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A new symbolic method for solving linear two-point boundary value problems on the level of operators☆

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Abstract

We present a new method for solving regular boundary value problems for linear ordinary differential equations with constant coefficients (the case of variable coefficients can be adopted readily but is not treated here). Our approach works directly on the level of operators and does not transform the problem to a functional setting for determining the Green's function.

We proceed by representing operators as noncommutative polynomials, using as indeterminates basic operators like differentiation, integration, and boundary evaluation. The crucial step for solving the boundary value problem is to understand the desired Green's operator as an oblique Moore–Penrose inverse. The resulting equations are then solved for that operator by using a suitable noncommutative Gröbner basis that reflects the essential interactions between basic operators.

We have implemented our method as a Mathematica™ package, embedded in the TH∃OREM∀ system developed in the group of Prof. Bruno Buchberger. We show some computations performed by this package.

Keywords: Boundary value problems; Differential equations; Operator calculus; Noncommutative Gröbner bases

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1. Introduction

1.1. Two-point boundary value problems

In this article, we consider *boundary value problems* (BVPs) of the following type.[1] Given a forcing function $f \in C^\infty[a, b]$, we want to solve

$$\begin{aligned} Tu &= f, \\ B_0 u = u_0, \ldots, B_{n-1} u &= u_{n-1} \end{aligned} \tag{1}$$

for the unknown function $u \in C^\infty[a, b]$. Here $[a, b]$ is a finite interval of $\mathbb{R}$; T is a linear differential operator of order n; $B_0, \ldots, B_{n-1}$ are boundary operators; and $u_0, \ldots, u_{n-1}$ are constants of $\mathbb{C}$. The differential operator T is given by

$$Tu = c_n \, u^{(n)} + \cdots + c_1 \, u' + c_0 \, u \tag{2}$$

with coefficient functions $c_0, \ldots, c_n \in C^\infty[a, b]$, and the boundary operators B_i are specified by

$$\begin{aligned} B_i u = \; & p_{i,n-1} \, u^{(n-1)}(a) + \cdots + p_{i,1} \, u'(a) + p_{i,0} \, u(a) \\ & + q_{i,n-1} \, u^{(n-1)}(b) + \cdots + q_{i,1} \, u'(b) + q_{i,0} \, u(b), \end{aligned} \tag{3}$$

where the coefficients p_{ij}, q_{ij} are again from $\mathbb{C}$. Note that initial conditions are covered by the special choice of p being the identity matrix and q being the zero matrix.

Analytically, the operator T acts on the Banach space $(C[a, b], || \cdot ||_\infty)$ with dense domain of definition $C^n[a, b]$; see for example (Engl and Nashed, 1981). For our purposes, however, it is better to maintain a *purely algebraic viewpoint*, where the domain of T is the complex vector space $C^\infty[a, b]$, without any prescribed topology.

One can view BVPs as inhomogeneous linear ordinary differential equations (LODEs) that are parametrized in the forcing function f. The occurrence of the parameter f is crucial: it means that one really faces an operator problem—given T and $B_0, \ldots, B_{n-1}$ with $u_0, \ldots, u_{n-1}$, the goal is to find an operator G such that $u = Gf$ fulfills (1). In the literature (Stakgold, 1979), this G is known as the *Green's operator* of the BVP. In the important case of semi-inhomogeneous problems (see Section 2.1), (1) is equivalent to $TG = 1$, $B_0 G = \cdots = B_{n-1} G = 0$; thus G is characterized as a right inverse of T that is annihilated by all the B_i.

1.2. An operator-based approach

Since we have to solve an operator problem, it seems natural to ask for a method that works on the operator level, i.e. one that yields the desired Green's operator G for (1) by performing calculations on various *operators* related to it.

Alternatively, one may also translate the problem to a *functional setting* as done by the standard methods in the literature (Kamke, 1983, pp. 188–190). The crucial idea here is the

[1] For the sake of clarity, we will restrict ourselves to the smooth setting in the sense that all functions involved are C^∞. See the remarks in Section 2.5 for passing to the C^n or distributional setting.

following: For BVPs of the form (1), G can always be written as an integral operator having the so-called Green's function g as its kernel; see Coddington and Levinson (1955). So

$$Gf\,(x) = \int_a^b g(x,\xi)\, f(\xi)\,\mathrm{d}\xi \tag{4}$$

for all $f \in C^\infty[a,b]$ and $x \in [a,b]$. Hence the problem of searching for the *operator* G is reduced to finding the *function* g. (As we will see in the next section, our method also extracts the Green's function g in a postprocessing step. However, this step is optional and may be seen as a translation to the functional formulation of BVPs.)

While the classical translation approach does have its merits, we would like to point out some *advantages* of our new approach:

- It has a greater *potential of generalization*. For example, the whole theory of Green's functions presupposes linear differential operators, and it is far less perspicuous for partial differential equations. (Of course, our method cannot be applied to these problems in the form presented here. However, we can already see some possibilities for adapting it; see Section 4 for a brief discussion of generalizations.)
- From a *conceptual point of view*, it is more satisfying to solve a problem at the level where it is actually stated. Even though one can often solve a problem by transforming it to different domains, a uniform solution method has the additional benefit of structural simplicity and clarity.
- Besides this, our method may be superior in terms of *complexity*. We have not yet embarked on a rigorous analysis of this issue, but there are some indications pointing in this direction: The formula given in Kamke (1983, p. 189) involves Gaussian elimination with functional entries. At least for the important special case of constant-coefficient LODEs considered in this article, our approach avoids that.[2]

1.3. Previous work

The present article summarizes the essential points of the author's *Ph.D. thesis* (Rosenkranz, 2003a) supervised by Bruno Buchberger (first advisor) and Heinz W. Engl (second advisor). It originated in the stimulating atmosphere of the symbolic-numeric "Hilbert Seminars" organized jointly by the two advisors. Some early ideas were published in Rosenkranz et al. (2003a), using a purely heuristic approach without implementation: noncommutative Gröbner bases were computed by the MMA package *NCAlgebra* from UCSD (Helton and Miller, 2004; Helton et al., 1998) on a per-problem basis rather than using a fixed Gröbner basis. A sketchy overview of the thesis was also presented in a poster at ISSAC'03, to be published as a four-page survey in Rosenkranz (2003).

Exact *solution methods* for linear BVPs are of course not new as we have already pointed out (Kamke, 1983; Coddington and Levinson, 1955; Stakgold, 1979). But as far as

[2] Note that the matrix inverse in Lemma 2 involves only numbers.

we know, all these methods typically work on a functional level in the sense discussed in Section 1.2.

Originally we got the inspiration for our method from the paper Helton et al. (1998), which describes the use of noncommutative Gröbner bases for *simplifying* huge terms arising in operator control theory. Using a lexicographic term ordering, however, it is clear that Gröbner bases can do more than that—solving systems of operator equations. And this is essentially what we did on a per-problem basis in our early paper Rosenkranz et al. (2003a); for details, see the remarks at the end of Section 2.1 and the explanations after Theorem 4 in Section 2.5.

Operator-based methods are routinely used in symbolic summation and integration of *holonomic functions*; see Zeilberger (1990), Chyzak and Salvy (1997) and Paule and Strehl (2003). Noncommutative Gröbner bases are applied there for elimination in Ore algebras of operators. But to all our knowledge, the case of BVPs has not yet been analyzed in this frame; we believe that such an investigation could be very profitable. In fact, we plan to come back to this issue in extending our method—see the discussion in Section 4.

1.4. Structure of the article

In Section 2, we describe our new method in detail: Section 2.1 introduces the key concept used in our approach—the noncommutative polynomial ring modeling the relevant operators; besides that we clarify some issues of notation. The fundamental tool to be employed for solving the BVP is the oblique Moore–Penrose inverse; we discuss it in Section 2.2. As we will see there, one can take care of the given boundary conditions by choosing an appropriate nullspace projector for the Moore–Penrose inverse; this is carried out systematically in Section 2.3. For actually solving the given BVP, we will end up with the problem of right inversion, which is treated in Section 2.4. Finally, we will have to simplify the resulting solution operator; as explained in Section 2.5, this will eventually drive us to a convergent term rewriting system or—in other words—to a noncommutative Gröbner basis; we conclude this subsection with a correctness proof of the solution algorithm.

In Section 3, we solve several sample BVPs by our implementation. In Section 3.1, we start out with a brief description of the overall program structure. The first example, presented at some length in Section 3.2, is the classical problem of steady heat conduction in a homogeneous rod. As an example with an exponential Green's function we consider damped oscillations in Section 3.3. A fourth-order equation is treated in Section 3.4, where the physical background is the description of the transverse deflection in a beam.

In the *Conclusion*, we will address various potential generalizations of our method. On a rather direct line of thought, one may consider relaxing several restrictions inherent in the presentation given here—vector versus scalar equations, partial versus ordinary, nonlinear versus linear, underdetermined versus regular problems, integro-differential equations versus purely differential ones. Beyond these direct generalizations, however, we will sketch the contours of what could be a whole new field of computer algebra—a field that we have called "symbolic functional analysis".

2. The solution method

2.1. General set-up

The solution method to be described applies to BVPs of the form (1), subject to the following restrictions:

- We assume that the BVP is *regular* in the sense that there must be a unique solution. This implies that the boundary conditions must be consistent and linearly independent. (See the end of Section 3.5 for a short example of what happens otherwise.)
- We will only cover the *semi-inhomogeneous* case, meaning that $u_0, \ldots, u_{n-1}$ are zero. This involves no loss of generality because any fully inhomogeneous problem can be decomposed into such a semi-inhomogeneous one and a rather trivial BVP with homogeneous differential equation and inhomogeneous boundary conditions; see Stakgold (1979, p. 43).
- In this article we focus on linear differential operators with constant coefficients, moving entirely along the lines of Rosenkranz (2003a). However, our method also works for linear differential operators with variable coefficients: All the results stated here remain valid, with the notable exception of Section 2.4, where we will briefly indicate the necessary modifications. For a more detailed treatment, we refer the reader to the technical report (Rosenkranz, 2003b).

