GRÖBNER-SHIRSHOV BASES FOR REPRESENTATIONS OF LIE ALGEBRAS AND HECKE ALGEBRAS OF TYPE A

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Abstract. In this paper we discuss the Gröbner-Shirshov basis theory for representations of Lie algebras and Hecke algebras of type A. We describe Gröbner-Shirshov pairs and monomial bases for the Weyl modules over the special linear Lie algebras, and present the structure of the Specht modules over the Hecke algebras through Gröbner-Shirshov pairs and monomial bases.

0. Introduction

An efficient way to understand a (commutative, associative or Lie) algebra defined by generators and relations is to find a Gröbner-Shirshov basis for the algebra. Once a Gröbner-Shirshov basis is determined it automatically gives a linear basis of the algebra and an algorithm to write down an element into a linear combination of basis elements. The Gröbner-Shirshov basis theory was developed by Buchberger for commutative algebras ([8]) and by Shirshov for Lie algebras ([19]). It was also generalized by Bergman ([1]) and Bokut ([2]) to the case of associative algebras.

The theory can also be used to study representations of associative (and Lie) algebras. Among the representations the irreducible ones play the role of building blocks. For the irreducible representations of an algebra, it is enough to consider the following situation. Let $\mathcal{A}$ be a free associative algebra and let (S,T) be a pair of subsets of monic elements of $\mathcal{A}$. Let J be the two-sided ideal of $\mathcal{A}$ generated by S and $A = \mathcal{A}/J$ be the quotient algebra. We denote by I the left (or right) ideal of A generated by (the image of) T. Then the left (or right) A-module $M = A/I$ is said to be *defined by the pair* (S,T). Then the pair (S,T) is closed under composition if and only if the set of (S,T)-*standard monomials* forms a linear basis of M. In [13], such a pair (S,T) was called a *Gröbner-Shirshov pair* for M.

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The purpose of this work is to describe Gröbner-Shirshov bases for representations of Lie algebras and Hecke algebras of type A. In the first section, we recall the Gröbner-Shirshov basis theory for (cyclic) representations of associative algebras. We discuss in the next section the application of the theory to the Weyl modules over the special linear Lie algebra sl_{n+1}. In [14] Gröbner-Shirshov pairs were determined for the Weyl modules over the ground field $\mathbb{C}$. We modify those Gröbner-Shirshov pairs to have coefficients in $\mathbb{Z}$, and obtain from them linear bases of the Weyl modules over $sl_{n+1}\mathbb{F}$ for any field $\mathbb{F}$ of characteristic not equal to 2. The last section is devoted to describe the Specht modules over Hecke algebras of type A using Gröbner-Shirshov pairs. For each partition λ of n, there exists a Specht module. In [15] the set of *cozy* tableaux was identified with the set of standard monomials of the corresponding Specht module. We introduce the notion of *semicozy* tableau and identify the set of semicozy tableaux with the set of standard monomials of the Hecke algebra $\mathcal{H}_n(q)$. Hereby we explain the results of [15] in a much simpler way.

1. Gröbner-Shirshov pair

In this section we present the Gröbner-Shirshov basis theory for (cyclic) representations of associative algebras. Although we deal with left modules in this section, the theory can be easily modified to be fit for the right modules.

Let X be a set and let X^* be the set of associative monomials on X including the empty monomial 1. We denote the *length* of a monomial u by $l(u)$ with a convention $l(1) = 0$.

Definition 1.1. A well-ordering $\prec$ on X^* is called a *monomial order* if $x \prec y$ implies $axb \prec ayb$ for all $a, b \in X^*$.

Example 1.2. Let $X = \{x_1, x_2, \cdots\}$ be the set of alphabets and let

$$u = x_{i_1}x_{i_2}\cdots x_{i_k}, \quad v = x_{j_1}x_{j_2}\cdots x_{j_l} \in X^*.$$

We define $u \prec_{\text{deg-lex}} v$ if and only if $k < l$ or $k = l$ and $i_r < j_r$ for the first r such that $i_r \neq j_r$; it is a monomial order on X^* called the *degree lexicographic order*.

Fix a monomial order $\prec$ on X^* and let $\mathcal{A}_X$ be the free associative algebra generated by X over a field $\mathbb{F}$. Given a nonzero element $p \in \mathcal{A}_X$, we denote by $\overline{p}$ the maximal monomial appearing in p under the ordering $\prec$. Thus $p = \alpha\overline{p} + \sum \beta_i w_i$ with $\alpha, \beta_i \in \mathbb{F}$, $w_i \in X^*$, $\alpha \neq 0$ and $w_i \prec \overline{p}$. If $\alpha = 1$, p is said to be *monic*.

Let (S,T) be a pair of subsets of monic elements of $\mathcal{A}_X$, let J be the two-sided ideal of $\mathcal{A}_X$ generated by S, and let I be the left ideal of the algebra $A = \mathcal{A}_X/J$ generated by (the image of) T. Then we say that the algebra $A = \mathcal{A}_X/J$ is *defined by* S and that the left A-module $M = A/I$ is *defined by the pair* (S,T). The images of $p \in \mathcal{A}_X$ in A and in M under the canonical quotient maps will also be denoted by p.

Definition 1.3. Given a pair (S,T) of subsets of monic elements of $\mathcal{A}_X$, a monomial $u \in X^*$ is said to be (S,T)-*standard* if $u \neq a\overline{s}b$ and $u \neq c\overline{t}$ for any $s \in S$, $t \in T$ and $a, b, c \in X^*$. Otherwise, the monomial u is said to be (S,T)-*reducible*. If $T = \emptyset$, we simply say that u is S-*standard* or S-*reducible*.

Lemma 1.4. ([13, 14]) *Every $p \in \mathcal{A}_X$ can be expressed as*

$$p = \sum \alpha_i a_i s_i b_i + \sum \beta_j c_j t_j + \sum \gamma_k u_k, \tag{1.1}$$

where $\alpha_i, \beta_j, \gamma_k \in \mathbb{F}$, $a_i, b_i, c_j, u_k \in X^*$, $s_i \in S$, $t_j \in T$, $a_i\overline{s_i}b_i \preceq \overline{p}$, $c_j\overline{t_j} \preceq \overline{p}$, $u_k \preceq \overline{p}$ and u_k are (S,T)-standard.

