section 4 we show that the method to determine whether an endomorphism of $K[X]$ is an automorphism which was the starting point of the study in Section 2, is based on a general fact for the ideals of arbitrary free algebras and, with some modifications, works also in the case of other algebraic systems as groups, semigroups etc. The advantage in the polynomial case is that the ideals of $K[X]$ are finitely generated and there exist effective procedures (as the Gröbner bases techniques) to find a minimal system of generators of a given ideal.

2. Gröbner bases and face functions

A Gröbner basis is a (finite) set of special generators of an ideal of a polynomial algebra. Fixing some ordering of the monomials in $K[x_1, \ldots, x_n]$, the leading term of any element from the ideal is divisible by a leading term of some element from the Gröbner basis. The basis is reduced, if no term of its elements is divisible by the leading term of another element. The Gröbner bases have many attractive computational properties. The method is widely used in computational commutative algebra and in computational algebraic geometry. The reader is referred to [3]. The theory of Gröbner bases was used for deciding when a polynomial map $F=(f_1, \ldots, f_n)$ is an automorphism. The algorithm uses "tag" variables in a fashion related to the method for finding the relations among the f_i's. We introduce tag variables, indeterminates $y_1, \ldots, y_n$, one for each f_i.

Let $K[x_1,\dots,x_n,y_1,\dots,y_n]$ have a monomial ordering, where

$$K[x_1,\dots,x_n]\succ K[y_1,\dots,y_n]$$

which means that each monomial in $K[x_1,\dots,x_n]$ is larger than any monomial which lies in $K[y_1,\dots,y_n]$. We call it a "tag" monomial ordering.

Proposition 2.1. *Let $F=(f_1,\dots,f_n)$ be an endomorphism of $K[x_1,\dots,x_n]$. Then F is an automorphism if and only if the ideal I of $K[x_1,\dots,x_n,y_1,\dots,y_n]$ generated by*

$$\{y_1-f_1(x_1,\dots,x_n),\dots,y_n-f_n(x_1,\dots,x_n)\}$$

has a generating set

$$\{x_1-g_1(y_1,\dots,y_n),\dots,x_n-g_n(y_1,\dots,y_n)\}.$$

Moreover, $G:=G(X):=(g_1(X),\dots,g_n(X))$ is the inverse of $F=(f_1,\dots,f_n)$ and the reduced Gröbner basis of the ideal I with respect to any "tag" monomial ordering $\succ$ such that $K[x_1,\dots,x_n]\succ K[y_1,\dots,y_n]$ is precisely $\{x_1-g_1(y_1,\dots,y_n),\dots,x_n-g_n(y_1,\dots,y_n)\}$.

There are several proofs of this result, by van den Essen [9] and Shannon and Sweedler [20] for fields and Abhyankar and Li [1] for arbitrary commutative rings K.

It has been proved by several authors (see [10, 14, 16, 24], that a polynomial map $F=(f_1,\dots,f_n)\in \operatorname{Aut}K[x_1,\dots,x_n]$ is completely determined by its n^2 face polynomials

$$f_1(0,x_2,\dots,x_n),\dots,f_1(x_1,\dots,x_{n-1},0),$$
$$\dots$$
$$f_n(0,x_2,\dots,x_n),\dots,f_n(x_1,\dots,x_{n-1},0).$$

Now we present a new result concerning the reduced Gröbner bases of its face functions. It is more general than that by van den Essen [9].

Proposition 2.2. *Let K be any field and $F=(f_1,\dots,f_n)\in \operatorname{Aut}K[x_1,\dots,x_n]$. Let*

$$G=\{x_1-g_1(y_1,\dots,y_n),\dots,x_n-g_n(y_1,\dots,y_n)\}$$

be the reduced Gröbner basis of the ideal $I=(y_1-f_1(x_1,\dots,x_n),\dots,y_n-f_n(x_1,\dots,x_n))$ in $K[x_1,\dots,x_n,y_1,\dots,y_n]$ with respect to any "tag" monomial ordering $\succ$ such that $K[x_1,\dots,x_n]\succ K[y_1,\dots,y_n]$. Then, for any integer $j=1,\dots,n$, the set

$$H=\{x_1-g_1(y_1,\dots,y_n),\dots,x_{j-1}-g_{j-1}(y_1,\dots,y_n),g_j(y_1,\dots,y_n),$$
$$x_{j+1}-g_{j+1}(y_1,\dots,y_n)\dots,x_n-g_n(y_1,\dots,y_n)\}$$

is a Gröbner basis of the ideal J generated by

$$\{y_1-f_1(x_1,\dots,x_{j-1},0,x_{j+1},\dots,x_n),\dots,y_n-f_n(x_1,\dots,x_{j-1},0,x_{j+1},\dots,x_n)\}.$$

In general, H is not the reduced Gröbner basis. The reduced Gröbner basis is obtained after reduction of all polynomials $g_i(y_1,\dots,y_n)$, $i=1,\dots,\hat{j},\dots,n$, by $g_j(y_1,\dots,y_n)$. (Here $\hat{j}$ means that the corresponding number j does not participate in the expression.)