Before we proceed, we establish the following *implicit lambda convention*. Whenever we use a term τ (usually but not necessarily containing a free occurrence of x) in place of a function, we mean the mapping $x \mapsto \tau$ or, in computer-science notation,[3] the lambda term $\lambda x.\tau$. Hence the differentiation operator D acting on functions actually means ∂/∂_x.

In order to apply computer algebra methods, we will eventually model operators by noncommutative polynomials, so let us try to write the operators involved in a polynomial form. For example, consider the *differential operator* informally represented by $T = x^3 D^2 + \mathrm{e}^x D + \sin x$. The coefficient functions $c_2 = x^3$, $c_1 = \mathrm{e}^x$, $c_0 = \sin x$ can be seen as *multiplication operators* in the following sense:[4] any $f \in C^\infty[a, b]$ induces an operator M_f defined by $M_f\, u = f u$ for all $u \in C^\infty[a, b]$. Using this notation, the above operator has to be written as $T = \lceil x^3 \rceil D^2 + \lceil \mathrm{e}^x \rceil D + \lceil \sin x \rceil$, where juxtaposition denotes operator composition (note that this is consistent with the power notation for differentiation) and $\lceil f \rceil$ is a shorthand for M_f. In this way, any linear differential operator can be written as a noncommutative polynomial in the indeterminates D and M_f with f ranging over a certain function domain yet to be fixed.

Turning to *boundary operators*, we have to introduce two more indeterminates. For the above operator T, a typical boundary operator could be $B_0 u = 2u'(a) - 3u(a) + 7u'(b)$. Let us write L and R for evaluation at the left and right boundary, respectively, so $Lu = u(a)$

[3] If necessary, we will designate mappings by the notation $x \mapsto \tau$ rather than $\lambda x.\tau$, so any further occurrences of λ do not have the meaning of the lambda quantifier.

[4] Note that in the following equality juxtaposition on the left-hand side denotes operator application, whereas it denotes the pointwise multiplication of functions on the right-hand side—an abuse of language commonly found in the literature.

and $Ru = u(b)$ for all $u \in C^\infty[a, b]$. (Note that by the implicit lambda convention, these boundary operators actually map functions to functions, namely the constant functions having the corresponding boundary value.) With this notation, the boundary operator B_0 is represented by the noncommutative polynomial $2LD - 3L + 7RD$.

It is now clear how to formulate the differential and boundary operators of (1) in terms of noncommutative polynomials in the indeterminates D, $\lceil f \rceil$, L, R. But this will clearly not be sufficient for representing the operator G supposed to solve (1), since the latter must involve *integration*. Hence we introduce the following operator A for computing the antiderivative

$$Af = \int_a^x f(\xi)\,\mathrm{d}\xi$$

of any function $f \in C^\infty[a, b]$. Since we know that the n-th derivative of the Green's function jumps along the diagonal, we will also include the dual of A, namely the operator

$$Bf = \int_x^b f(\xi)\,\mathrm{d}\xi,$$

such that the integral (4) can be patched by adding A and B portions (see Section 3 for examples).

Let us now formally introduce the underlying *polynomial ring*. The domain $\mathfrak{F}$ used for parametrizing the multiplication operators will be introduced in Section 2.5. For the moment, it is sufficient to think of it as the $\mathbb{C}$-algebra $\mathfrak{Exp}$ with basis $\mathfrak{Exp}^\# = \{x^n e^{\lambda x} \mid n \in \mathbb{N} \wedge \lambda \in \mathbb{C}\}$; we call this the polyexponential algebra $\mathfrak{Exp}$. (Every algebra $\mathfrak{A}$ considered here is assumed to include the notion of a distinguished basis referred to as $\mathfrak{A}^\#$.)

Definition 1. Let $\mathfrak{F}$ be an analytic algebra. Then the noncommutative polynomial ring

$$\mathbb{C}\langle\{D, A, B, L, R\} \cup \{\lceil f \rceil \mid f \in \mathfrak{F}^\#\}\rangle$$

is called the ring of *analytic polynomials* over $\mathfrak{F}$, denoted by $\mathfrak{An}(\mathfrak{F})$.

Strictly speaking, we should from now on distinguish between the *formal operators* in $\mathfrak{An}(\mathfrak{F})$ and the *actual operators* in $L(C^\infty[a, b], C^\infty[a, b])$. Most of the time, however, it is either clear which of the two concepts we mean or a certain statement is true for both of them. In order not to overload notation, we will therefore abstain from making this difference explicit—except for Theorem 5, where it is really crucial. If the reader desires a more rigorous treatment, she may want to consult Rosenkranz (2003a).

Using the ring $\mathfrak{An}(\mathfrak{F})$, the operator-theoretic formulation of (1) can be written as a system of polynomial equations. However, this implies also that all the basic operators occurring as indeterminates are void of any analytic meaning. Therefore we have to add appropriate *interaction equalities* for algebraically capturing their essential properties. For example, the interaction between differentiation and multiplication operators is stated in the well-known "Leibniz equality". For other operator interactions, the corresponding equalities are less obvious, and completeness questions (confluence, termination, adequacy) become urgent.

For the moment, however, let us postpone these issues to Section 2.5, where we show the full polynomial system along with the corresponding completeness theorems. So we assume we have an appropriate reduction system, which we can employ for solving the given polynomial system $TG = 1$ and $B_0G = \cdots = B_{n-1}G = 0$. In principle, we could merge these equations with the interaction equalities, impose a lexicographic term order, and feed the resulting system into a noncommutative Gröbner basis solver; this is essentially what we have done in Rosenkranz et al. (2003a). However, we can do much better than that by using a *generic preprocessing strategy* that avoids the costly computation of a new Gröbner basis for each BVP of type (1).

2.2. *The Moore–Penrose inverse*

The key to solving the given polynomial system is the so-called *Moore–Penrose inverse*, also known as generalized inverse: Introduced by Moore in Moore (1920), the concept of generalized inverse received almost no attention until its rediscovery by Penrose in Penrose (1955, 1956); see for example Nashed (1976) and Engl et al. (1996) for a modern treatment. The Moore–Penrose inverse provides a substitute for inverting a nonbijective linear operator in any vector space—including the space $C^\infty[a, b]$ used in our case.

Why would we want to do this? For a linear differential operator T, we have to solve $TG = 1$ for G, subject to the additional conditions $B_0G = \cdots = B_{n-1}G = 0$, which serve to determine the solution uniquely. So we seek a special right inverse G of T. The usual way of seeing this is that G is the *full inverse* (not just right inverse) of the operator T by restricting the domain of the latter to those functions in $C^\infty[a, b]$ that fulfill the given boundary conditions.

Though theoretically elegant, this interpretation is not adequate for our purposes since it encodes the boundary conditions in the domain, which is not readily available for computation. It is therefore more promising to see the given operator T as nonbijective, having all of $C^\infty[a, b]$ as its domain—just like the basic operators $D, A, B, L, R, \lceil f \rceil$. Doing this, we can employ the Moore–Penrose theory for finding *generalized inverses* of T. In general, there will be many such inverses, so we must find some means of singling out the unique one that fulfills the given boundary conditions.

This can be achieved by using *oblique Moore–Penrose inverses* (Nashed, 1976, pp. 57–61). The idea is the following: An arbitrary linear operator T between two vector spaces X and Y may fail to be injective, so its nullspace N is typically nontrivial. In order to cure this, one takes a complement M: choose a projector P onto N and set $M = (1 - P)X$. The operator $T|_M$ is then invertible as a map from X to the range R. Furthermore, T may fail to be surjective, so R will typically not exhaust all of Y. For repairing this, one chooses a projector Q onto R, calls the corresponding complement $S = (1 - Q)Y$ and adjoins S as a nullspace to $(T|_M)^{-1}$. The resulting operator is called the oblique Moore–Penrose inverse of T with respect to the nullspace projector P and range projector Q; it is denoted by $T^\dagger_{P,Q}$. The freedom in choosing these projectors is crucial for incorporating the boundary conditions.

What makes the Moore–Penrose inverse particularly attractive for symbolic computation is that it can be characterized uniquely by the four so-called *Moore–Penrose equations*.[5] Let us briefly recall them here for reference purposes.

Theorem 1. *Let X and Y be vector spaces, T a linear operator from X to Y. Choose projectors P and Q to the nullspace and range of T, respectively, and let M and S be the corresponding complements. Then the oblique Moore–Penrose inverse is characterized uniquely as the linear operator $T^\dagger$ from Y to X that fulfills the equations*

$$TT^\dagger T = T, \tag{5}$$

$$T^\dagger T T^\dagger = T^\dagger, \tag{6}$$

$$T^\dagger T = 1 - P, \tag{7}$$

$$TT^\dagger = Q. \tag{8}$$

Furthermore, $T^\dagger$ has nullspace S and range M.

In our setting, it is already clear that Q must be the identity operator 1, because any linear differential operator is surjective on $C^\infty[a, b]$. But then (5) and (6) obviously follow from (8). So we are left with the two Eqs. (7) and (8). It turns out, however, that we can even restrict ourselves to (7) because (8) follows from it as we will show now.