Remark. The proof of the above lemma actually gives us an algorithm of writing an element p of $\mathcal{A}_X$ into the form of (1.1).

The term $\sum \gamma_k u_k$ in the expression (1.1) is called a *normal form* of p with respect to the pair (S,T) (and with respect to the monomial order $\prec$). In general, a normal form is not unique.

As an immediate corollary of Lemma 1.4, we obtain:

Corollary 1.5. ([13, 14]) *The set of (S,T)-standard monomials spans the left A-module $M = A/I$ defined by the pair (S,T).*

Definition 1.6. A pair (S,T) of subsets of monic elements of $\mathcal{A}_X$ is a *Gröbner-Shirshov pair* if the set of (S,T)-standard monomials forms a linear basis of the left A-module $M = A/I$ defined by the pair (S,T). In this case, we say that (S,T) is a *Gröbner-Shirshov pair* for the module M defined by (S,T). If a pair $(S,\emptyset)$ is a Gröbner-Shirshov pair, then we also say that S is a *Gröbner-Shirshov basis* for the algebra $A = \mathcal{A}_X/J$ defined by S.

Let p and q be monic elements of $\mathcal{A}_X$ with leading terms $\overline{p}$ and $\overline{q}$. We define the *composition* of p and q as follows.

Definition 1.7. (a) If there exist a and b in X^* such that $\overline{p}a = b\overline{q} = w$ with $l(\overline{p}) > l(b)$, then the *composition of intersection* is defined to be $(p,q)_w = pa - bq$. Furthermore, if $a = 1$, the composition $(p,q)_w$ is called *right-justified*.

(b) If there exist a and b in X^* such that $a \neq 1$, $a\overline{p}b = \overline{q} = w$, then the *composition of inclusion* is defined to be $(p,q)_w = apb - q$.

Let $p, q \in \mathcal{A}_X$ and $w \in X^*$. We define a *congruence relation* on $\mathcal{A}_X$ as follows: $p \equiv q \mod (S,T;w)$ if and only if $p-q = \sum \alpha_i a_i s_i b_i + \sum \beta_j c_j t_j$, where $\alpha_i, \beta_j \in \mathbb{F}$, $a_i, b_i, c_j \in X^*$, $s_i \in S$, $t_j \in T$, $a_i\overline{s_i}b_i \prec w$, and $c_j\overline{t_j} \prec w$. When $T = \emptyset$, we simply write $p \equiv q \mod (S;w)$.

Definition 1.8. A pair (S,T) of subsets of monic elements in $\mathcal{A}_X$ is said to be *closed under composition* if

(i) $(p,q)_w \equiv 0 \mod (S;w)$ for all $p,q \in S$, $w \in X^*$ whenever the composition $(p,q)_w$ is defined,

(ii) $(p,q)_w \equiv 0 \mod (S,T;w)$ for all $p,q \in T$, $w \in X^*$ whenever the right-justified composition $(p,q)_w$ is defined,

(iii) $(p,q)_w \equiv 0 \mod (S,T;w)$ for all $p \in S$, $q \in T$, $w \in X^*$ whenever the composition $(p,q)_w$ is defined.

If $T = \emptyset$, we say that S is closed under composition.

The following lemma is a variation of *Composition-Diamond Lemma* ([1, 2, 19]).

Lemma 1.9. ([13]) *Let (S,T) be a pair of subsets of monic elements in the free associative algebra $\mathcal{A}_X$ generated by X, let $A = \mathcal{A}_X/J$ be the associative algebra defined by S, and let $M = A/I$ be the left A-module defined by (S,T). If (S,T) is closed under composition and the image of $p \in \mathcal{A}_X$ is trivial in M, then the word $\overline{p}$ is (S,T)-reducible.*

As an immediate consequence, we obtain:

Theorem 1.10. ([14]) *Let (S,T) be a pair of subsets of monic elements in $\mathcal{A}_X$. Then the following conditions are equivalent :*

(a) *(S,T) is a Gröbner-Shirshov pair.*

(b) *(S,T) is closed under composition.*

(c) *For each $p \in \mathcal{A}_X$, the normal form of p is unique.*

2. Special linear Lie algebras

In this section we apply the Gröbner-Shirshov basis theory developed in the previous section to the Weyl modules over the special linear Lie algebra $sl_{n+1}\mathbb{F}$. The special linear Lie algebra $sl_{n+1}\mathbb{F}$ is the Lie algebra of $(n+1)\times(n+1)$ matrices with trace 0. We assume that the base field $\mathbb{F}$ is not of characteristic 2. We fix our monomial order to be the degree lexicographic order $\prec_{\text{deg-lex}}$ and denote it by $\prec$ for simplicity.

2.1. **Weyl modules.** To begin with we fix the notations. Recall that the Lie algebra $sl_{n+1}\mathbb{F}$ is generated by $\{e_i, h_i, f_i | 1 \le i \le n\}$ with defining relations

$$\begin{aligned} W \ &: \ [h_ih_j] \ \ (i>j), \quad [e_if_j]-\delta_{ij}h_i, \\ &\quad\ [e_ih_j]+a_{ji}e_i, \quad [h_if_j]+a_{ij}f_j, \\ S_+ \ &: \ [e_{i+1}[e_{i+1}e_i]], \quad [[e_{i+1}e_i]e_i] \ \ (1\le i\le n-1), \\ &\quad\ [e_ie_j] \ \ (i>j+1), \\ S_- \ &: \ [f_{i+1}[f_{i+1}f_i]], \quad [[f_{i+1}f_i]f_i] \ \ (1\le i\le n-1), \\ &\quad\ [f_if_j] \ \ (i>j+1), \end{aligned} \tag{2.1}$$

where the Cartan matrix $(a_{ij})_{1\le i,j\le n}$ is given by

$$a_{ii}=2, \quad a_{i+1,i}=a_{i,i+1}=-1, \quad a_{ij}=0 \ \text{ for } |i-j|>1. \tag{2.2}$$

Let U be the universal enveloping algebra of $sl_{n+1}\mathbb{F}$ and let U^+ (resp. U^-) be the subalgebra of U with 1 generated by $E=\{e_1,\cdots,e_n\}$ (resp. $F=\{f_1,\cdots,f_n\}$). Thus the algebra U^+ (resp. U^-) is the associative algebra defined by the set S_+ (resp. S_-) of relations in the free associative algebra $\mathcal{A}_E$ generated by E (resp. $\mathcal{A}_F$ generated by F).