Proof. The sets

$$\{y_i - f_i(x_1,\dots,x_n) \mid i=1,\dots,n\},\ \{x_i - g_i(y_1,\dots,y_n) \mid i=1,\dots,n\}$$

generate the same ideal I of $K[x_1,\dots,x_n,y_1,\dots,y_n]$. Substituting $x_j=0$, we obtain that the set H generates in $K[x_1,\dots,\hat{x}_j,\dots,x_n,y_1,\dots,y_n]$ the same ideal as

$$\{y_1 - f_1(x_1,\dots,x_{j-1},0,x_{j+1},\dots,x_n),\dots,y_n - f_n(x_1,\dots,x_{j-1},0,x_{j+1},\dots,x_n)\}.$$

Hence, these two sets generate also the same ideal in $K[x_1,\dots,x_n,y_1,\dots,y_n]$. The set H is a Gröbner basis, because every element from the ideal J is a linear combination of polynomials with leading term containing $x_1,\dots,\hat{x}_j,\dots,x_n$ and of polynomials divisible by $g_j(y_1,\dots,y_n)$. After reduction of all polynomials $g_i(y_1,\dots,y_n)$, $i=1,\dots,\hat{j},\dots,n$, by $g_j(y_1,\dots,y_n)$, we obtain a set $\bar{H}$ such that no monomial of an element from $\bar{H}$ is divisible by a leading term of another polynomial, i.e. $\bar{H}$ is the reduced Gröbner basis. □

In others words, our result states that the specialization of the Gröbner basis works for this particular kind of ideals J, but in general the specialization is not the reduced Gröbner basis. In the case $n=2$, we prove the following stronger result.

Theorem 2.3. *Let K be any field and let $F=(f_1(x,y),f_2(x,y))\in \operatorname{Aut} K[x,y]$. Then the reduced Gröbner bases of its face functions produce automorphisms, where the Gröbner bases are computed with respect to any "tag" monomial ordering $\succ$ such that $K[x,y]\succ K[s,t]$, i.e.*

$$\text{Gbasis}\,(s-f_1(0,y),t-f_2(0,y)) = \{g_1(s,t), y - g_2(s,t)\},$$
$$\text{Gbasis}\,(s-f_1(x,0),t-f_2(x,0)) = \{h_1(s,t), x - h_2(s,t)\},$$

where $(g_1(s,t),g_2(s,t))$ and $(h_1(s,t),h_2(s,t))$ are elements of the group $\operatorname{Aut} K[s,t]$. Conversely, let $p(t),q(t)\in K[t]$ such that one of them is nonconstant and the reduced Gröbner basis G of the ideal $I=(p(t)-x,q(t)-y)$ with respect to an ordering $\succ$ such that $K[t]\succ K[x,y]$ is

$$G=\{f_1(x,y),t-f_2(x,y)\}.$$

Then (f_1,f_2) is an automorphism, $(p(t),q(t))$ are face functions of an automorphism (g_1,g_2) whose inverse is exactly (f_1,f_2), and in this case $K[p(t),q(t)]=K[t]$.

Proof. Let $P=(p_1(x,y),p_2(x,y))$ be the inverse of F. Since

$$(s-f_1(x,y),t-f_2(x,y)) = (x - p_1(s,t), y - p_2(s,t)),$$

substituting $x=0$, we obtain that $\{(p_1(s,t), y-p_2(s,t))\}$ is a Gröbner basis of the ideal $(s-f_1(0,y), t-f_2(0,y))$ in $K[s,t,y]$ under any ordering which satisfies $K[y]\succ K[s,t]$. If $\{p_1(s,t), y-p_2(s,t)\}$ is already reduced, the uniqueness of the reduced Gröbner basis gives that

$$(g_1(s,t), g_2(s,t)) = (p_1(s,t), p_2(s,t)) \in \operatorname{Aut} K[s,t].$$

If there are further reductions, the only possibility is to reduce $p_2(s,t)$ modulo $p_1(s,t)$. We apply the algorithm for decomposing an automorphism into a product of elementary automorphisms (see [6, Theorem 6.8.5, p. 348]). It follows from this algorithm that the reduction

$$\{p_1(s,t), p_2(s,t)\} \to \{p_1(s,t), g_2(s,t)\}$$

gives an automorphism (p_1, g_2) of $K[s,t]$. Hence, the reduced Gröbner basis of the ideal $(p_1(s,t), y-p_2(s,t))$ is $\{p_1(s,t), y-g_2(s,t)\}$ and again

$$(g_1(s,t), g_2(s,t)) = (p_1(s,t), g_2(s,t)) \in \operatorname{Aut} K[s,t].$$

Conversely, since

$$I = (p(t)-x, q(t)-y) = (f_1(x,y), t-f_2(x,y)),$$

there exist polynomials $u_i, v_i \in K[x,y,t]$, $i=1,2$, such that

$$f_1(x,y) = (p(t)-x)u_1(x,y,t) + (q(t)-y)u_2(x,y,t),$$
$$t-f_2(x,y) = (p(t)-x)v_1(x,y,t) + (q(t)-y)v_2(x,y,t).$$

Substituting $x=p(t)$, $y=q(t)$, we obtain

$$f_1(p(t),q(t)) = 0, \quad f_2(p(t),q(t)) = t.$$

By [3, Theorem 2.3.4], $\{f_1(x,y)\}$ is the Gröbner basis of the ideal of $K[x,y]$

$$I \cap K[x,y] = \{f(x,y) \in K[x,y] \mid f(p(t),q(t)) = 0\}.$$

On the other hand, at least one of the polynomials $p(t)$ and $q(t)$ essentially depends on t and $I\cap K[x,y]$ is a principal ideal generated by an irreducible polynomial. Hence $f_1(x,y)$ is irreducible. By the Abhyankar-Moh Theorem [2], there exists a polynomial $h(x,y)\in K[x,y]$ such that $F_1=(f_1, f_2+hf_1)$ is an automorphism of $K[x,y]$. Let $F_1^{-1}=(p_1(x,y), q_1(x,y))$ be the inverse of F_1. We have

$$(p_1(f_1,f_2), q_1(f_1,f_2)) = (x,y)$$

and

$$(p_1(0,t), q_1(0,t)) = (p(t), q(t)).$$

By Proposition 2.2, $\{f_1(x,y), t-f_2(x,y)-h(x,y)f_1(x,y)\}$ is a (nonreduced) Gröbner basis of the ideal $(p(t)-x, q(t)-y)$ under any ordering that satisfies $K[t]\succ K[x,y]$, the reduced Gröbner basis under the same ordering is $\{f_1(x,y), t-f_2(x,y)\}$. Arguing as in the first part of the proof, i.e., applying again the algorithm for decomposing of F_1 into a product of elementary automorphisms we obtain that $F=(f_1,f_2)$ is also an automorphism with $F^{-1}(0,t)=(p(t),q(t))$. □

It is important to remark that, even if Question 1.1 has a positive answer for $n \geq 3$, the above proof of Theorem 2.3 holds only for two variables.