Lemma 1. *The operator equation $TG = 1$ follows from $GT = 1 - P$, where P is some nullspace projector for the linear differential operator T.*

Proof. Let T^* be any right inverse of T (there is always a right inverse or—in other words—a fundamental solution for T, and we will construct a particular one in Section 2.4). Then premultiplying $GT = 1 - P$ by T and postmultiplying by T^* yields $TGTT^* = TT^* - TPT^*$. Now by the choice of T^* we have $TT^* = 1$; and since P projects onto the nullspace of T, we have $TP = 0$. Hence $TG = 1$ as claimed. □

As a consequence, we need only consider the equation $GT = 1 - P$, but we must take care to choose P in such a way that the *boundary conditions* $B_0\, G = \cdots = B_{n-1}\, G = 0$ are fulfilled. Then we can be sure that G is actually the Green's operator: Since it is uniquely determined, it must coincide with the single Moore–Penrose inverse of T corresponding to that choice of P that incorporates the boundary conditions.

2.3. Computation of the nullspace projector

For that purpose, we use the fact mentioned at the end of Theorem 1, namely that the range of G is given by

$$(1 - P)\, C^\infty[a, b] = \{v - P\, v \mid v \in C^\infty[a, b]\}.$$

So if we want to ensure that the solution $u = Gf$ respects the boundary conditions $B_0 u = \cdots = B_{n-1} u = 0$ for any $f \in C^\infty[a, b]$, it suffices to construct P in such a

[5] Quoting (Steinberg, private communcation): "Functional analysis was developed to make analysis look like algebra (usually algebras of operators looking like matrices), so using functional analysis to do analysis problems in computer algebra is natural".

way that all the $v - Pv$ respect them—so we have to require

$$\begin{aligned} B_0 P v &= B_1 v \\ &\cdots \\ B_{n-1} P v &= B_{n-1} v \end{aligned} \tag{9}$$

for all $v \in C^\infty[a, b]$. This amounts to a small *linear interpolation* problem, to be solved in the next lemma.

For the sake of convenience, let us introduce some *matrix notation* (we will use overhat symbols for denoting vectors and matrices). We write $\hat{D}_n$ for the operator-valued vector $(1, D, D^2, \ldots, D^{n-1})$. With this notation, the vector boundary operator $\hat{B} = (B_0, \ldots, B_{n-1})$ can be written as $(L\hat{l} + R\hat{r})\hat{D}_n$ for suitable coefficient matrices $\hat{l}, \hat{r} \in \mathbb{R}^{n\times n}$. In fact, using the notation of (3), these matrices are given by

$$\hat{l} = \begin{pmatrix} p_{1,0} & p_{1,1} & \cdots & p_{1,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n,0} & p_{n,1} & \cdots & p_{n,n-1} \end{pmatrix}, \qquad \hat{r} = \begin{pmatrix} q_{1,0} & q_{1,1} & \cdots & q_{1,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ q_{n,0} & q_{n,1} & \cdots & q_{n,n-1} \end{pmatrix}.$$

We are now ready to state a concise *formula for computing the nullspace projector* in terms of $\hat{l}, \hat{r}$ and a fundamental matrix for T.

Lemma 2. *Let $\hat{w}$ be a fundamental matrix for the linear differential operator T, and let $\hat{l}, \hat{r}$ be the boundary matrices corresponding to $B_0, \ldots, B_{n-1}$ as introduced above. Compute*

$$\mathrm{Proj}_{\hat{w}}(\hat{l}, \hat{r}) = \lceil \hat{w}_1 \rceil\, (\hat{l}\hat{w}^{\leftarrow} + \hat{r}\hat{w}^{\rightarrow})^{-1}(L\hat{l} + R\hat{r})\hat{D}_n,$$

where $\hat{w}_1$ denotes the first row of $\hat{w}$ and $\hat{w}^{\leftarrow}$ and $\hat{w}^{\rightarrow}$ arise from $\hat{w}$ by evaluation at a and b, respectively. Then $\mathrm{Proj}_{\hat{w}}(\hat{l}, \hat{r})$ is a projector onto the nullspace of T that fulfills (9).

Proof. Let T be an operator of the form (2) and let $B_0, \ldots, B_{n-1}$ be boundary operators of the form (3) with corresponding boundary matrices $\hat{l}, \hat{r}$. Furthermore, let $\varphi_1, \ldots, \varphi_n$ be a fundamental system for T; hence the fundamental matrix $\hat{w}$ has rows $(\varphi_1^{(i)}, \ldots, \varphi_n^{(i)})$ for $i = 0, \ldots, n-1$.

We will now set up a generic linear operator P that projects onto the nullspace of T and then fit it against the conditions (9). Take an arbitrary $v \in C^\infty[a, b]$. Since the nullspace of T is spanned by $\varphi_1, \ldots, \varphi_n$, we must have $Pv = c_1(v)\varphi_1 + \cdots + c_n(v)\varphi_n$ for some coefficients $c_1, \ldots, c_n \in \mathbb{C}$ depending on v. Writing this in vector form, we have $Pv = \hat{w}_1\hat{c}(v)$, which yields the matrix equation $\hat{B}\hat{w}_1\hat{c}(v) = \hat{B}v$ upon substitution in (9). Now

$$\hat{B}\hat{w}_1 = (L\hat{l} + R\hat{r})\hat{D}_n\hat{w}_1 = (L\hat{l} + R\hat{r})\hat{w} = \hat{l}\hat{w}^{\leftarrow} + \hat{r}\hat{w}^{\rightarrow},$$

so $\hat{c}(v) = (\hat{l}\hat{w}^{\leftarrow} + \hat{r}\hat{w}^{\rightarrow})^{-1}\hat{B}v$, which yields $P = \mathrm{Proj}_{\hat{w}}(\hat{l}, \hat{r})$ as claimed in the lemma. □

Note that that the *matrix inversion* occurring in the Lemma 2 involves only a matrix of numerical constants rather than functional terms. This is crucial for complexity considerations.

2.4. Right inversion

We have now reduced the BVP (1) to the single equation $GT = 1 - P$, where P is the nullspace projector $\mathrm{Proj}_{\hat{w}}(\hat{l}, \hat{r})$ specified in Lemma 2 with $\hat{w}$ the fundamental matrix for T and $\hat{l}, \hat{r}$ the boundary matrices corresponding to $B_0, \ldots, B_{n-1}$. In order to solve this equation for G, it suffices to find a *right inverse* T^* of T; then G is obtained as $(1 - P)T^*$. We will construct one particular such right inverse, which we will denote by $T^{\blacklozenge}$.

It turns out that one can always find right inverses of T that can be written in a form analogous to (4) with a binary function g^*; in the literature (Kamke, 1983, p. 74), this function is known as the *fundamental solution* of the inhomogeneous differential equation $Tu = f$. The fundamental solution plays a role somewhat similar to the Green's function: When applying the corresponding integral operator to the forcing function f, it yields a solution u of the inhomogeneous equation, but it does not incorporate boundary conditions. In Section 2.5, we will show how to recover such a fundamental solution from the right inverse $T^{\blacklozenge}$ considered here.

As announced in Section 2.1, we will stick to the important special case of linear differential operators with *constant coefficients* along the lines of Rosenkranz (2003a). The general case of variable coefficients is treated in full detail in Rosenkranz (2003b), and we will also make a few remarks about it here.

Lemma 3. *If T is of the form (2) with constant coefficient functions $c_0, \ldots, c_n$, the operator*

$$T^{\blacklozenge} = \prod_{i=1}^{n} \lceil e^{\lambda_i x} \rceil A \lceil e^{-\lambda_i x} \rceil$$

is a right inverse, if $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ are the roots of the characteristic polynomial of T (repeated according to their multiplicities).

Proof. For arbitrary $\lambda \in \mathbb{C}$, the differential operator $D - \lambda$ has $\lceil e^{\lambda x} \rceil A \lceil e^{-\lambda x} \rceil$ as a right inverse as one can see by straightforward computation, using the product rule of differentiation and the fundamental theorem of calculus (see Section 2.5 for a precise listing of admissible reduction rules). The formula then follows since

$$T = (D - \lambda_1) \cdots (D - \lambda_n)$$

and operator composition is associative. □

As mentioned before, it is also possible to derive a similar though somewhat more complicated formula for linear differential operators with *variable coefficients*; see Rosenkranz (2003b) for the details. The crucial idea is to iterate a procedure that is typically called "reduction of order" in the literature (Coddington and Levinson, 1955, p. 84). As opposed to the case of constant coefficients, though, the analytic algebra needed for the formulation of $T^{\blacklozenge}$ will in general go beyond the polyexponential algebra $\mathfrak{Exp}$.

It should be emphasized, however, that the formula given above is particularly *simple*, taking advantage of the special structure of linear differential operators with constant coefficients. There seems to be no such advantage when applying the procedure from Kamke (1983) to this important special case.

2.5. The reduction system

Using the above results, we can compute the desired Green's operator G as $(1 - P)T^{\blacklozenge}$, where P is again $\mathrm{Proj}_{\hat{w}}(\hat{l}, \hat{r})$ as in Lemma 2 and $T^{\blacklozenge}$ is the right inverse specified in Lemma 3. However, we might obtain G in a somewhat *unconventional form*: for example, in the BVP for the heat equation (see Section 3.2), we have $T = D^2$ and $B_0 = L$, $B_1 = R$. In this case, Lemma 2 yields $P = \lceil 1 - x \rceil L + \lceil x \rceil R$, while Lemma 3 gives us of course $T^{\blacklozenge} = A^2$. Hence we have $G = (1 - \lceil 1 - x \rceil\, L + \lceil x \rceil\, R)\, A^2$. Written in this form, the Green's operator G uses double integration, and we cannot compare it with the classical kernel representation (4) for reading off the Green's function g associated with it.