For $i \ge j$, we define

$$\begin{aligned} &[e^{ji}] = [[[\cdots[e_je_{j+1}]\cdots]e_{i-1}]e_i], \quad e^{ji}=e_je_{j+1}\cdots e_i, \\ &[f_{ij}] = [f_i[f_{i-1}[\cdots[f_{j+1}f_j]\cdots]]], \quad f_{ij}=f_if_{i-1}\cdots f_j, \\ &\ h_{ij}=h_i+h_{i-1}+\cdots+h_j. \end{aligned} \tag{2.3}$$

Hence $[e^{ii}]=e^{ii}=e_i$ and $[f_{ii}]=f_{ii}=f_i$. We also set $[e^{i+1,i}]=e^{i+1,i}=1$ and $[f_{i,i+1}]=f_{i,i+1}=1$. We define $f_i \prec f_j$ if and only if $i<j$, and $(i,j)>(k,l)$ if and only if $i>k$ or $i=k$, $j>l$.

In [4], Bokut and Klein extended the set S_- to obtain a Gröbner-Shirshov basis $\mathcal{S}_-$ for the algebra U^- as given in the following proposition.

Proposition 2.1. ([4], [16]) *Assume that* char $\mathbb{F} \neq 2$ *and let*

$$\mathcal{S}_- = \{[[f_{ij}],[f_{kl}]] \,|\, (i,j) > (k,l),\ k \neq j-1\}. \tag{2.4}$$

Then $\mathcal{S}_-$ is a Gröbner-Shirshov basis for the algebra U^-. In addition, in U^-, we have

$$[[f_{ij}],[f_{j-1,k}]] = [f_{ik}]. \tag{2.5}$$

Remark. The Gröbner-Shirshov bases for classical Lie algebras first appeared in the paper of Lalonde and Ram ([16]), and were completely determined by Bokut and Klein in a series of papers ([4, 5, 6]). (See also [9].) For classical Lie superalgebras, the Gröbner-Shirshov bases were determined in [3].

For the time being we work over $\mathbb{C}$ and $\mathbb{Z}$, and then we will pass to an arbitrary field by extension of base. The following proposition is well-known and standard.

Proposition 2.2. ([10, 20]) *The set*

$$\{[e^{ji}], h_i, [f_{ij}] \,|\, 1 \le j \le i \le n\}$$

is a Chevalley basis of $sl_{n+1}\mathbb{C}$.

We use the following abbreviation: for $m \in \mathbb{Z}_{\ge 0}$

$$[f_{ij}]^{(m)} = \frac{[f_{ij}]^m}{m!}.$$

We fix the Chevalley basis given in the above proposition. Let $U^-_{\mathbb{Z}}$ be the corresponding $\mathbb{Z}$-form of U^-, that is, the subring of U^- (with 1) generated by all $[f_{ij}]^{(m)}$ $(m \in \mathbb{Z}_{\ge 0}, 1 \le j \le i \le n)$ over $\mathbb{Z}$.

Recall that the finite dimensional irreducible representations of $sl_{n+1}\mathbb{C}$ are indexed by partitions with at most n nonzero parts. Let $\lambda = (\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n \ge \lambda_{n+1} = 0)$ be a partition with at most n nonzero parts and let $V_{\mathbb{C}}(\lambda)$ denote the finite dimensional irreducible representation of $sl_{n+1}\mathbb{C}$ with highest weight λ. Set

$$m_i = \lambda_i - \lambda_{i+1} \quad \text{for } i = 1, 2, \cdots, n. \tag{2.6}$$

It is well-known (see [12], for example) that the $sl_{n+1}\mathbb{C}$-module $V_{\mathbb{C}}(\lambda)$ can be regarded as a U^--module defined by the pair (S_-, T_λ), where

$$T_\lambda = \{f_i^{m_i+1} | i = 1, \cdots, n\}. \tag{2.7}$$

We define $V_{\mathbb{Z}}(\lambda)$ to be the $U_{\mathbb{Z}}$-submodule of $V_{\mathbb{C}}(\lambda)$ generated by the highest wight vector $1 \in V_{\mathbb{C}}(\lambda)$. Thus we have $V_{\mathbb{Z}}(\lambda) = U^-_{\mathbb{Z}} \cdot 1$ and $V_{\mathbb{Z}}(\lambda)$ is an admissible lattice.

Now we accomplish the passage to an arbitrary field and define the Weyl modules over $sl_{n+1}\mathbb{F}$. We fix a partition λ having at most n nonzero parts. We define

$$V_{\mathbb{F}}(\lambda) = \mathbb{F} \otimes_{\mathbb{Z}} V_{\mathbb{Z}}(\lambda).$$

Then $V_{\mathbb{F}}(\lambda)$ naturally becomes an $sl_{n+1}\mathbb{F}$-module (cf. [10]).

Definition 2.3. The $sl_{n+1}\mathbb{F}$-module $V_{\mathbb{F}}(\lambda)$ is called the *Weyl module* of highest weight λ over $sl_{n+1}\mathbb{F}$.

2.2. **The natural relations in $V_{\mathbb{Z}}(\lambda)$.** As we mentioned in the introduction, Gröbner-Shirshov pairs for $V_{\mathbb{C}}(\lambda)$ were determined in [14]. We modify the relations in the Gröbner-Shirshov pairs for $V_{\mathbb{C}}(\lambda)$ so that they may hold in $V_{\mathbb{Z}}(\lambda)$. The proofs are similar to those in [14] and we omit them.

The following relations in $U_{\mathbb{Z}}^-$ play an important role in deriving the other relations in $V_{\mathbb{Z}}(\lambda)$.