Example 2.4. We consider the Nagata automorphism $N=(n_1,n_2,n_3)$, where

$$n_1 = x - 2y(zx+y^2) - z(zx+y^2)^2,$$
$$n_2 = y + z(zx+y^2), \qquad n_3 = z.$$

Using MAPLE 5.2 we have computed the reduced Gröbner bases of all its face functions, i.e. the reduced Gröbner basis of the ideals

$$I_x = (s - n_1(0,y,z), t - n_2(0,y,z), u - n_3(0,y,z)),$$
$$I_y = (s - n_1(x,0,z), t - n_2(x,0,z), u - n_3(x,0,z)),$$
$$I_z = (s - n_1(x,y,0), t - n_2(x,y,0), u - n_3(x,y,0)),$$

with respect to the lexicographic ordering $x > y > z > s > t > u$. We obtain that

$$\begin{aligned} \text{Gbasis}(I_x) &= \{y - t + ut^2 + u^2 s, z - u, -s - 2t^3 - 2tus + ut^4 + 2u^2t^2s + u^3s^2\} \\ &= \{y - (t - u(t^2+us)), z - u, -(s + 2t(t^2+us) - u(t^2+us)^2)\}, \\ \text{Gbasis}(I_y) &= \{x - s - tus - t^3, z - u, -t + ut^2 + u^2 s\} \\ &= \{x - (s + t(t^2+us), z - u, -(t - u(t^2+us))\}, \\ \text{Gbasis}(I_z) &= \{x - s - 2t^3, y - t, u\} = \{x - (s + 2t^3), y - t, u\}. \end{aligned}$$

Hence, the Gröbner basis of I_x produces the inverse of the Nagata automorphism

$$N^{-1} = (x + 2y(zx+y^2) - z(zx+y^2)^2, y - z(zx+y^2), z).$$

The Gröbner basis of I_z produces the automorhism $(x+2y^3, y, z)$, but the Gröbner basis of I_y produces the endomorphism

$$(x + y(y^2+xz), y - z(y^2+xz), z)$$

which is NOT an automorphism because its Jacobian matrix is not invertible. Moreover, it is also very easy to check that the above Gröbner bases are independent of the "tag" ordering of the monomials $\succ$, such that $K[x,y,z] \succ K[s,t,u]$.

The above example shows that the case $n=3$ differs from the case $n=2$. One may suggest the following problem.

Question 2.5. Let K be a field and $F=(f_1,\ldots,f_n) \in T(n,K)$. Do the reduced Gröbner bases for any "tag" monomial ordering of its face functions give rise to automorphisms?

By Theorem 2.3 this question has a positive answer if $n=2$. If $n>3$ the answer is negative because Martha Smith [21] showed that the Nagata automorphism is stably tame. If we extend it to $K[x,y,z,v]$, assuming that it acts identically on the new variable v, it becomes tame, i.e. (n_1,n_2,n_3,v) is a tame automorphism. The computations in $K[x,y,z,v]$ give the same result as in $K[x,y,z]$ and in this way the Nagata automorphism gives a counterexample to our question for $n>3$. The following example shows that the answer is negative also for $n=3$. The automorphism we consider below was discovered by Freudenburg [11] and is an example of a tame automorphism with nonlinear base locus. There was a conjecture that all such automorphisms are wild. So, although the automorphism from [11] is tame, it has some "wild" properties.

Example 2.6. Let K be a field of characteristic zero and let $L=(y,x,z)$, $A=(x,y,z+x^3-y^2)$, $B=(x,y+x^2,z+x^3+(3/2)xy)$ be elements of $T(3,K)$. We consider the automorphism $F=A\circ L\circ B\circ L^{-1}$, the functional composition of the polynomial maps A,L,B and L^{-1}, the inverse of L. Then, $F=(f_1,f_2,f_3)$ is an element of $T(3,K)$, where

$$f_1=x+z^2,\qquad f_2=y+z^3+(3/2)xz,$$

$$f_3=z+x^3-y^2-3xyz+(1/4)z^2(3x^2-8yz).$$

We compute the reduced Gröbner bases of all its face functions, i.e. the reduced Gröbner bases of the ideals

$$I_x=(s-f_1(0,y,z),t-f_2(0,y,z),u-f_3(0,y,z)),$$

$$I_y=(s-f_1(x,0,z),t-f_2(x,0,z),u-f_3(x,0,z)),$$

$$I_z=(s-f_1(x,y,0),t-f_2(x,y,0),u-f_3(x,y,0))$$

with respect to the lexicographic ordering $x>y>z>s>t>u$. Wez get that all Gröbner bases produce tame automorphisms and one of them is the inverse of F.
Now, we consider $F^{-1}=G=(g_1,g_2,g_3)$, the inverse of F, which is also in $T(3,K)$:

$$g_1=-x^6+x-z^2+2zx^3-2zy^2+2x^3y^2-y^4,$$

$$g_2=\tfrac{1}{2}(-x^9+2y-3xz+z^3-3z^2x^3+3z^2y^2+3zx^6-6x^3zy^2+3zy^4$$
$$+3x^4+3x^6y^2-3x^3y^4-3xy^2+y^6),$$

$$g_3=z-x^3+y^2.$$

We also compute the reduced Gröbner bases of all its face functions, i.e. the reduced Gröbner bases of the ideals

$$I_x=(s-g_1(0,y,z),t-g_2(0,y,z),u-g_3(0,y,z)),$$

$$I_y=(s-g_1(x,0,z),t-g_2(x,0,z),u-g_3(x,0,z)),$$

$$I_z=(s-g_1(x,y,0),t-g_2(x,y,0),u-g_3(x,y,0)),$$

with respect to the same lexicographic ordering $x > y > z > s > t > u$. As a result we obtain

$$\text{Gbasis}(I_x) = \{-2t + 2y + u^3, -4u + 4z + 4t^2 - 4tu^3 + u^6, s + u^2\},$$

$$\text{Gbasis}(I_y) = \{-s + x - u^2, 3z - 3s^3 - 4t^2 - 2tu^3 - u^6 - 3u, 2t + 2u^3 + 3us\},$$

$$\text{Gbasis}(I_z) = \{-s + x - u^2, 2y - 2t - 2u^3 - 3us, 4u + 4s^3$$
$$+3s^2u^2 - 4t^2 - 8tu^3 - 12tus\}.$$