Using the obvious simplification $LA = 0$, we can also rewrite G into $A^2 + \lceil x \rceil RA^2$. The representation via the Green's function in Section 3.2 is a third possibility. In general, there are many different polynomials in $\mathfrak{An}(\mathfrak{Exp})$ with the same interpretation as an operator on $C^\infty[a, b]$. Our goal is to organize rewriting in such a way that there is always a *unique* final result, which will moreover correspond to the classical kernel representation.

But before doing so, we would like to point out that the issue of representations is actually peripheral to the original problem of solving a BVP of the form (1): whatever representation of G we take, when we apply it to a given forcing function f, we will end up with the unique solution $u = Gf$ of the BVP—as long as the reduction system is *sound* in the sense to be discussed now.

In order to realize our goal, we have to set up an appropriate reduction system on the ring of analytic polynomials. As usual, the reductions are first specified for a set of monomials and then extended in the obvious way—see for example Bergman (1978). The reduction system should have the following five *key properties*:

- It must be *sound* in the sense that each polynomial equality becomes a valid identity of operators when interpreted as discussed before.
- It must be *adequate* in the sense that it provides "enough" reductions for algebraizing all the analytic knowledge relevant here.
- In order to solve the problem of unique representation addressed above, we require it to be *confluent*: there is no more than one normal form.
- Besides this, every simplification should terminate, i.e. the reduction system must be *noetherian*: there is at least one normal form.
- The normal forms of the reduction system should correspond exactly to the Green's functions of the classical kernel representation (4). Hence we will also refer to these normal forms as *Green's polynomials*.

The reduction system in Table 1—we have called it the Green's system—fulfills all these requirements. For a complete *proof* of this statement, see Rosenkranz (2003a); here we will only give a rough outline of the main steps in this proof.

First of all, let us clarify the role of the *analytic algebra* $\mathfrak{F}$ already mentioned in Definition 1; the variables f and g in Table 1 range over its basis $\mathfrak{F}^{\#}$. Analytic algebras are simply algebras with a few additional operations fufilling certain axioms that make them behave similarly to their analytic models—just like differential algebras, which can be seen as halfway between plain algebras and analytic algebras.

Definition 2. An algebra $\mathfrak{F}$ is called an *analytic algebra* iff it has five linear operations:[6] differentation $'$: $\mathfrak{F} \to \mathfrak{F}$, integral $\int^* : \mathfrak{F} \to \mathfrak{F}$, cointegral $\int_* : \mathfrak{F} \to \mathfrak{F}$, left boundary value $^\leftarrow : \mathfrak{F} \to \mathbb{C}$, right boundary value $^\rightarrow : \mathfrak{F} \to \mathbb{C}$ such that the seven axioms

$$\begin{aligned}
(fg)' &= f'g + fg',\\
\textstyle\int^* f' &= f - f^\leftarrow,\\
\textstyle\int_* f' &= f^\rightarrow - f,\\
(\textstyle\int^* f)' &= f,\\
(\textstyle\int_* f)' &= -f,\\
(fg)^\leftarrow &= f^\leftarrow g^\leftarrow,\\
(fg)^\rightarrow &= f^\rightarrow g^\rightarrow
\end{aligned}$$

are fulfilled.

We observe that the above *axioms* are very natural:[7] the first is the product rule for differentiation, thus making analytic algebras a special case of differential algebras (where this axiom is usually called the Leibniz rule). The next four axioms state that the integral and the negative cointegral are oblique Moore–Penrose inverses of differentiation, having as nullspace projectors the left and right boundary value, respectively (with trivial range projectors in both cases); cf. the Moore–Penrose equations in Theorem 1. So the operations $^\leftarrow$ and $^\rightarrow$ serve to choose among the oblique Moore–Penrose inverses by fixing the integration constant. The last two axioms stipulate that $f \mapsto (x \mapsto f^\leftarrow)$ and $f \mapsto (x \mapsto f^\rightarrow)$ be homomorphisms in the algebra $\mathfrak{F}$.

As mentioned before, a typical choice for $\mathfrak{F}$ is the polyexponentials $\mathfrak{Exp}$. It can easily be verified that they form an analytic algebra. Of course its operations will in general transform basis elements to nonbasis elements; for example, $x\mathrm{e}^x \in \mathfrak{Exp}^\#$ becomes $\mathrm{e}^x + x\mathrm{e}^x \in \mathfrak{Exp} \setminus \mathfrak{Exp}^\#$ under differentiation. So strictly speaking, the right-hand sides of Table 1 may not be polynomials of $\mathfrak{An}(\mathfrak{F})$. Therefore the reduction rules must be understood as containing an implicit *basis reduction* after applying them: Any occurrence of a monomial $\cdots \lceil f \rceil \cdots$ with $f \in \mathfrak{F} \setminus \mathfrak{F}^\#$ is replaced by $\sum c_i \cdots \lceil f_i \rceil \cdots$, where $\sum c_i f_i$ is the basis expansion of f with nonzero coefficients $c_i \in \mathbb{C}$ and basis functions $f_i \in \mathfrak{F}^\#$.

The axioms for analytic algebras play a crucial role in *establishing the confluence* of the Green's system. What we have actually proved is that for every analytic algebra $\mathfrak{F}$, the system of Table 1 establishes a confluent reduction on the ring of analytic polynomials $\mathfrak{An}(\mathfrak{F})$. It is enough to consider the case $\mathfrak{F}^\# = \mathfrak{F}$, as one can easily see. By Lemma 1.2 of Bergman (1978), it suffices to prove that all overlap ambiguities of the reduction system are resolvable (in general, one also has to consider inclusion ambiguities, but by inspecting

[6] Note that these operations correspond—in the given order—to the indeterminates D, A, B, L, R of $\mathfrak{An}(\mathfrak{F})$, while each element $f \in \mathfrak{F}$ corresponds to the multiplication operator $\lceil f \rceil$.

[7] We have obtained these axioms by starting the confluence proof with an empy list of axioms, gradually adding whatever properties we needed in order to overcome failing proofs. In the end, we simplified the resulting requirements, coming up with the above axioms. This procedure is an instance of what Bruno Buchberger has called the *Lazy Thinking Pardadigm*. It is implemented in TH∃OREM∀ for various provers on natural numbers and tuples; see Buchberger (2003).

Table 1
The Green's system

Equalities for algebraic simplification:	*Equalities for contracting integration operators:*
$\lceil f \rceil \lceil g \rceil \to \lceil fg \rceil$	$A \lceil f \rceil A \to \lceil \int^* f \rceil A - A \lceil \int^* f \rceil$
Equalities for isolating differential operators:	$A \lceil f \rceil B \to \lceil \int^* f \rceil B + A \lceil \int^* f \rceil$
	$B \lceil f \rceil A \to \lceil \int_* f \rceil A + B \lceil \int_* f \rceil$
$D\,A \to 1$	$B \lceil f \rceil B \to \lceil \int_* f \rceil B - B \lceil \int_* f \rceil$
$D\,B \to -1$	$A\,A \to \lceil \int^* 1 \rceil A - A \lceil \int^* 1 \rceil$
$D \lceil f \rceil \to \lceil f \rceil D + \lceil f' \rceil$	$A\,B \to \lceil \int^* 1 \rceil B + A \lceil \int^* 1 \rceil$
$D\,L \to 0$	$B\,A \to \lceil \int_* 1 \rceil A + B \lceil \int_* 1 \rceil$
$D\,R \to 0$	$B\,B \to \lceil \int_* 1 \rceil B - B \lceil \int_* 1 \rceil$
Equalities for isolating boundary operators:	*Equalities for absorbing integration operators:*
$L\,A \to 0$	$A \lceil f \rceil D \to -f^{\leftarrow} L + \lceil f \rceil - A \lceil f' \rceil$
$R\,A \to A + B$	$B \lceil f \rceil D \to f^{\rightarrow} R - \lceil f \rceil - B \lceil f' \rceil$
$L\,B \to A + B$	$A\,D \to -L + 1$
$R\,B \to 0$	$B\,D \to R - 1$
$L \lceil f \rceil \to f^{\leftarrow} L$	$A \lceil f \rceil L \to \lceil \int^* f \rceil L$
$R \lceil f \rceil \to f^{\rightarrow} R$	$B \lceil f \rceil L \to \lceil \int_* f \rceil L$
$L\,L \to L$	$A \lceil f \rceil R \to \lceil \int^* f \rceil R$
$L\,R \to R$	$B \lceil f \rceil R \to \lceil \int_* f \rceil R$
$R\,L \to L$	$A\,L \to \lceil \int^* 1 \rceil L$
$R\,R \to R$	$B\,L \to \lceil \int_* 1 \rceil L$
	$A\,R \to \lceil \int^* 1 \rceil R$
	$B\,R \to \lceil \int_* 1 \rceil R$

Table 1 we see that we do not have any inclusions). We do this in the usual manner by showing that the S-polynomial $w_2 p_1 - p_2 w_1$ reduces to 0 for any pair of rules $w w_1 \to p_1$ and $w_2 w \to p_2$.

It turns out that there are 233 S-polynomials to be considered, and the task of doing all these reductions is rather daunting. It is therefore preferable to *automate the proof*. As we have implemented the whole algorithm for computing Green's operators in the TH∃OREM∀ system (see Section 3.1 for some details), it seems natural to do this also in TH∃OREM∀—a neat example of how this system offers support on various levels: here, on the object level of computation (using the reduction system for computing as explained below) as well as on the meta level of proof (verifying properties of the system, like confluence in our case). For the general philosophy of treating object and meta levels, see Buchberger (1999).