Lemma 2.4. ([14]) *The following relations hold in $U_{\mathbb{Z}}^-$:*

$$[f_{i,j}][f_{j-1,k}]^{(m)} = [f_{j-1,k}]^{(m-1)}[f_{i,k}] + [f_{j-1,k}]^{(m)}[f_{i,j}] \quad (m \geq 1). \tag{2.8}$$

We fix a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq \lambda_{n+1} = 0)$ with at most n nonzero parts, and set $m_i = \lambda_i - \lambda_{i+1}$ $(1 \leq i \leq n)$ as before. For $1 \leq j \leq i \leq n$ and $a_{s,t}, r_s \in \mathbb{Z}$, we define

$$B_{i,j} = \prod_{s=i}^{n} \binom{a_{s,j+1} + r_s}{r_s}, \quad |r|_i = \sum_{s=i}^{n} r_s,$$

$$b_{i,j} = m_j + 1 + \sum_{s=i+1}^{n} (a_{s,j+1} - a_{s,j}) \qquad \text{and}$$

$$H_{i,j} = \prod_{s=i}^{n} \left(\prod_{t=1}^{j-1} [f_{s,t}]^{(a_{s,t})} \right) [f_{s,j}]^{(a_{s,j}-r_s)} [f_{s,j+1}]^{(a_{s,j+1}+r_s)} \left(\prod_{t=j+2}^{s} [f_{s,t}]^{(a_{s,t})} \right).$$

Lemma 2.5. *The following relations hold in $V_{\mathbb{Z}}(\lambda)$ for $k \geq 0$.*

(1)

$$f_n^{(b_{n,n}+k)} = 0.$$

(2) *For $1 \le j < n$ and $a_{s,t} \in \mathbb{Z}_{\ge 0}$,*

$$\sum_{r_n=0}^{b_{n,j}+k} f_j^{(r_n)} [f_{n,j}]^{(b_{n,j}+k-r_n)} [f_{n,j+1}]^{(a_{n,j+1}+r_n)} \prod_{t=j+2}^{n} [f_{n,t}]^{(a_{n,t})} = 0.$$

(3) *For $1 \le j < n$ and $a_{s,t} \in \mathbb{Z}_{\ge 0}$ such that $b_{j,j} > 0$,*

$$\sum_{r_n=0}^{a_{n,j}} \cdots \sum_{r_{j+1}=0}^{a_{j+1,j}} B_{j+1,j} f_j^{(b_{j,j}+k+|r|_{j+1})} H_{j+1,j} = 0.$$

(4) *For $1 \le j < i < n$ and $a_{s,t} \in \mathbb{Z}_{\ge 0}$ such that $b_{i,j} > 0$,*

$$\sum_{r_n=0}^{a_{n,j}} \cdots \sum_{r_{i+1}=0}^{a_{i+1,j}} \sum_{r_i=-a_{i,j+1}}^{b_{i,j}+k} B_{i+1,j} f_j^{(|r|_i)}$$
$$\times \quad [f_{i,j}]^{(b_{i,j}+k-r_i)} [f_{i,j+1}]^{(a_{i,j+1}+r_i)} \left(\prod_{t=j+2}^{i} [f_{i,t}]^{(a_{i,t})} \right) H_{i+1,j} = 0,$$

where the summand is 0 whenever $|r|_i < 0$.

2.3. **Monomial bases for $V_{\mathbb{F}}(\lambda)$.** In this subsection we determine a Gröbner-Shirshov pair and the standard monomial basis for the Weyl module $V_{\mathbb{C}}(\lambda)$, and give a 1-1 correspondence between the standard monomial basis and the set of semistandard tableaux of given shape. Modifying the standard monomial basis for $V_{\mathbb{C}}(\lambda)$, we obtain bases of $V_{\mathbb{Z}}(\lambda)$ and $V_{\mathbb{F}}(\lambda)$ consisting of monomials in divided powers of the elements in the negative part of the Chevalley basis.

Since $V_{\mathbb{Z}}(\lambda)$ is a submodule of $V_{\mathbb{C}}(\lambda)$, all the relations in $V_{\mathbb{Z}}(\lambda)$ hold also in $V_{\mathbb{C}}(\lambda)$. Hence the relations in Lemma 2.5 hold in $V_{\mathbb{C}}(\lambda)$. We will see that those relations (with $k = 0$) actually consist of a Gröbner-Shirshov pair for $V_{\mathbb{C}}(\lambda)$.

Recall that a Gröbner-Shirshov pair consists of monic elements, so we need to make the relations in Lemma 2.5 to be monic by multiplying appropriate scalars. Let $\mathcal{T}_\lambda$ be the subset of $\mathcal{A}_X$ consisting of the following monic elements, which are just scalar multiples of the left-hand side of the relations with $k = 0$ in Lemma 2.5:

(1)

$$\kappa(n,n) f_n^{(b_{n,n})},$$

(2)

$$\kappa(n,j)\sum_{r_n=0}^{b_{n,j}} f_j^{(r_n)}[f_{n,j}]^{(b_{n,j}-r_n)}[f_{n,j+1}]^{(a_{n,j+1}+r_n)}\prod_{t=j+2}^{n}[f_{n,t}]^{(a_{n,t})}$$

for $1 \le j < n$ and $a_{s,t} \in \mathbb{Z}_{\ge 0}$,

(3)

$$\kappa(j,j)\sum_{r_n=0}^{a_{n,j}}\cdots\sum_{r_{j+1}=0}^{a_{j+1,j}} B_{j+1,j} f_j^{(b_{j,j}+|r|_{j+1})} H_{j+1,j}$$

for $1 \le j < n$ and $a_{s,t} \in \mathbb{Z}_{\ge 0}$ such that $b_{j,j} > 0$, and

(4)

$$\kappa(i,j)\sum_{r_n=0}^{a_{n,j}}\cdots\sum_{r_{i+1}=0}^{a_{i+1,j}}\sum_{r_i=-a_{i,j+1}}^{b_{i,j}} B_{i+1,j} f_j^{(|r|_i)}$$
$$\times \quad [f_{i,j}]^{(b_{i,j}-r_i)}[f_{i,j+1}]^{(a_{i,j+1}+r_i)}\left(\prod_{t=j+2}^{i}[f_{i,t}]^{(a_{i,t})}\right)H_{i+1,j}$$

for $1 \le j < i < n$ and $a_{s,t} \in \mathbb{Z}_{\ge 0}$ such that $b_{i,j} > 0$,

where

$$\kappa(i,j) = b_{i,j}!\prod_{t=j+1}^{i} a_{i,t}!\prod_{s=i+1}^{n}\prod_{t=1}^{s} a_{s,t}!$$

for $1 \le j \le i \le n$.

Note that the maximal monomial of each relation in $\mathcal{T}_\lambda$ appears when $r_s = 0$ for all s, and that it is of the form

$$f_{i,j}^{b_{i,j}}\prod_{t=j+1}^{i} f_{i,t}^{a_{i,t}}\prod_{s=i+1}^{n}\prod_{t=1}^{s} f_{s,t}^{a_{s,t}} \quad (1 \le j \le i \le n).$$

Our observation yields the following proposition.