Again, it turns out that $\text{Gbasis}(I_z)$ gives F, the inverse of G. Besides, $\text{Gbasis}(I_x)$ produces an automorhism, but $\text{Gbasis}(I_y)$ does NOT correspond to an automorphism. However, if we change the "tag" monomial ordering, e.g. consider the lexicographic ordering $x > y > z > t > s > u$, the reduced Gröbner bases of I_x, I_y or I_z produce tame automorphisms.

The above example indicates that the Gröbner bases of the faces functions of a tame automorphism F with respect to a "tag" monomial ordering may produce tame automorphisms, but with the same "tag" monomial ordering, the inverse F^{-1} may not. It also suggests to modify slightly Question 2.5 for the case $n = 3$.

Conjecture 2.7. *Let $F = (f_1, f_2, f_3) \in T(3, K)$, i.e., F is a tame automorphism. There exists at least one "tag" monomial ordering which satisfies the condition $K[x, y, z] \succ K[s, t, u]$ such that the reduced Gröbner bases of the face functions of F under this ordering produce automorphisms.*

It is clear that the positive answer to Conjecture 2.7 implies that the Nagata automorphism is not tame. We have made a lot of experiments with tame automorphisms of $K[x, y, z]$, with different length of the decomposition into a product of affine and triangular automorphisms. In all the cases we have obtained confirmations of the conjecture. The above computations also suggest to study the following problems.

Question 2.8. Let $F = (f_1, \dots, f_n) \in \text{Aut}\, K[X]$. Fixing a "tag" monomial ordering, then:

(i) Does at least one of the reduced Gröbner bases of the face functions of F give the inverse of the automorphism F?

(ii) Does at least one of the reduced Gröbner bases of its face functions give an automorphism?

Just taking any linear automorphism involving all variables in the image of every variable, we obtain a negative answer to Question 2.8(i). Another non-trivial counterexample is the following automorphism from the book by Cohn [6]:

$$C = (x + y(xy - yz), y, z + y(xy - yz)).$$

We have checked that no reduced Gröbner bases of its face functions produce the inverse of C. Moreover, C is a tame automorphism, in fact, it is the composition of tame automorphisms $L \circ G \circ L$, where $L=(x,y,x-z)\in A(3,K)$ and $G=(x+zy^2,y,z)\in J(3,K)$. Finally, Question 2.8(ii) is still an open problem for $n>2$.

3. Lifting the Nagata automorphism

Let $K\langle X\rangle$ be the free associative algebra freely generated by the set $X=\{x_1,\dots,x_n\}$. One may consider this algebra as the algebra of polynomials in n noncommuting variables. The notion of tame and wild automorphisms of $K\langle X\rangle$ is introduced as in the case of $K[X]$. Every automorphism of $K\langle X\rangle$ induces an automorphism of $K[X]$ and every tame automorphism of $K[X]$ can be obtained in this way, i.e. can be lifted to an automorphism of $K\langle X\rangle$.

The result of Makar-Limanov [15] and Czerniakiewicz [7] states that every automorphism of $K\langle x,y\rangle$ is tame. Again, the answer is still unknown for $n>2$. Therefore, if an automorphism of $K[X]$ can be lifted to an automorphism of $K\langle X\rangle$, it may be regarded "not wilder" than the corresponding automorphism of the free algebra.

Problem 3.1. Can every automorphism ϕ of $K[X]$ be lifted to an automorphism ψ of $K\langle X\rangle$? If ϕ fixes the variables $x_1,\dots,x_m$, can the lifting ψ be chosen in such a way that to fix the same variables in $K\langle X\rangle$? Can the Nagata automorphism be lifted?

If an automorphism ϕ of $K[X]$ cannot be lifted to an automorphism of $K\langle X\rangle$, we may ask the question "how far" can be lifted ϕ. The natural way to measure the "distance" is in the language of T-ideals.

Definition 3.2. An ideal of $K\langle X\rangle$ is called a T-ideal if it is invariant under all endomorphisms of $K\langle X\rangle$.

The T-ideals of $K\langle X\rangle$ have important properties which make them very convenient for applications to automorphisms.

Proposition 3.3. (i) *For every T-ideal U of $K\langle X\rangle$ there exists an algebra R such that U coincides with the ideal $T(R)$ of all polynomial identities in n variables of R, i.e. $f(x_1,\dots,x_n)\in U$ if and only if $f(r_1,\dots,r_n)=0$ for all $r_1,\dots,r_n\in R$. The factor algebra $F(R)=K\langle X\rangle/T(R)$ is called the relatively free algebra of rank n in the variety of algebras generated by R.*

(ii) *Every automorphism of $K\langle X\rangle$ induces an automorphism of $K\langle X\rangle/U$.*

Problem 3.4. For a given automorphism ϕ of $K[X]$ find the T-ideals U with the property that ϕ can be lifted to $K\langle X\rangle/U$.