Table 2
Fragment of the confluence proof

The rules DA and AMA yield the S-polynomial:

$$\lceil f \rceil A - D \left\lceil \int^* f \right\rceil A + DA \left\lceil \int^* f \right\rceil \stackrel{(\ldots)}{\stackrel{\downarrow}{=}}$$

$$\lceil f \rceil A - D \left\lceil \int^* f \right\rceil A + \boxed{DA} \left\lceil \int^* f \right\rceil \stackrel{(DA)}{\stackrel{\downarrow}{=}}$$

$$\left\lceil \int^* f \right\rceil + \lceil f \rceil A - \boxed{D \left\lceil \int^* f \right\rceil} A \stackrel{(DM)}{\stackrel{\downarrow}{=}}$$

$$\left\lceil \int^* f \right\rceil + \lceil f \rceil A - \left\lceil \boxed{\left(\int^* f\right)'} \right\rceil A - \left\lceil \int^* f \right\rceil DA \stackrel{(da)}{\stackrel{\downarrow}{=}}$$

$$\left\lceil \int^* f \right\rceil - \left\lceil \int^* f \right\rceil \boxed{DA} \stackrel{(DA)}{\stackrel{\downarrow}{=}}$$

$0 \quad \square$

...

The rules DA and AMA yield the S-polynomial:

$$A \lceil f \rceil A + B \lceil f \rceil A - R \left\lceil \int^* f \right\rceil A + RA \left\lceil \int^* f \right\rceil \stackrel{(\ldots)}{\stackrel{\downarrow}{=}}$$

$$A \lceil f \rceil A + B \lceil f \rceil A - R \left\lceil \int^* f \right\rceil A + \boxed{RA} \left\lceil \int^* f \right\rceil \stackrel{(RA)}{\stackrel{\downarrow}{=}}$$

$$A \left\lceil \int^* f \right\rceil + B \left\lceil \int^* f \right\rceil + A \lceil f \rceil A + B \lceil f \rceil A - \boxed{R \lceil \int^* f \rceil} A \stackrel{(RM)}{\stackrel{\downarrow}{=}}$$

$$A \left\lceil \int^* f \right\rceil + B \left\lceil \int^* f \right\rceil - \boxed{\left(\int^* f\right)^{\rightarrow}} RA + A \lceil f \rceil A + B \lceil f \rceil A \stackrel{(ra)}{\stackrel{\downarrow}{=}}$$

$$A \left\lceil \int^* f \right\rceil + B \left\lceil \int^* f \right\rceil - \left(\oint f\right) \boxed{RA} + A \lceil f \rceil A + B \lceil f \rceil A \stackrel{(RA)}{\stackrel{\downarrow}{=}}$$

$$-\left(\oint f\right) A - \left(\oint f\right) B + A \left\lceil \int^* f \right\rceil + B \left\lceil \int^* f \right\rceil + \boxed{A \lceil f \rceil A} + B \lceil f \rceil A \stackrel{(AMA)}{\stackrel{\downarrow}{=}}$$

$$-\left(\oint f\right) A - \left(\oint f\right) B + B \left\lceil \int^* f \right\rceil + \left\lceil \int^* f \right\rceil A + \boxed{B \lceil f \rceil A} \stackrel{(BMA)}{\stackrel{\downarrow}{=}}$$

$$-\left(\oint f\right) A - \left(\oint f\right) B + B \left\lceil \int^* f \right\rceil + B \left\lceil \boxed{\int_* f} \right\rceil + \left\lceil \int^* f \right\rceil A + \left\lceil \boxed{\int_* f} \right\rceil A \stackrel{(b)}{\stackrel{\downarrow}{=}}$$

$0 \quad \square$

For the automated proof, we had to hand-prove some auxiliary equalities that are valid in any analytic algebra $\mathfrak{F}$. These equalities are mainly integral theorems like

$$\int^* \left(f \; \left(\int^* f\right)\right) = \tfrac{1}{2} \left(\int^* f\right)^2;$$

see Rosenkranz (2003a) for details. Tables 2 and 3 show a small fragment of the *actual confluence proof* (everything in these tables is verbatim TH∃OREM∀ output), which

Table 3
Fragment of the confluence proof (cont'd)

The rules BR and RR yield the S-polynomial:

$$-BR + \lceil \textstyle\int_* 1 \rceil R^2 \overset{(\ldots)}{\stackrel{\downarrow}{=}}$$

$$-BR + \left\lceil \boxed{\textstyle\int_* 1} \right\rceil R^2 \overset{(b)}{\stackrel{\downarrow}{=}}$$

$$-BR + (\textstyle\oint 1) \boxed{R^2} - \lceil \textstyle\int^* 1 \rceil R^2 \overset{(RR)}{\stackrel{\downarrow}{=}}$$

$$(\textstyle\oint 1)R - BR - \lceil \textstyle\int^* 1 \rceil \boxed{R^2} \overset{(RR)}{\stackrel{\downarrow}{=}}$$

$$(\textstyle\oint 1)R - \boxed{BR} - \lceil \textstyle\int^* 1 \rceil R \overset{(BR)}{\stackrel{\downarrow}{=}}$$

$$(\textstyle\oint 1)R - \left\lceil \boxed{\textstyle\int_* 1} \right\rceil R - \lceil \textstyle\int^* 1 \rceil R \overset{(b)}{\stackrel{\downarrow}{=}}$$

$$0 \quad \square$$

☑ Computed 233 S-polynomials in 129 seconds.
☑ Reduced them in 3144 seconds.
☑ All of them reduced to zero!
□

Table 4
Grammar of Green's polynomials

Production rule	Name
$\mathcal{M} ::= \mathcal{AIA} \mid \mathcal{AD} \mid \mathcal{ABD}$	Monomial operator
$\mathcal{I} ::= A \mid B$	Integral operator
$\mathcal{A} ::= 1 \mid \lceil f \rceil$	Algebraic operator
$\mathcal{B} ::= L \mid R$	Boundary operator
$\mathcal{D} ::= 1 \mid D\mathcal{D}$	Differential operator

covers approximately 2000 lines altogether. In every intermediate expression, the redex is framed by the system in order to improve readability. The uppercase letters above the equality symbol refer to the corresponding rules of Table 1 (the names are derived from the monomial on the left-hand side, with multiplication operators generically denoted by the letter M); the lowercase letters refer to the auxiliary equalities. The expression $\oint f$, with $f \in \mathfrak{F}$, is an abbreviation for the "definite integral" $\int^* f + \int_* f$.

For establishing the *termination* of the Green's system, we have given two different proofs in Rosenkranz (2003a). The more intuitive proof uses the idea of various termination terms associated with the rules. For example, several rules decrease the "differential weight" (the number of occurrences of the indeterminate D), whereas none of the rules increases it. The other proof proceeds on a more algebraic line: We set up a suitable graded lexicographic ordering on the word monoid Ω^* over $\Omega = \{D, A, B, L, R, M\}$, which is then extended to a well-ordering on the system of finite subsets of Ω^*. This well-ordering induces a noetherian strict partial order on $\mathfrak{An}(\mathfrak{F})$ by identifying all $\lceil f \rceil$ with M and taking the support of the resulting polynomial. Hence it suffices to prove that the reductions are compatible with this induced order—which is easily achieved.

Summarizing the previous two results, we have proved convergence (confluence and termination) for the Green's system.

Theorem 2. *For any analytic algebra $\mathfrak{F}$, the system in Table 1 constitutes a convergent rewrite system on the ring of analytic polynomials $\mathfrak{An}(\mathfrak{F})$.*

As mentioned before, we can also characterize the normal forms (which always exist and are unique by the preceding theorem), and they will turn out to be precise analogs of the Green's functions.

Definition 3. A polynomial of $\mathfrak{An}(\mathfrak{F})$ is said to be a *Green's polynomial* iff all its monomials are produced by the rule $\mathcal{M}$ of the grammar in Table 4. We denote the set of Green's polynomials by $\mathfrak{Gr}_{\downarrow}(\mathfrak{F})$.

Theorem 3. *The normal forms of $\mathfrak{An}(\mathfrak{F})$ with respect to the reduction system specified in Table 1 are precisely the Green's polynomials $\mathfrak{Gr}_{\downarrow}(\mathfrak{F})$.*

The proof of the preceding theorem is rather straightforward, albeit slightly technical. It is easy to see that any Green's polynomial is indeed irreducible. For proving the converse, one takes an arbitrary monomial $p \in \mathfrak{An}(\mathfrak{F}) \setminus \mathfrak{Gr}_{\downarrow}(\mathfrak{F})$ and shows that it is reducible, using a case distinction on the first letters of p. Despite its rather technical proof, the statement of the theorem is actually *very intuitive*: Any linear integro-differential-boundary operator must be a superposition of purely integral or differential or boundary operators (algebraic operators can be seen as zero-order differential operators). This is clear: on the monomial level, integration and differentiation cancel each other, and boundary evaluation collapses the functional range to a single point.

It is now easy to see why a Green's polynomial allows us to read off the corresponding *Green's function*. Since we know that the "differential weight" is invariant under the Green's system, the normal form of a Green's operator cannot be of type $\mathcal{AD}$ or $\mathcal{ABD}$; hence it must be of type $\mathcal{AIA}$. So each monomial has the form $\lceil f \rceil A \lceil g \rceil$ or $\lceil f \rceil B \lceil g \rceil$, where f or g may also be 1; it contributes the term $f(x)g(\xi)$ to the "upper" or "lower" part of a Green's function defined by the case distinction

$$g(x, \xi) = \begin{cases} \text{upper}(x, \xi) & \text{if } a \le \xi \le x \le b, \\ \text{lower}(x, \xi) & \text{if } a \le x \le \xi \le b, \end{cases}$$

reflecting the characteristic jump on the diagonal of $[a, b] \times [a, b]$.