Proposition 2.6. ([14]) *The set $G_W(\lambda)$ of $(\mathcal{S}_-, \mathcal{T}_\lambda)$-standard monomials is given by*

$$G_W(\lambda) = \{\prod_{i=1}^{n}\prod_{j=1}^{i} f_{i,j}^{a_{i,j}} \mid 0 \le a_{i,j} \le b_{i,j}-1\}, \tag{2.9}$$

where $b_{i,j} = m_j + 1 + \sum_{s=i+1}^{n}(a_{s,j+1} - a_{s,j})$.

The *diagram* of a composition $\lambda = (\lambda_1, \lambda_2, \cdots)$ is defined to be the set

$$[\lambda] = \{(i,j) | 1 \le j \le \lambda_i \text{ and } i \ge 1\}.$$

If λ is a composition of m then a λ-*tableau* is a map $t : [\lambda] \to \{1, 2, \cdots, m\}$; we also denote by $[t]$ the diagram $[\lambda]$ corresponding to the tableau t. Given a partition λ, a λ-tableau t is *semistandard* if

$$t(i,j) \le t(i,j+1) \quad \text{and} \quad t(i,j) < t(i+1,j) \quad \text{for all } i \text{ and } j.$$

As usual, we can present a semistandard tableau by an array of colored boxes.

Suppose that λ is a partition with at most n nonzero parts. Let $SST(\lambda)$ denote the set of all semistandard tableaux of shape λ. We define a map $\Psi : G_W(\lambda) \to SST(\lambda)$ as follows.

(a) Let $\Psi(1)$ be the semistandard tableau τ^λ defined by $\tau^\lambda(i,j) = i$ for all i and j.

(b) Let $w = \prod_{i=1}^{n} \prod_{j=1}^{i} f_{i,j}^{a_{i,j}}$ be an $(\mathcal{S}_-, \mathcal{T}_\lambda)$-standard monomial in $G_W(\lambda)$. We define $\Psi(w)$ to be the semistandard tableau τ obtained from τ^λ by applying the words $f_{i,j}$'s successively (as a left U^--action) in the following way:

the word $f_{i,j}$ changes the rightmost occurrence of the box $\overline{j}$ in the j-th row of τ^λ to the box $\overline{i+1}$.

For example, for the monomial $w = f_1 f_2^2 f_{3,1} f_3^2$ in $G_W((4,3,2))$, $\Psi(w)$ is the semistandard tableau $\begin{array}{cccc} 1 & 1 & 2 & 4 \\ 2 & 3 & 3 & \\ 4 & 4 & & \end{array}$.

It is now straightforward to verify that Ψ is a bijection between $G_W(\lambda)$ and $SST(\lambda)$. We have just proved the following proposition.

Proposition 2.7. *The set $G_W(\lambda)$ is in 1-1 correspondence with the set $SST(\lambda)$ of all semistandard tableaux of shape λ.*

By Corollary 1.5, $G_W(\lambda)$ is a spanning set of $V_{\mathbb{C}}(\lambda)$, and since $\dim V_{\mathbb{C}}(\lambda) = \#SST(\lambda)$, it is actually a linear basis of $V_{\mathbb{C}}(\lambda)$. Therefore, we conclude:

Theorem 2.8. ([14]) *The pair $(\mathcal{S}_-, \mathcal{T}_\lambda)$ is a Gröbner-Shirshov pair for the irreducible $sl_{n+1}\mathbb{C}$-module $V_{\mathbb{C}}(\lambda)$ with highest weight λ, and the set $G_W(\lambda)$ is a monomial basis for $V_{\mathbb{C}}(\lambda)$.*

Now we can easily obtain a basis of $V_{\mathbb{Z}}(\lambda)$ and $V_{\mathbb{F}}(\lambda)$ from the set $G_W(\lambda)$. We define

$$\hat{G}_W(\lambda) = \{\prod_{i=1}^{n}\prod_{j=1}^{i}[f_{i,j}]^{(a_{i,j})} \mid 0 \le a_{i,j} \le b_{i,j} - 1\},$$

where $b_{i,j} = m_j + 1 + \sum_{s=i+1}^{n}(a_{s,j+1} - a_{s,j})$. Since the leading monomial of each element in $\hat{G}_W(\lambda)$ is just the corresponding element in $G_W(\lambda)$, the set $\hat{G}_W(\lambda)$ is another basis of $V_{\mathbb{C}}(\lambda)$. Furthermore, we have the following theorem.

Theorem 2.9. *The set $\hat{G}_W(\lambda)$ is a basis of $V_{\mathbb{Z}}(\lambda)$, and also a basis of $V_{\mathbb{F}}(\lambda)$.*

Proof. Since $\hat{G}_W(\lambda)$ is linearly independent over $\mathbb{C}$, it is also linearly independent over $\mathbb{Z}$. We show that $\hat{G}_W(\lambda)$ spans $V_{\mathbb{Z}}(\lambda)$ over $\mathbb{Z}$. From the definition of $V_{\mathbb{Z}}(\lambda)$ and the relations in $\mathcal{S}_-$, it is enough to consider elements f_ζ of the form $f_\zeta = \prod_i \prod_j [f_{i,j}]^{(a_{i,j})}$. We use induction. If $f_\zeta = 1$, there is nothing to prove. Assume that $f_\zeta \succ 1$ and that all the elements f_η of the form $f_\eta = \prod_i \prod_j [f_{i,j}]^{(a'_{i,j})}$ with $f_\eta \prec f_\zeta$ are $\mathbb{Z}$-linear combination of elements in $\hat{G}_W(\lambda)$. If $a_{i,j} < b_{i,j}$ for all i, j, then $f_\zeta \in \hat{G}_W(\lambda)$. Otherwise we can use relations in Lemma 2.5 to write f_ζ into a $\mathbb{Z}$-linear combination of elements $f_\eta = \prod_i \prod_j [f_{i,j}]^{(a'_{i,j})}$ with $f_\eta \prec f_\zeta$. By induction f_ζ is a $\mathbb{Z}$-linear combination of elements in $\hat{G}_W(\lambda)$. Thus $\hat{G}_W(\lambda)$ spans $V_{\mathbb{Z}}(\lambda)$ over $\mathbb{Z}$. The second assertion immediately follows from the first. □

3. Hecke algebras

In this section we apply the Gröbner-Shirshov basis theory to the Specht modules over the Hecke algebras $\mathcal{H}_n(q)$ of type A. As in the previous section, we take $\prec_{\text{deg-lex}}$ as our monomial order and denote it simply by $\prec$. From now on we consider only right modules.