For a background on PI-algebras see the book by Rowen [19]. The theorem of Amitsur that over an infinite field the ideals $T(M_m(K))$ of polynomial identities of the $m \times m$ matrix algebra $M_m(K)$ are the only prime T-ideals and the Razmyslov–Kemer–Braun theorem [5, 13, 18] for the nilpotence of the radical of a finitely generated PI-algebra give that, over infinite fields, for every T-ideal U there exist positive integers m and k such that

$$T(M_m(K)) \supseteq U \supseteq T^k(M_m(K)).$$

Since our final purpose is to see whether an automorphism of $K[X]$ can be lifted to an automorphism of $K\langle X\rangle$, it is natural to consider liftings to $K\langle X\rangle/T^k(M_m(K))$ only. The following statement shows that the difficult part of the lifting is to $K\langle X\rangle/T(M_m(K))$ and $K\langle X\rangle/T^2(M_m(K))$ only.

Proposition 3.5. *For $k>2$ every endomorphism of $K\langle X\rangle/T^k(M_m(K))$ which induces an automorphism of $K\langle X\rangle/T^2(M_m(K))$ is an automorphism.*

Proof. We repeat the arguments from [8] for $K\langle X\rangle/C^k$, where C is the commutator ideal of $K\langle X\rangle$. Let $U = T(M_m(K))$, $A = K\langle X\rangle/U^k$. We use the same symbols (e.g. X, U, etc.) for the images of objects from $K\langle X\rangle$ in A. Let $\bar{A} = A/U^2$. If $\phi \in \operatorname{End} A$ induces an automorphism on $\bar{A}$, then there exists an endomorphism ψ of A such that $\phi \circ \psi$ and $\psi \circ \phi$ both induce the identity map on $\bar{A}$. Therefore, it is sufficient to prove that every endomorphism θ of A which induces the identity map on $\bar{A}$, is an automorphism. Let

$$\theta(x_i) = x_i + f_i, \quad f_i \in U^2, \quad i = 1, \ldots, n.$$

Consider any element $g \in U^p$. Since $f_i \in U^2$, it is easy to see that $\theta(g) - g \in U^{p+1}$. Hence, θ induces the identity map on the factors U^p/U^{p+1}, $p = 0, 1, \ldots, k-1$. Since $U^k = 0$ in A, as a vector space A is isomorphic to

$$\bar{A} \oplus U^2/U^3 \oplus \cdots \oplus U^{k-1}/U^k$$

and θ acts identically on all these factors. Hence θ is invertible, i.e., an automorphism. □

Umirbaev has shown that every automorphism of $K[X]$ can be lifted to an automorphism of the “free metabelian” associative algebra. This is the algebra $K\langle X\rangle/C^2$, where $C = T(M_1(K)) = T(K)$ is the commutator ideal of $K\langle X\rangle$. Therefore it can be also lifted to $K\langle X\rangle/C^k$ for any $k>2$. If the answer to Problem 3.1 is negative, we may ask its more precise version.

Problem 3.6. Can every automorphism ϕ of $K[X]$ be lifted to $\operatorname{Aut} K\langle X\rangle/T(M_m(K))$ and $\operatorname{Aut} K\langle X\rangle/T^2(M_m(K))$ for $m>1$? If ϕ fixes $x_1, \ldots, x_m$, can the lifting be chosen with the same property? How far can the Nagata automorphism be lifted with and without fixing one of the variables?

The construction of Martha Smith [21] showing that the Nagata automorphism is stably tame, is the following.

A derivation ∂ of $K[X]$ is called locally nilpotent if for any $f \in K[X]$ there exists a positive integer p such that $\partial^p(f)=0$. The derivation ∂ is triangular, if $\partial(x_i)=f(x_{i+1},\ldots,x_n)$, $i=1,\ldots,n$. For ∂ locally nilpotent (and if the characteristic of K is equal to 0), the map $\exp\partial : K[X] \to K[X]$ defined by

$$\exp\partial(f)=f+\frac{\partial}{1!}f+\frac{\partial^2}{2!}f+\cdots, \qquad f\in K[X],$$

is an automorphism of $K[X]$. If $w \in K[X]$ is in the kernel of ∂, then $\Delta = w\partial$ is also a locally nilpotent derivation.

The Nagata automorphism is equal to $\exp\Delta$, for the locally nilpotent derivation $\Delta = w(x,$
$y,z)\partial$, where ∂ is the derivation of $K[x,y,z]$ defined by

$$\partial(x)=-2y, \quad \partial(y)=z, \quad \partial(z)=0,$$

and $w(x,y,z)=y^2+xz$ is from the kernel of ∂. We extend the action of ∂ to $K[x,y,z,v]$ by $\partial(v)=0$. Then the automorphism $\exp(w\partial)$ of $K[x,y,z,v]$ acts identically on v and is tame because

$$\exp(w\partial)=\tau^{-1}\circ\exp^{-1}(v\partial)\circ\tau\circ\exp(v\partial),$$

where τ is the tame automorphism fixing x,y,z and $\tau(v)=v+w(x,y,z)$. Besides, $\exp(v\partial)$ is also tame and acts on x,y,z,v by

$$\exp(v\partial)(x)=x-2vy-v^2z, \quad \exp(v\partial)(y)=y+vz,$$
$$\exp(t\partial)(z)=z, \quad \exp(v\partial)(v)=v.$$

Considering the same automorphisms of $K\langle x,y,z,v\rangle$, we see that $\exp(v\partial)$ is not anymore an automorphism because v does not commute with the variables. But we replace $\exp(v\partial)$ with the tame automorphism θ of $K\langle x,y,z,v\rangle$ defined by

$$\theta(x)=x-2vy-v^2z, \quad \theta(y)=y+vz, \quad \theta(z)=z, \quad \theta(v)=v.$$