One can also extract a binary function h from the right inverse $T^{\blacklozenge}$ of the given differential operator T just as one extracts the Green's function g from the corresponding Green's operator G. In the literature, the function h is known as the *fundamental solution* of the differential equation $Tu = f$. Its role is similar to g, except that it ignores boundary conditions: for any forcing function f, the convolution defined by (4), with h instead of g, yields *some* solution u of the differential equation $Tu = f$. Comparing this with the relation $G = (1 - P)T^{\blacklozenge}$, we gain a new interpretation of the fundamental solution: it is the "Green's function" associated with the trivial nullspace projector $P = 0$ (which can never arise from the boundary conditions of a regular BVP).

Before clarifying the relations between the actual operators acting on $C^\infty[a, b]$ and their formal counterparts in the algebraic structure $\mathfrak{An}(\mathfrak{F})$, let us investigate the latter just a bit more: it is highly instructive to interpret the results about the Green's system from a purely *ring-theoretic perspective*.

Definition 4. Let $\mathfrak{F}$ be an analytic algebra. Then $\mathfrak{Gr}_0(\mathfrak{F})$ denotes the *Green's system* over $\mathfrak{F}$, i.e. the set of all polynomials $l - r$ where $l \to r$ is a rule of the reduction system in Table 1 and the variables f, g range over all of $\mathfrak{F}^\#$. Furthermore, $\mathfrak{Gr}(\mathfrak{F})$ denotes the two-sided ideal generated by $\mathfrak{Gr}_0(\mathfrak{F})$ in $\mathfrak{An}(\mathfrak{F})$; we call it the *Green's ideal* over $\mathfrak{F}$.

Theorem 4. *For any analytic algebra $\mathfrak{F}$, the Green's system $\mathfrak{Gr}_0(\mathfrak{F})$ constitutes a noncommutative Gröbner basis for the ideal $\mathfrak{Gr}(\mathfrak{F})$ in $\mathfrak{An}(\mathfrak{F})$.*

The notion of *Gröbner bases* was originally introduced in the "classical" context of commutative polynomials by Bruno Buchberger in his Ph.D. Thesis (Buchberger, 1965); see also the journal version Buchberger (1970) and a concise treatment in Buchberger (1998). As discovered by Mora (1986, 1988), the computation of Gröbner bases can be transferred to noncommutative rings in a straightforward way (though it may not terminate in all cases). Actually, there are several variations on the notion of noncommutative Gröbner bases; our usage is in harmony with Theorem 8 of Ufnarovski (1998). In the present context, the essential idea of Gröbner bases is the confluence of the induced reduction—which we have considered just before.

This leads us back to our remarks at the close of Section 2.1: it is now clear why we can avoid the costly computation of a Gröbner basis for each new problem as in Rosenkranz et al. (2003a): We *have* already a Gröbner basis, namely $\mathfrak{Gr}_0(\mathfrak{F})$, and it need not be changed for the different instances of BVPs considered. Of course, $\mathfrak{Gr}_0(\mathfrak{F})$ is not a finite Gröbner basis since the variables f and g in Table 1 range over all of $\mathfrak{F}^\#$; however, it is *finitary* in the sense of being described by finitely many parametrized polynomials.

Finally we can now address the questions of soundness and adquacy—how the formal operators are related to the actual ones. For this, let us first clarify the correspondence between *formal and actual operators*.

Definition 5. Let $\mathfrak{F}$ be an analytic algebra, $\mathfrak{A}$ an algebra containing $\mathfrak{F}$, and $\mathfrak{L}$ a subalgebra of the algebra of all linear operators on $\mathfrak{A}$. A homomorphism $I : \mathfrak{An}(\mathfrak{F}) \to \mathfrak{L}$ will be called an *interpretation of* $\mathfrak{An}(\mathfrak{F})$ *in* $\mathfrak{L}$ if $I(\lceil f \rceil)\, a = f a$ for all $f \in \mathfrak{F}$ and $a \in \mathfrak{A}$. It is called *sound* if all the equalities of Table 1 (where $\to$ is now regarded as $=$) are valid.

If $\mathfrak{L}$ is the algebra of all linear operators on the algebra of smooth functions $C^\infty[a, b]$, we define the *smooth interpretation* sm of $\mathfrak{An}(\mathfrak{F})$ in $\mathfrak{L}$ by setting

$$\mathrm{sm}(D)(u) = u',$$
$$\mathrm{sm}(A)(u) = x \mapsto \int_a^x u(\xi)\, d\xi,$$
$$\mathrm{sm}(B)(u) = x \mapsto \int_x^b u(\xi)\, d\xi,$$
$$\mathrm{sm}(L)(u) = x \mapsto u(a),$$

$$\text{sm}(R)(u) = x \mapsto u(b),$$
$$\text{sm}(\lceil f \rceil)(u) = fu,$$

where u ranges over $C^\infty[a,b]$, x over $[a,b]$, and f over $\mathfrak{F}$. It is easy to check that sm is indeed sound. (Actually, the equalities of Table 1 were extracted from relations in $\mathfrak{L}$ in the first place!) In a similar fashion, one may also define a distributional interpretation by using the algebra of boundary-valued distributions $C_0^{-\infty}[a,b]$ instead of $C^\infty[a,b]$. In fact, all the statements formulated here for the smooth interpretation carry over to the distributional setting, which allows for strong, weak and distributional solutions; see Rosenkranz (2003a, p. 45) for details.

Finally we arrive now at the summit of this treatise: the correctness theorem for our method of computing the Green's operator—at the same time asserting the *adequacy* of the Green's system in Table 1. The smooth interpretation of an analytic polynomial p will be denoted by $\underline{p}$.

Theorem 5. *Assume we have*

- *a linear differential operator $\underline{T}$ of order n with constant coefficients,*
- *n boundary operators $\underline{B_0}, \ldots, \underline{B_{n-1}}$*
- *such that the resulting BVP (1) has a unique solution,*

Now compute

- *the nullspace projector P according to Lemma 2,*
- *the right inverse $T^\blacklozenge$ of T as in Lemma 3,*
- *and finally the normal form G of $(1-P)T^\blacklozenge$, reduced with respect to the Green's system in Table 1.*

Then $\underline{G}$ is the Green's operator of the BVP, and G represents the corresponding Green's function g of (4).

Proof. By Lemma 2, $\underline{P}$ is indeed a projector onto the nullspace of $\underline{T}$. Since $\underline{T}$ is always surjective, $\underline{1}$ is the only possible projector onto the range of $\underline{T}$. Now there is a unique oblique Moore–Penrose inverse of T having these projectors; we will write it as $\underline{G}$ for some $G \in \mathfrak{An}(\mathfrak{F})$ yet to be determined.

By Theorem 1, $\underline{G}$ is also determined uniquely by the four Moore–Penrose Eqs. (5)–(8). As explained after Theorem 1, we can restrict ourselves to (7) and (8); finally, Lemma 1 reduces everything to (7), which reads $\underline{GT} = \underline{1-P}$. Since $\underline{TT^\blacklozenge} = \underline{1}$ by Lemma 3, postmultiplying by $\underline{T^\blacklozenge}$ yields $\underline{G} = \underline{(1-P)T^\blacklozenge}$. Hence we may choose the normal form of $(1-P)T^\blacklozenge$ for G, and its interpretation $\underline{G}$ will be the desired Moore–Penrose inverse.

For any $f \in C^\infty[a,b]$, the image $u = \underline{G}f$ fulfills the given differential equation $\underline{T}u = f$ because of the fourth Moore–Penrose Eq. (8). The range of $\underline{G}$ is $\underline{1-P}\,C^\infty[a,b]$ by Theorem 1, and every function contained in this range fulfills the given boundary conditions by Lemma 2. Hence $\underline{G}f$ fulfills the given BVP for any $f \in C^\infty[a,b]$, and $\underline{G}$ must coincide with the desired Green's operator due to the regularity assumption. Moreover, G represents the Green's function g since G is a Green's polynomial; see the discussion after Theorem 3. □

3. Sample computations

3.1. About the implementation

As mentioned before, we have implemented our method in TH∃OREM∀—a mathematical software system developed at RISC under the supervision of Prof. Bruno Buchberger. Based on Mathematica™, this system offers support for *proving, computing and solving* in various mathematical domains. As explained above, our implementation is actually a good example for the interplay between these three fundamental activities in mathematics: for *solving* a BVP, we *compute* the Green's operator by a reduction system that is *proved* confluent (see Tables 2 and 3 for a screen shot displaying two typical cases selected from the total of 233 cases that occur in the proof generated automatically by our proof algorithm).

The *core machinery* for computing the Green's operator by our method is concerned with handling noncommutative polynomials—this is mainly addition, subtraction, multiplication, reduction to normal form. We have implemented these operations as a separate "basic evaluator" named **ReduceNoncommutativePolynomial**. Based on TH∃OREM∀, it benefits also from the neat notation facilities available there: One may write the noncommutative polynomials exactly as one would on paper (e.g. denoting multiplication by juxtaposition rather than $**$ as in plain Mathematica™).

The basic evaluator for noncommutative polynomials is used for computing the nullspace projector as in Lemma 2, the right inverse as in Lemma 3, and finally the Green's function as in Theorem 5. All these *applied operations* are implemented in another basic evaluator named **GreenEvaluator**. In the next section, we will show some computations carried out by this evaluator (note that all the input and output printed[8] there is verbatim).