3.1. **Specht Modules.** Let $\mathbb{F}$ be a field, and fix $n \in \mathbb{Z}_{>0}$ and a nonzero $q \in \mathbb{F}^\times$. The *Hecke algebra $\mathcal{H}_n(q)$ of type A* is defined to be the associative algebra over $\mathbb{F}$ generated by $X = \{T_1, T_2, \cdots, T_{n-1}\}$ with defining relations

$$R_H : \quad \begin{aligned} &T_iT_j = T_jT_i && \text{for } i > j+1, \\ &T_i^2 = (q-1)T_i + q && \text{for } 1 \le i \le n-1, \\ &T_{i+1}T_iT_{i+1} = T_iT_{i+1}T_i && \text{for } 1 \le i \le n-2. \end{aligned} \tag{3.1}$$

We write $T_{i,j} = T_iT_{i-1}\cdots T_j$ for $i \ge j$ (hence $T_{i,i} = T_i$) with a convention $T_{i,i+1} = 1$ ($i \ge 0$).

Proposition 3.1. ([15]) *A Gröbner-Shirshov basis for $\mathcal{H}_n(q)$ with respect to the monomial order $\prec$ is given by*

$$\mathcal{R}_H \; : \; \begin{array}{ll} T_iT_j - T_jT_i & \text{for } \; i > j+1, \\ T_i^2 - (q-1)T_i - q & \text{for } \; 1 \le i \le n-1, \\ T_{i+1,j}T_{i+1} - T_iT_{i+1,j} & \text{for } i \ge j. \end{array} \tag{3.2}$$

Hence the set of $\mathcal{R}_H$-standard monomials is

$$\mathcal{B}_H = \{T_{1,a_1}T_{2,a_2}\cdots T_{n-1,a_{n-1}} | 1 \le a_k \le k+1, k = 1, 2, \cdots, n-1\}.$$

Suppose that λ is a composition. A λ-tableau t is *row standard* if the entries in t increase from left to right in each row. The set of row standard λ-tableaux will be denoted by $RS(\lambda)$. A row standard λ-tableau t is *standard* if λ is a partition, t is bijective as a map and the entries in t also increase from top to bottom in each column. We denote the set of standard λ-tableaux by $ST(\lambda)$.

Let μ and ν be compositions. Write $\mu \trianglerighteq \nu$ if

$$\sum_{j=1}^{i} \mu_j \ge \sum_{j=1}^{i} \nu_j \qquad \text{for all } i \ge 1.$$

Given a row standard tableau t and an integer $m \le n$, let $t \downarrow m$ be the tableau obtained from t by deleting all of entries greater than m. Note that $t \downarrow m$ is also row standard. Given a row standard λ-tableaux s and t, write $s \trianglerighteq t$ if $[s \downarrow m] \trianglerighteq [t \downarrow m]$ for $m = 1, 2, \cdots, n$.

Given a composition $\lambda = (\lambda_1, \lambda_2, \cdots)$ of n, we define

$$0_\lambda = 0 \quad \text{and} \quad i_\lambda = \sum_{l=1}^{i} \lambda_l \quad \text{for } i \ge 1.$$

For example, if $\lambda = (3, 4, 1) \models 8$, then we have

$$0_\lambda = 0, \; 1_\lambda = 3, \; 2_\lambda = 7 \; \text{ and } \; 3_\lambda = 8.$$

We call a λ-tableau *semicozy* if $1 \le t(i, j) \le (i-1)_\lambda + j$. The set of semicozy λ-tableau will be denoted by $SC(\lambda)$. For a fixed composition λ, we identify an element $T_{1,a_1}T_{2,a_2}\cdots T_{n-1,a_{n-1}} \in \mathcal{B}_H$ with the semicozy λ-tableau defined by $(i, j) \mapsto a_{(i-1)_\lambda + j - 1}$ $(a_0 = 1)$. In this way the set $SC(\lambda)$ of semicozy λ-tableau is identified with the set $\mathcal{B}_H$.

Example 3.2. If $\lambda = (3, 2, 1) \vdash 6$, then

$$T_{1,2}T_{2,3}T_{3,4}T_{4,5}T_{5,6} = 1 \quad \longleftrightarrow \quad \begin{array}{ccc} \hline 1 & 2 & 3 \\ 4 & 5 & \\ 6 & & \end{array} \; ,$$

$$\text{and} \quad T_{1,2}T_{2,3}T_{3,1}T_{4,2}T_{5,5} = T_{3,1}T_{4,2}T_5 \quad \longleftrightarrow \quad \begin{array}{ccc} \hline 1 & 2 & 3 \\ 1 & 2 & \\ 5 & & \end{array} .$$

Let t^λ be the tableau defined by

$$t^\lambda(i,j) = (i-1)_\lambda + j \quad \text{for all } i \text{ and } j.$$

For example, $t^{(5,3,2,1)}$ is the tableau given below :

$$\begin{array}{ccccc} \hline 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & & \\ 9 & 10 & & & \\ 11 & & & & \end{array}$$

For a given partition $\lambda = (\lambda_1, \lambda_2, \cdots)$ and for each $i \geq 2$ and $j \geq 1$, we define the (i,j)-*Garnir tableau* $t^\lambda_{i,j}$ by

$$t^\lambda_{i,j}(a,b) = \begin{cases} t^\lambda(a,b) & \text{if } a \neq i \text{ or } b > j, \\ (i-2)_\lambda + j + b - 1 & \text{otherwise.} \end{cases}$$

We also define $\sum(t^\lambda_{i,j}) \in \mathcal{A}_X$ to be the sum of λ-tableau t (or corresponding monomials) such that

$$t \in RS(\lambda) \cap SC(\lambda) \quad \text{and} \quad t^\lambda \trianglelefteq t \trianglelefteq t^\lambda_{i,j}.$$

Let e be the order of $q \in \mathbb{F}^\times$, so $e \in \mathbb{N} \cup \{\infty\}$. We define the $\mathcal{H}_n(q)$-module $\hat{S}^\lambda_q$ by the pair (R_H, R^λ_H), where R^λ_H is the set of elements:

$$\begin{array}{ll} T_i - q & (i \neq l_\lambda, l \geq 1), \\ \sum(t^\lambda_{i,1}) & (i \geq 2), \\ \sum(t^\lambda_{i,ke}) & (e < \infty,\ i \geq 2,\ k \geq 1). \end{array} \tag{3.3}$$

The module $\hat{S}^\lambda_q$ will turn out to be isomorphic to the Specht module. To do this, we use Murphy's construction ([17]) of Specht modules. So let us recall the construction.