Considering the automorphism $\tau=(x,y,z,v+w)$ of $K\langle x,y,z,v\rangle$ with $w=y^2+xz$, we obtain the automorphism $\psi=\tau^{-1}\circ\theta^{-1}\circ\tau\circ\theta$ which induces an automorphism ϕ of $K[x,y,z,v]$ acting as the Nagata automorphism on x, y, z and ϕ fixing v. Unfortunately, we are not able to restrict ψ to an automorphism of $K\langle x,y,z\rangle$ because the variable v is not fixed by ψ and the images of x and y involve v. For example, if $[a,b]=ab-ba$ denotes the commutator of a and b, then

$$\psi(v)=v+(v-w)[y,z]+[v-w,y]z-(v-w)[v-w,z]z.$$

Our problem would be solved if we are able to find an automorphism ρ of $K\langle x,y,z,v\rangle$, which induces the identity map on $K[x,y,z,v]$ and acts on v as ψ. Then $\rho^{-1}\circ\psi$ will also induce the Nagata automorphism and will fix v. Therefore, if we factorize $K\langle x,y,z,v\rangle$

modulo the ideal generated by v we will obtain a lifting of the Nagata automorphism to an automorphism of $K\langle x,y,z\rangle$. Hence, it is worthy to try to find a better lifting for the Nagata automorphism to $\operatorname{Aut} K\langle x,y,z,v\rangle$.

We have succeeded to give an explicit lifting of the Nagata automorphism to $K\langle x,y,z\rangle/C^2$ which still fixes z. We hope that this lifting will be useful for further attempts to lift the Nagata automorphism to an automorphism of $K\langle x,y,z\rangle$. We have followed the procedure from the proof of the result of Umirbaev [22].

We work in the free metabelian algebra $M(X)=K\langle X\rangle/C^2$. We fix two sets of variables $Y=\{y_1,\dots,y_n\}$ and $Z=\{z_1,\dots,z_n\}$ and consider the polynomial algebra $K[Y\cup Z]$. Changing a little the notation of Umirbaev, we define formal partial derivatives $\partial/\partial x_i$ assuming that

$$\frac{\partial x_i}{\partial x_i}=1,\quad \frac{\partial x_j}{\partial x_i}=0,\ j\neq i,$$

and, for a monomial $u=x_{i_1}\dots x_{i_m}\in M(X)$

$$\frac{\partial u}{\partial x_i}=\sum_{k=1}^{m} y_{i_1}\cdots y_{i_{k-1}}z_{i_{k+1}}\cdots z_{i_m}\frac{\partial x_{i_k}}{\partial x_i}.$$

The Jacobian matrix of an endomorphism ϕ of $M(X)$ is

$$J(\phi)=\left(\frac{\partial\phi(x_j)}{\partial x_i}\right),$$

which is a matrix with entries from $K[Y\cup Z]$. One of the main results in [22] is that the Jacobian matrix $J(\phi)$ is invertible (as a matrix with entries from $K[Y\cup Z]$) if and only if ϕ is an automorphism of $M(X)$. Clearly, the invertibility of $J(\phi)$ is equivalent to $0\neq\det(J(\phi))\in K$.

Proposition 3.7. *The following endomorphism σ of $K\langle x,y,z\rangle/C^2$ is an automorphism which induces the Nagata automorphism of $K[x,y,z]$:*

$$\sigma(x)=x-(y^2+xz)y-y(y^2+xz)-(y^2+xz)^2z,\sigma(y)=y+(y^2+xz)z,\sigma(z)=z.$$

Proof. For simplicty of notation we assume that

$$X=\{x,y,z\},\quad Y=\{x_1,y_1,z_1\},\quad Z=\{x_2,y_2,z_2\}.$$

Clearly, the endomorphism σ of $M(X)$ induces the Nagata automorphism of $K[X]$. In order to show that σ is an automorphism it is sufficient to calculate the determinant of its Jacobian matrix. The partial derivatives of $\sigma(x)$, $\sigma(y)$ and $\sigma(z)$ are

$$\frac{\partial\sigma(x)}{\partial x}=1-(y_1+y_2)z_2-(y_1^2+x_1z_1+y_2^2+x_2z_2)z_2^2,$$

$$\frac{\partial\sigma(x)}{\partial y}=-(y_1+y_2)^2-(1+(y_1+y_2)z_2)(y_1^2+x_1z_1+y_2^2+x_2z_2),$$

$$\frac{\partial\sigma(y)}{\partial x}=z_2^2,\ \frac{\partial\sigma(y)}{\partial y}=1+(y_1+y_2)z_2,$$

$$\frac{\partial\sigma(z)}{\partial x}=\frac{\partial\sigma(z)}{\partial y}=0,\ \frac{\partial\sigma(z)}{\partial z}=1.$$

Hence, the Jacobian determinant $|J(\sigma)|$ is equal to

$$\begin{vmatrix} 1-(y_1+y_2)z_2-uz_2^2 & z_2^2 \\ -(y_1+y_2)^2-(1+(y_1+y_2)z_2)u & 1+(y_1+y_2)z_2 \end{vmatrix}=1,$$

where $u=y_1^2+x_1z_1+y_2^2+x_2z_2$, and σ is an automorphism. □

We sketch how, using the proof in [22], we have found the lifting of the Nagata automorphism. We hope that our considerations may be useful for attemps to lift other automorphisms suspected to be wild. Let $v=(f_1(X),f_2(X),z)$ be the Nagata automorphism, where

$$f_1(X)=x-2(y^2+xz)y-(y^2+xz)^2z,\ f_2(X)=y+(y^2+xz)z.$$

Tracing the proof in [22], we know that the ideals of $K[Y\cup Z]$ generated by

$$f_1(Y)-f_1(Z),\ f_2(Y)-f_2(Z),\ z_1-z_2 \text{ and } x_1-x_2,\ y_1-y_2,\ z_1-z_2$$

coincide and there is an invertible matrix $R=(r_{ij})\in K[Y\cup Z]$ such that

$$(f_1(Y)-f_1(Z),f_2(Y)-f_2(Z),z_1-z_2)=(x_1-x_2,y_1-y_2,z_1-z_2)R.$$

We do not know if there is an algorithm to find the matrix R but nevertheless we have found one matrix with this property,

$$R=\begin{pmatrix} 1-z_2(y_1+y_2)-z_2^2u & z_2^2 & 0 \\ -(y_1+y_2)^2-(1+z_2(y_1+y_2))u & 1+z_2(y_1+y_2) & 0 \\ -x_1(y_1+y_2)-(y_1^2+x_1z_1)^2-x_1z_2u & y_1^2+x_1(z_1+z_2) & 1 \end{pmatrix}.$$