3.2. A simple classical example

The following problem seems to be one of the classical examples that are most often used for introducing the concepts of ordinary linear BVPs (Stakgold, 1979, p. 42). It can be interpreted as describing *one-dimensional steady heat conduction in a homogeneous rod*. In its functional formulation (after scaling everything to unity), it means that we have to solve

$$u'' = f,$$
$$u(0) = u(1) = 0$$

for the temperature $u \in C^\infty[0, 1]$ with a given heat source $f \in C^\infty[0, 1]$.

In this example, we have the differential operator $T = D^2$, so the nullspace is $\{\alpha x + \beta \mid \alpha, \beta \in \mathbb{C}\}$, and finding the *nullspace projector* P reduces to the following linear interpolation problem: given a function $v \in C^\infty[0, 1]$, find a linear function Pv that agrees with v at the grid points 0 and 1. In our case we can do this automatically:

```
In[13]:=  Compute[Proj_ŵ, by → GreenEvaluator,
            using → KnowledgeBase["ClassicalHeatConduction"]]
Out[13]=  L − ⌈x⌉L + ⌈x⌉R.
```

8 For aesthetic reasons, however, we have displayed Euler's number as e instead of using Mathematica's standard form ⅇ.

The other crucial step is to find the *right inverse* $(D^2)^\blacklozenge$. Trivially, this is A^2 in our case, but this is not in normal form. Computing it by our system returns the normal form:

In[14]:= $\mathrm{Compute}[(D^2)^\blacklozenge, \mathrm{by} \to \mathrm{GreenEvaluator}]$

Out[14]= $-A\lceil x\rceil + \lceil x\rceil A.$

Now it is easy to find the *Green's operator* G by computing $(1 - P)\, T^\blacklozenge$ in its normal form:

In[15]:= $\mathrm{Compute}[(1 - L + \lceil x\rceil L - \lceil x\rceil R)(-A\lceil x\rceil + \lceil x\rceil A),$
$\mathrm{by} \to \mathrm{GreenEvaluator}]$

Out[15]= $-A\lceil x\rceil - \lceil x\rceil B + \lceil x\rceil A\lceil x\rceil + \lceil x\rceil B\lceil x\rceil.$

Of course, we could also compute the Green's operator *immediately* (by specifying the given differential operator together with the list of boundary operators):

In[16]:= $\mathrm{Compute}[\mathrm{Green}[D^2, \langle L, R\rangle, \mathrm{by} \to \mathrm{GreenEvaluator}]$

Out[16]= $-A\lceil x\rceil - \lceil x\rceil B + \lceil x\rceil A\lceil x\rceil + \lceil x\rceil B\lceil x\rceil.$

Using the translation procedure described after Theorem 3, this corresponds to the *Green's function*

$$g(x, \xi) = \begin{cases} (x - 1)\xi & \text{if} \quad 0 \le \xi \le x \le 1, \\ x(\xi - 1) & \text{if} \quad 0 \le x \le \xi \le 1. \end{cases}$$

3.3. *Damped oscillations*

For a slightly more complicated problem, we take Example 2 in the textbook (Krall, 1986, p. 109). The differential operator of this BVP has *damped oscillations* as its eigenfunctions; see Krall (1986, p. 107). Stated in our terminology, the problem reads as follows: Given $f \in C^\infty[0, \pi]$, find $u \in C^\infty[0, \pi]$ such that

$$u'' + 2u' + u = f,$$
$$u(0) = u(\pi) = 0.$$

This time, we will immediately compute the *Green's operator*:

In[17]:= $\mathrm{Compute}[\mathrm{Green}[D^2 + 2D + 1, \langle L, R\rangle],$
$\mathrm{by} \to \mathrm{GreenEvaluator}]$

Out[17]= $(1 - \pi^{-1})\lceil \mathrm{e}^{-x}x\rceil A\lceil \mathrm{e}^{x}\rceil - \lceil \mathrm{e}^{-x}\rceil A\lceil \mathrm{e}^{x}x\rceil + \pi^{-1}\lceil \mathrm{e}^{-x}x\rceil A\lceil \mathrm{e}^{x}x\rceil$
$- \pi^{-1}\lceil \mathrm{e}^{-x}x\rceil B\lceil \mathrm{e}^{x}\rceil + \pi^{-1}\lceil \mathrm{e}^{-x}x\rceil B\lceil \mathrm{e}^{x}x\rceil.$

Written in the language of *Green's functions*, this means that

$$g(x, \xi) = \begin{cases} \frac{1}{\pi}(\pi - x)\xi\, \mathrm{e}^{\xi - x} & \text{if } 0 \le \xi \le x \le \pi, \\ \frac{1}{\pi}(\pi - \xi)x\, \mathrm{e}^{\xi - x} & \text{if } 0 \le x \le \xi \le \pi. \end{cases}$$

3.4. *Transverse beam deflection*

For one more example, let us now do a fourth-order problem (Stakgold, 1979, p. 49) that describes the *transverse deflection* $u \in C^\infty[0, 1]$ of a homogeneous beam with given distributed transversal load $f \in C^\infty[0, 1]$, simply supported at both ends. Using a linear elasticity model, one ends up with

$$u'''' = f,$$
$$u(0) = u(1) = u''(0) = u''(1) = 0.$$

Again computing the *Green's operator* directly, we arrive at:

In[18]:= Compute[Green[D^4, $\langle L, R, LD^2, RD^2\rangle$],
by → GreenEvaluator]

Out[18]= $\frac{1}{3}\lceil x\rceil A\lceil x\rceil - \frac{1}{6}A\lceil x^3\rceil - \frac{1}{2}\lceil x^2\rceil A\lceil x\rceil + \frac{1}{6}\lceil x\rceil A\lceil x^3\rceil + \frac{1}{6}\lceil x^3\rceil A\lceil x\rceil + \frac{1}{3}\lceil x\rceil B\lceil x\rceil - \frac{1}{2}\lceil x\rceil B\lceil x^2\rceil - \frac{1}{6}\lceil x^3\rceil B + \frac{1}{6}\lceil x\rceil B\lceil x^3\rceil + \frac{1}{6}\lceil x^3\rceil B\lceil x\rceil.$

This corresponds to the *Green's function*

$$g(x, \xi) = \begin{cases} \frac{1}{3}x\xi - \frac{1}{6}\xi^3 - \frac{1}{2}x^2\xi + \frac{1}{6}x\xi^3 + \frac{1}{6}x^3\xi & \text{if } 0 \le \xi \le x \le \pi, \\ \frac{1}{3}x\xi - \frac{1}{2}x\xi^2 - \frac{1}{6}x^3 + \frac{1}{6}x\xi^3 + \frac{1}{6}x^3\xi & \text{if } 0 \le x \le \xi \le \pi. \end{cases}$$

3.5. Nonunique solutions

As stated in Section 2.1, our method handles BVPs of the form (1) which are regular in the sense that for each forcing function $f \in C^\infty[a, b]$, a solution $u \in C^\infty[a, b]$ exists,[9] and this solution is unique. However, it is often desirable to compute solutions that exist only for certain choices of f; in this case, there are necessarily several independent solutions—this is made precise by the Alternative Theorem for BVPs, see Stakgold (1979, p. 210). In such a situation, one can compute something like a Green's function that allows us to transform any *admissible* forcing function f to *some* solution u; this is what a *modified Green's function* is used for, see Stakgold (1979, p. 216).

Let us look at the following illuminating *example*;[10] see e.g. Equation (5.1) in Stakgold (1979, p. 215): given $f \in C^\infty[0, 1]$, find $u \in C^\infty[0, 1]$ such that

$$-u'' = f,$$
$$u'(0) = u'(1) = 0. \tag{10}$$

Integrating the differential equation, one sees immediately that a solution u can only exist if f fulfills the *solvability condition* $\int_0^1 f(x)\,dx = 0$. In this case, one boundary condition implies the other because we have $u'(1) = u'(0) + \int_0^1 f(x)\,dx = u'(0)$.

Computing the nullspace projector via Lemma 2 does not work since the matrix $\hat{l}\hat{w}^\leftarrow + \hat{r}\hat{w}^\rightarrow$ is singular, reflecting the fact that one of the boundary conditions is superfluous. Obviously we cannot apply the standard method described in this article.

[9] As noted at the end of Section 2.5, the smooth setting used in this article can readily be extended to the more general distributional setting.

[10] I am grateful to my referee Stanly Steinberg for pointing me to this example.

However, we will now show how we can still solve (10) by *transforming it to a regular problem*.

We can ensure *uniqueness* by imposing the condition that the L^2 norm of u be minimal. Since the nullspace of (10) are the constant functions, the norm of u can always be made zero. Hence we add the integral condition $\int_0^1 u(\xi)\,\mathrm{d}\xi = 0$ to the two given boundary conditions, see Equation (5.12) in Stakgold (1979, p. 216).

Next we enforce *existence* by projecting the given f onto the subspace of admissible forcing functions, $\mathfrak{A} = \{f \in C^\infty[0,1] \mid \int_0^1 f(\xi)\,\mathrm{d}\xi = 0\}$. In general there are many projectors, but the canonical choice is to take the space of constant functions as the complement of $\mathfrak{A}$. In this case we have to use the projector $1 - A - B : C^\infty[0,1] \to \mathfrak{A}$ that maps f to $f - \int_0^1 f(\xi)\,\mathrm{d}\xi$. Observe that this projector maps the constant function 1 to zero, hence the generalized Green's operator maps 1 to the unique minimum-norm solution of $u'' = 0, u'(0) = u'(1) = 0$, which is again zero. This fact is used to single out the modified Green's operator among all other generalized Green's operators, see Equation (5.4) in Stakgold (1979, p. 216).