Let S_n be the symmetric group on n letters and let $\tau_i = (i, i+1)$ be the transposition of i and $i+1$. We define for $T = T_{i_1}T_{i_2}\cdots T_{i_k} \in \mathcal{H}_n(q)$, $T^* = T_{i_k}T_{i_{k-1}}\cdots T_{i_1}$ and extend $*$ to an anti-automorphism of $\mathcal{H}_n(q)$ by linearity. For a reduced expression $w = \tau_{i_1}\tau_{i_2}\cdots\tau_{i_k} \in S_n$, we define $T_w \in \mathcal{H}_n(q)$ to be

$$T_w = T_{i_1}T_{i_2}\cdots T_{i_k}.$$

Then T_w is well-defined and $T^*_w = T_{w^{-1}}$.

The symmetric group S_n acts naturally from the right on the set of all λ-tableaux which is bijective as a map. If t is row-standard and bijective, we denote by $d(t)$ the element of S_n for which $t = t^\lambda d(t)$. We denote by W_λ the group of row stabilizers of t^λ. Let λ be a composition of n. By a *λ-pair* we mean a pair (s,t) of bijective row standard λ-tableaux. A λ-pair is called *standard* if both s and t are standard.

For a composition λ of n and for any λ-pair (s,t), we define

$$x_\lambda = \sum_{w \in W_\lambda} T_w \quad \text{and} \quad x_{st} = T^*_{d(s)} x_\lambda T_{d(t)}. \tag{3.4}$$

Hence $x_{t^\lambda t^\lambda} = x_\lambda$. From now on, whenever the subscript is t^λ, we will abbreviate it to λ. For example, we will write $x_{t^\lambda t^\lambda} = x_{\lambda\lambda} = x_\lambda$ and $x_{t^\lambda s} = x_{\lambda s}$.

For a partition $\lambda \vdash n$, let N^λ (resp. $\bar{N}^\lambda$) be the $\mathbb{F}$-submodule of $\mathcal{H}_n(q)$ spanned by x_{rs}, where (r,s) runs over all standard μ-pair for a partition $\mu \vdash n$ with $\mu \trianglerighteq \lambda$ (resp. $\mu \triangleright \lambda$). Let $M^\lambda = x_\lambda \mathcal{H}_n(q)$ be the cyclic $\mathcal{H}_n(q)$-module generated by x_λ and set $\bar{M}^\lambda = M^\lambda \bigcap \bar{N}^\lambda$.

Definition 3.3. The $\mathcal{H}_n(q)$-module $S_q^\lambda = M^\lambda / \bar{M}^\lambda$ is called the *Specht module* over $\mathcal{H}_n(q)$ corresponding to the partition λ.

Proposition 3.4. ([17]) *The Specht module S_q^λ has a basis consisting of the vectors $x_{\lambda s} + \bar{M}^\lambda$, where s runs over all standard λ-tableaux.*

3.2. **Gröbner-Shirshov pairs for the Specht modules.** In this subsection we determine Gröbner-Shirshov pairs for the modules $\hat{S}_q^\lambda$. It is achieved by showing that $\hat{S}_q^\lambda$ is isomorphic to the Specht module S_q^λ. The monomial basis for $\hat{S}_q^\lambda$ is identified with the set of cozy λ-tableaux.

A connection between $\hat{S}_q^\lambda$ and S_q^λ is made in the following proposition.

Proposition 3.5. ([15, 17]) *In S_q^λ the following relations hold.*

(1)
$$x_\lambda (T_i - q) = 0 \quad (i \neq l_\lambda, l \geq 1).$$

(2)
$$x_\lambda \big(\sum (t^\lambda_{i,j}) \big) = 0 \quad (i \geq 2, j \geq 1).$$

The above proposition gives us a surjection from $\hat{S}_q^\lambda$ to S_q^λ. Indeed, we define an $\mathcal{H}_n(q)$-module homomorphism $\Psi : \hat{S}_q^\lambda \to S_q^\lambda$ by $1 \mapsto x_\lambda$. Then it is assured in the above proposition that Ψ is well-defined. Naturally, Ψ is surjective.

To obtain a Gröbner-Shirshov pair for $\hat{S}_q^\lambda$ we need more relations holding in $\hat{S}_q^\lambda$.

Lemma 3.6. ([15])

(1) *For each $t \in SC(\lambda)$ such that $t(i,j) \geq t(i,j+1)$ for some i,j, we have in $\hat{S}_q^\lambda$*

$$t = qt',$$

where

$$t'(a,b) = \begin{cases} t(i,j+1) & \text{if } (a,b) = (i,j), \\ t(i,j)+1 & \text{if } (a,b) = (i,j+1), \\ t(a,b) & \text{otherwise.} \end{cases}$$

(2) *For each $t \in SC(\lambda) \cap RS(\lambda)$ such that $t(i,j) + j > t(i+1,j)$ for some i,j, we can write in $\hat{S}_q^\lambda$*

$$t = \sum_{s \prec t} a_{t,s} s \quad (s \in SC(\lambda), a_{t,s} \in \mathbb{F}).$$

Let $\mathcal{R}_H^\lambda$ be the set of relations from Lemma 3.6. That is,

$$\mathcal{R}_H^\lambda : \begin{array}{ll} t - qt' & (t \in SC(\lambda), t(i,j) \geq t(i,j+1) \text{ for some } i,j), \\ t - \displaystyle\sum_{s \prec t} a_{t,s} s & (t \in SC(\lambda) \cap RS(\lambda), t(i,j) + j > t(i+1,j) \text{ for some } i,j). \end{array}$$

A semicozy λ-tableau $t \in SC(\lambda)$ is said to be *cozy* if it is row standard and satisfies the condition $t(i,j) + j \leq t(i+1,j)$ for all i and j. We denote by $CZ(\lambda)$ the set of cozy λ-tableaux. For example, the following tableaux are cozy :