The next step in the proof is to calculate the Jacobian matrix of any endomorphism ϕ of $M(X)$ which induces v on $K[X]$ and to compute $R-J(\phi)$. We start with

$$\phi=(x-2(y^2+xz)y-(y^2+xz)^2z,y+(y^2+xz)z,z),$$

$$R-J(\phi)=\begin{pmatrix} z_2(y_2-y_1) & 0 & 0 \\ x_1z_1-x_2z_2 & 0 & 0 \\ x_1(y_2-y_1) & 0 & 0 \end{pmatrix}.$$

Again, by the proof in [22], there exist elements w_1,w_2,w_3 from the commutator ideal of $M(X)$ such that their derivatives are exactly the columns of $R-J(\phi)$. In our case

$$w_1=[x,y]z+x[z,y]=xzy-yxz,w_2=w_3=0$$

and one automorphism σ of $M(X)$ which lifts the Nagata automorphism is

$$\sigma = (\phi(x) + w_1,\ \phi(y) + w_2,\ \phi(z) + w_3).$$

Starting with the automorphism σ we can find a better lifting of the Nagata automorphism to an automorphism of $K\langle x,y,z,v\rangle$. Let $\theta^*, \tau \in \operatorname{Aut} K\langle x,y,z,v\rangle$, where

$$\theta^* = (x - vy - yv - v^2 z, y + vz, z, v),$$

$$\tau = (x, y, z, v + w),\ w = y^2 + xz.$$

Concrete calculations show that

$$\psi^* = \tau^{-1} \circ (\theta^*)^{-1} \circ \tau \circ \theta^*$$

is also a lifting of the Nagata automorphism which fixes the variable v modulo the commutator ideal and

$$\psi^*(v) = v + (v - w)[y,z].$$

Again, the elements $\psi^*(x)$ and $\psi^*(y)$ depend on v, for example,

$$\psi^*(y) = y + wz + (v - w)[y,z]z.$$

Although the expression for $\psi^*(v)$ is simpler than $\psi(v)$, obtained above, we do not know how to fix the variable v in $K\langle x,y,z,v\rangle$.

The effective lifting of the automorphisms of $K[X]$ to automorphisms of the free metabelian algebra $M(X)$ depends on a result of Artamonov [4] on ideals of polynomial algebras: If the ideals $(p_1,\dots,p_k)$ and $(q_1,\dots,q_k)$ of $K[X]$ coincide, then there exists an invertible $k \times k$ matrix R with entries from $K[X]$ such that the following matrix equality holds

$$(p_1,\dots,p_k) = (q_1,\dots,q_k)R.$$

We expect that the following stronger version of the theorem of Artamonov holds.

Conjecture 3.8. *Let* $\{p_1,\dots,p_k,x_1,\dots,x_m\}$ *and* $\{q_1,\dots,q_k,x_1,\dots,x_m\}$ *generate the same ideal of the polynomial algebra* $K[X]$. *Then there exists an invertible matrix* $R \in M_{k+m}(K[X])$ *such that*

$$(p_1,\dots,p_k,x_1,\dots,x_m)R = (q_1,\dots,q_k,x_1,\dots,x_m)$$

and R *has the block form*

$$\begin{pmatrix} * & 0 \\ * & E \end{pmatrix},$$

where 0 *is the* $k \times m$ *matrix with zero entries and* E *is the* $m \times m$ *identity matrix.*

Problem 3.9. Can the matrix R from the paper by Artamonov and from above conjecture be found effectively?

If Conjecture 3.8 is true, this would imply a stronger version of the theorem of Umirbaev [22], which we state as another conjecture.

Conjecture 3.10. *Let C be the commutator ideal of $K\langle X\rangle$. Every automorphism of $K[X]$ fixing the variables $x_1,\dots,x_m$ can be lifted to an automorphism of $K\langle X\rangle/C^2$ also fixing $x_1,\dots,x_m$.*

4. Ideals and automorphisms

In this section we show that the method given in Proposition 2.1 which determines whether an endomorphism of $K[X]$ is an automorphism and which was the starting point of the study in Section 2, is based on a general fact for the ideals of arbitrary free algebras and, with some modifications, works also in the case of other algebraic systems as groups, semigroups, etc.

Let K be an arbitrary field. We consider associative K-algebras, not necessarily finitely generated. For a subset X of the K-algebra R, we denote by R_X the subalgebra of R generated by X.

Lemma 4.1. *Let $X=\{x_i \mid i\in I\}\subset R$ and $Y=\{y_i \mid i\in I\}\subset R$ be subsets of the same cardinality such that $R=R_{X\cup Y}$ and let*

$$\phi: R_X\to R_Y,\quad \phi(x_i)=f_i(Y),\quad i\in I,$$

$$\psi: R_Y\to R_X,\quad \psi(y_i)=g_i(X),\quad i\in I,$$

be homomorphisms of K-algebras. Let U and V be the ideals of R generated, respectively, by $\{x_i-f_i(Y)\mid i\in I\}$, $\{y_i-g_i(X)\mid i\in I\}$.

(i) *If ϕ and ψ are isomorphisms and $\psi=\phi^{-1}$, then $U=V$.*

(ii) *If $U=V$ and $\bar{R}_X=R_X/(R_X\cap U)$, $\bar{R}_Y=R_Y/(R_Y\cap V)$ are, respectively, the factor algebras of R_X and R_Y modulo the ideals U and V, then ϕ and ψ induce isomorphisms*

$$\bar{\phi}: \bar{R}_X\to\bar{R}_Y,\ \bar{\psi}: \bar{R}_Y\to\bar{R}_X,$$

and $\bar{\psi}=\bar{\phi}^{-1}$.

Proof. (i) Let ϕ and ψ be isomorphisms and let $\psi=\phi^{-1}$. Working modulo U we denote by $\bar{X}$ and $\bar{Y}$ the images of X and Y in R/U. We have

$$\bar{x}_k=f_k(\bar{Y})=f_k(\bar{y}_{j_1},\dots,\bar{y}_{j_n}),\quad k\in I,$$

$$g_i(\bar{x}_{k_1},\dots,\bar{x}_{k_m})=g_i(f_{k_1}(\bar{Y}),\dots,f_{k_m}(\bar{Y})),\quad i\in I.$$

Since $\psi\circ\phi$ is the identity map of R_Y,

$$\bar{y}_i=\overline{\psi\circ\phi(y_i)}=g_i(f_{k_1}(\bar{Y}),\dots,f_{k_m}(\bar{Y}))=g_i(\bar{x}_{k_1},\dots,\bar{x}_{k_m})=g_i(\bar{X}),$$

and $\bar{y}_i=g_i(\bar{X})$ implies that $V\subseteq U$. Similarly we obtain that $U\subseteq V$, i.e. $U=V$.

(ii) Let $U=V$. Then, in $R/U=R/V$

$$\bar{y}_i=f_i(\bar{x}_{k_1},\dots,\bar{x}_{k_m})=f_i(g_{k_1}(\bar{X}),\dots,g_{k_m}(\bar{X})),$$

$$\overline{\phi\circ\psi(y_i)}=\bar{y}_i,\quad i\in I,$$

similarly

$$\overline{\psi\circ\phi(x_i)}=\bar{x}_i,\quad i\in I.$$

If id_S is the identity map of a set S, the equalities

$$\bar{\phi}\circ\tilde{\psi}=\mathrm{id}_{\bar{R}_Y},\ \tilde{\psi}\circ\bar{\phi}=\mathrm{id}_{\bar{R}_X}$$

give that $\bar{\phi}$ and $\tilde{\psi}$ are isomorphisms and $\tilde{\psi}=\bar{\phi}^{-1}$. □

We apply the above lemma to endomorphisms of relatively free algebras $K\langle X\rangle/T(R)$. The same statement holds if we consider relatively free Lie or other K-algebras.

Theorem 4.2. *Let $X=\{x_i\mid i\in I\}$ and $Y=\{y_i\mid i\in I\}$ be two disjoint sets of the same cardinality and let $T(R)$ be the ideal of $K\langle X\cup Y\rangle$ consisting of the polynomial identities of an associative algebra R. Let ϕ and ψ be endomorphisms of the relatively free algebra $K\langle X\rangle/(K\langle X\rangle\cap T(R))$,*

$$\phi(x_i)=f_i(X)=f_i(x_{j_1},\dots,x_{j_p}),\quad \psi(x_i)=g_i(X)=g_i(x_{k_1},\dots,x_{k_q}),\qquad i\in I.$$

Then ϕ is an automorphism and $\psi=\phi^{-1}$ if and only if the ideals $U=(x_i-f_i(Y)\mid i\in I)$ and $V=(y_i-g_i(X)\mid i\in I)$ of $K\langle X\cup Y\rangle/T(R)$ coincide.

Proof. It is sufficient to show that

$$K\langle Y\rangle\cong K\langle Y\rangle/(K\langle Y\rangle\cap U),\ K\langle X\rangle\cong K\langle X\rangle/(K\langle X\rangle\cap V)$$

and to apply Lemma 4.1. The homomorphism $\rho: K\langle X\cup Y\rangle/T(R)\to K\langle X\cup Y\rangle/T(R)$ defined by

$$\rho(x_i)=x_i-f_i(Y),\quad \rho(y_i)=y_i,\qquad i\in I,$$

is a triangular automorphism. Hence $\rho(X)\cup\rho(Y)$ is a system of free generators of $K\langle X\cup Y\rangle/T(R)$ and the endomorphism θ defined by

$$\theta(x_i)=f_i(Y),\quad \theta(y_i)=y_i,\qquad i\in I,$$

maps the generators $\rho(X)$ to 0 and $\rho(Y)$ identically to $\rho(Y)$. Hence, the ideal U is the kernel of θ, and $\operatorname{Ker}\theta\cap K\langle Y\rangle/(K\langle Y\rangle\cap T(R))=(0)$, i.e., $K\langle Y\rangle\cong K\langle Y\rangle/(K\langle Y\rangle\cap U)$. □

Replacing relatively free algebras with relatively free groups and ideals with normal subgroups, we obtain a result similar to Theorem 4.2. For semigroups we have to replace ideals with congruences. We state the result for free groups only.

Corollary 4.3. *Let $G(X)$ be the free group freely generated by a set $X=\{x_i \mid i\in I\}$. An endomorphism ϕ of $G(X)$ such that $\phi(x_i)=f_i(X)=f_i(x_{k_1},\dots,x_{i_m})$ is an automorphism if and only if there exists an endomorphism ψ of $G(X)$ such that $\psi(x_i)=g_i(X)=g_i(x_{k_1},\dots,x_{k_m})$ and the elements $\{x_i^{-1}f_i(Y) \mid i\in I\}$ and $\{y_i^{-1}g_i(X) \mid i\in I\}$ generate the same normal subgroup of the free group $G(X\cup Y)$.*

Remark 4.4. For some relatively free algebras the problem whether the ideal generated by the elements $x_i-f_i(Y)$, $i\in I$, has a set of generators of the form $y_i-g_i(X)$, $i\in I$, can be solved algorithmically. For example, in the case of the polynomial algebra $K[X]$, $|X|=n<\infty$, we can solve this problem effectively by the Gröbner basis method.

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