1	2	3	4	5	6	7
2	4	6	8	10		
3	6	9				
4	8	12				

1	2	3	4	5	6	7
3	4	6	9	10		
4	7	9				
7	10	15				

Lemma 3.7. ([15]) *The set $CZ(\lambda)$ of cozy λ-tableaux is in one-to-one correspondence with the set $ST(\lambda)$ of standard λ-tableaux.*

Proof. A cozy λ-tableau t is identified with a monomial $T_{i_1} \cdots T_{i_k}$ in $\mathcal{H}_n(q)$, and the monomial $T_{i_1} \cdots T_{i_k}$ corresponds to the element $\tau_{i_1} \cdots \tau_{i_k}$ in S_n. It is easy to check that $t^\lambda \tau_{i_1} \cdots \tau_{i_k}$ is a standard λ-tableau, and that this is a one-to-one correspondence between $CZ(\lambda)$ and $ST(\lambda)$. □

Now we state the main theorem of this section.

Theorem 3.8. ([15])

(a) *The set of* $(\mathcal{R}_H, \mathcal{R}_H^\lambda)$*-standard monomials is exactly the set of cozy* λ*-tableaux* $CZ(\lambda)$ *under the identification of* $\mathcal{B}_H$ *with* $SC(\lambda)$.

(b) *The module* $\hat{S}_q^\lambda$ *defined by* (R_H, R_H^λ) *is isomorphic to the Specht module* S_q^λ.

(c) *The pair* $(\mathcal{R}_H, \mathcal{R}_H^\lambda)$ *is a Gröbner-Shirshov pair for* $\hat{S}_q^\lambda$ *with respect to* $\prec$.

Proof. From the definition of a cozy λ-tableau, it is clear that the set of $(\mathcal{R}_H, \mathcal{R}_H^\lambda)$-standard monomials is exactly the set of cozy λ-tableaux $CZ(\lambda)$ under the identification of $\mathcal{B}_H$ with $SC(\lambda)$. Since there is a surjection Ψ from $\hat{S}_q^\lambda$ onto S_q^λ, we have $\dim \hat{S}_q^\lambda \geq \dim S_q^\lambda$. It follows from Corollary 1.5 that $CZ(\lambda)$ spans $\hat{S}_q^\lambda$ and so $\#CZ(\lambda) \geq \dim \hat{S}_q^\lambda$. On the other hand, $\dim S_q^\lambda = \#ST(\lambda)$ from Proposition 3.4. Thus from Lemma 3.7 we obtain

$$\#ST(\lambda) = \#CZ(\lambda) \geq \dim \hat{S}_q^\lambda \geq \dim S_q^\lambda = \#ST(\lambda).$$

Now $\hat{S}_q^\lambda$ is isomorphic to S_q^λ, and $CZ(\lambda)$ is a linear basis of $\hat{S}_q^\lambda$. The part (c) follows from the definition of a Gröbner-Shirshov pair. □

Remark. The monomial basis $CZ(\lambda)$ is mapped onto the linear basis of S_q^λ given in Proposition 3.4 under the isomorphism Ψ.

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References

[1] G. M. Bergman, *The diamond lemma for ring theory*, Adv. Math. **29** (1978), 178–218.

[2] L. A. Bokut, *Imbedding into simple associative algebras*, Algebra and Logic **15** (1976), 117–142.

[3] L. A. Bokut, S.-J. Kang, K.-H. Lee, P. Malcolmson *Gröbner-Shirshov bases for Lie superalgebras and their universal enveloping algebras*, J. Algebra **217** (1999), 461-495.

[4] L. A. Bokut, A. A. Klein, *Serre relations and Gröbner-Shirshov bases for simple Lie algebras I, II*, Internat. J. Algebra Comput. **6** (1996), 389 – 400, 401–412.

[5] L. A. Bokut, A. A. Klein, *Gröbner-Shirshov bases for the exceptional Lie algebras* E_6, E_7, E_8, To appear in the proceedings of ICAC (1997), Springer-Verlag.

[6] L. A. Borkut, A. A. Klein, *Gröbner-Shirshov bases for exceptional Lie algebras I*, J. Pure Appl. Algebra **133** (1998), 51–57.

[7] L. A. Bokut, P. Malcolmson, *Gröbner-Shirshov basis for a Lie algebra and its universal enveloping algebra*, To appear in the proceedings of ICAC (1997), Springer-Verlag.

[8] B. Buchberger, *An algorithm for finding a basis for the residue class ring of a zero-dimensional ideal*, Ph.D. thesis, University of Innsbruck, 1965.

[9] M. E. Hall, *Verma basis of modules for simple Lie algebras*, Ph.D. thesis, University of Wisconsin, Madison, 1987.

[10] J. E. Humphreys, *Introduction to Lie algebras and representation theory*, Springer-Verlag, New York, 1972.

[11] J. E. Humphreys, *Reflection groups and Coxeter groups*, Cambridge University Press, New York, 1990.

[12] V. G. Kac, *Infinite dimensional Lie algebras*, 3rd ed., Cambridge University Press, 1990.

[13] S.-J. Kang, K.-H. Lee, *Gröbner-Shirshov bases for representation theory*, J. Korean Math. Soc. **37** (2000), 55–72.

[14] S.-J. Kang, K.-H. Lee, *Gröbner-Shirshov basis theory for irreducible* sl_{n+1}*-modules*, J. Algebra **232** (2000), 1–20.

[15] S.-J. Kang, I.-S. Lee, K.-H. Lee, H. Oh, *Hecke algebras, Specht modules and Gröbner-Shirshov bases*, submitted.

[16] P. Lalonde, A. Ram, *Standard Lyndon bases of Lie algebras and enveloping algebras*, Trans. Amer. Math. Soc. **347** (1995), 1821–1830.

[17] G. E. Murphy, *The representations of Hecke algebras of type* A_n, J. Algebra **173** (1995), 97–121.

[18] E. N. Poroshenko, *Gröbner-Shirshov bases for Kac-Moody algebras* $A_n^{(1)}$, M.S. thesis, Novosibirsk State University, 1999.

[19] A. I. Shirshov, *Some algorithmic problems for Lie algebras*, Siberian Math. J. **3** (1962), 292–296.

[20] R. Steinberg, *Lectures on Chevalley groups*, Yale University, New Haven, 1968.

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