We are now confronted with the following *regular problem*: given $f \in C^\infty[0,1]$, find $u \in C^\infty[0,1]$ such that

$$\begin{aligned} &-u'' = (1 - F)\,f, \\ &u'(0) = u'(1) = \textstyle\int_0^1 u(\xi)\,\mathrm{d}\xi = 0. \end{aligned} \tag{11}$$

Here we have used the abbreviation $F \equiv A + B$ for denoting the operator of definite integration.

Though regular, problem (11) is still not in the scope of the method described in this paper: first, we have three conditions to fulfill (it is no problem that one of them is not a boundary condition), but the nullspace of $-D^2$ is only two dimensional. Second, the projector $1 - F$ prevents us from interpreting the differential equation as finding a right inverse of $-D^2$. We can knock out both problems at once with a simple trick—by differentiating one more time. Doing so, we arrive at the following *accessory problem*: given $f \in C^\infty[0,1]$, find $u \in C^\infty[0,1]$ such that

$$\begin{aligned} &-u''' = f', \\ &u'(0) = u'(1) = \textstyle\int_0^1 u(\xi)\,\mathrm{d}\xi = 0. \end{aligned} \tag{12}$$

Note that the projector has now disappeared because $D(1 - F) = 0$.

Problem (12) is *equivalent* to (11): the direction from (11) to (12) is trivial, so assume now u is a solution of (12). In order to obtain $-u'' = f$ from $-u''' = f'$, we have to integrate using the mean-value antiderivative $A - FA$ rather than the standard antiderivative A. Whereas the standard antiderivative

$$\int_0^x u'(\xi)\,\mathrm{d}\xi = u(x) - u(0)$$

takes the integration constant as the left boundary value, the mean-value antiderivative (slightly rewritten)

$$\int_0^1 \int_\tau^x u'(\xi)\,\mathrm{d}\xi\,\mathrm{d}\tau = u(x) - \int_0^1 u(\xi)\,\mathrm{d}\xi$$

takes it as the mean value. In operator notation, these equations are written in the succinct form $AD = 1 - L$ and $(A - FA)D = 1 - F$. Note that the former is part of the reduction system of Table 1, whereas the latter can easily be obtained from it. Applying now $A - FA$ to the differential equation of (12), we obtain

$$-u'' + \int_0^1 u''(\xi)\,\mathrm{d}\xi = f - \int_0^1 f(\xi)\,\mathrm{d}\xi,$$

which is indeed the differential equation of (11) since $\int_0^1 u''(\xi)\,\mathrm{d}\xi = u'(1) - u'(0) = 0$.

Hence we are left to problem (12). This time we can apply our standard method described in this article. The nullspace of $-D^3$ is given by the quadratic polynomials, so it has dimension 3. Hence we can choose a *projector* P onto it such that its complement consists exactly of those functions $u \in C^\infty[0,1]$ that fulfill the three side conditions of (12). Using the ansatz $Pu = \alpha_u x^2 + \beta_u x + \gamma_u$, one obtains immediately

$$P = F - \frac{1}{3}LD - \frac{1}{6}RD + \lceil x \rceil LD + \frac{1}{2}\lceil x^2 \rceil (RD - LD).$$

The corresponding Green's operator $\tilde{G}$ fulfills the third Moore–Penrose Eq. (7), so we have $-D^3\tilde{G} = 1 - P$. But note that $\tilde{G}$ has to be applied to f' rather than f. Hence the actual *Green's operator* that maps f to u is given by $G = \tilde{G}D$, and we have $-D^2G = 1 - P$. Now we can use the usual procedure of right inversion, giving $G = -(1 - P)A^2$. The final step is now to normalize this analytic polynomial by using the **GreenEvaluator** described in Section 3.1, yielding:

In[19]:= Compute[$-(1 - P)A^2$, by → GreenEvaluator]

Out[19]= $\frac{1}{2} A\lceil x^2 \rceil - B\lceil x \rceil + \frac{1}{2} B\lceil x^2 \rceil - \lceil x \rceil A + \frac{1}{2}\lceil x^2 \rceil B + \frac{1}{2}\lceil x^2 \rceil A + \frac{1}{3} A + \frac{1}{3} B.$

The standard translation procedure extracts from this the following *modified Green's function*

$$g(x,\xi) = \begin{cases} \frac{1}{3} - x + \frac{x^2+\xi^2}{2} & \text{if} \quad 0 \le \xi \le x \le 1, \\ \frac{1}{3} - \xi + \frac{x^2+\xi^2}{2} & \text{if} \quad 0 \le x \le \xi \le 1, \end{cases}$$

see Equation (5.5) in Stakgold (1979, p. 216).

3.6. *The generic Sturm problem*

The two-point BVPs treated in this article can be understood as inhomogeneous LODEs whose inhomogeneity is *parametrized* (plus side conditions). It is common practice to regard all other data as predetermined. Quoting (Stakgold, 1979, p. 51): "The differential operator and boundary operators appearing on the left sides ... are kept fixed; no one is proposing to solve all differential equations with arbitrary boundary conditions in one stroke!"

Following a recent proposal,[11] we will nevertheless attempt to do something in this direction: we are not going to solve *all* LODEs or *all* boundary conditions in one stroke, but we will consider the *generic Sturm Problem*, i.e. the general second-order BVP with unmixed boundary conditions (for a linear differential operator with constant coefficients), see Stakgold (1979, p. 191f).

So we deal with the following *problem*: Given $f \in C^\infty[0,1]$, find $u \in C^\infty[0,1]$ such that

$$\begin{aligned} &u'' + au' + bu = f \\ &\alpha u(0) + \beta u'(0) = \gamma u(1) + \delta u'(1) = 0. \end{aligned} \tag{13}$$

Note that we have assumed $[0,1]$ as the domain, which can always be enforced by rescaling. The coefficient of u'' is assumed to be nonzero (otherwise we would have a first-order problem), so it is divided out. Furthermore, we assume that the parameters fulfill $a, b, \alpha, \beta, \gamma, \delta \neq 0$ and that they make (13) a regular BVP. It is well-known (Coddington and Levinson, 1955, p. 291) that this is the case iff

$$\begin{vmatrix} \alpha\varphi(0) + \beta\varphi'(0) & \alpha\psi(0) + \beta\psi'(0) \\ \gamma\varphi(1) + \delta\varphi'(1) & \gamma\psi(1) + \delta\psi'(1) \end{vmatrix} \neq 0 \tag{14}$$

where $\{\varphi, \psi\}$ is any fundamental system for the homogeneous equation $u'' + au' + bu = 0$.

For solving (13), we proceed just as before. The only difference is that the scalar field underlying the analytic algebra $\mathfrak{Exp}$ is no longer $\mathbb{C}$ but rather the *rational-function field* $\mathbb{C}(a, b, \alpha, \beta, \gamma, \delta)$. All the computations described so far carry over without essential changes.

Let us denote the differential operator of problem (13) by $T \equiv D^2 + a\,D + b$, its two boundary operators by $M \equiv \alpha\,L + \beta\,LD$ and $N \equiv \gamma\,R + \delta\,RD$. With λ and μ being the roots of the characteristic equation $x^2 + ax + b = 0$, the differential operator factors as $T = (D - \lambda)(D - \mu)$ and has, by Lemma 3,

$$T^{\blacklozenge} = \lceil e^{\lambda x} \rceil A \lceil e^{(\mu-\lambda)x} \rceil A \lceil e^{-\mu x} \rceil \tag{15}$$

as a *right inverse*. Note that the middle factor disappears if $\lambda = \mu$. In the following, we will only treat the case $\lambda \neq \mu$; the case of a double root is completely analogous.

The next step is to compute a *nullspace projector*. In the notation used there, we have now $\hat{l} = \left(\begin{smallmatrix} \alpha & \beta \\ 0 & 0 \end{smallmatrix}\right)$, $\hat{r} = \left(\begin{smallmatrix} 0 & 0 \\ \gamma & \delta \end{smallmatrix}\right)$ and $(L\hat{l} + R\hat{r})\hat{D}_2 = \left(\begin{smallmatrix} M \\ N \end{smallmatrix}\right)$. We have to choose some fundamental system $\{\varphi, \psi\}$ of the homogeneous equation $Tu = 0$. We will follow the practice of Stakgold (1979, p. 195), selecting φ and ψ to fulfill the boundary conditions $M\varphi = 0$ and $N\psi = 0$, respectively. We do this by taking φ and ψ to be the unique solutions of the following initial-value problems for the differential equation $Tu = 0$. For φ, the initial conditions are taken as $\varphi(0) = \beta$, $\varphi'(0) = -\alpha$; for ψ, they are $\psi(1) = \delta$, $\psi'(1) = -\gamma$. A small computation (e.g. by the Mathematica command **DSolve**)

[11] This proposal was forwarded to me from my referee Stanly Steinberg, whom I would also like to thank here.

leads to

$$\varphi(x) = (\lambda - \mu)^{-1}\left((\alpha + \beta\lambda)\,\mathrm{e}^{\mu x} - (\alpha + \beta\mu)\,\mathrm{e}^{\lambda x}\right),$$
$$\psi(x) = (\lambda - \mu)^{-1}\left((\gamma + \delta\lambda)\,\mathrm{e}^{\mu(x-1)} - (\gamma + \delta\mu)\,\mathrm{e}^{\lambda(x-1)}\right).$$

Let $\hat{w}$ be the fundamental matrix of $\{\varphi, \psi\}$. Using the relation $M\varphi = N\psi = 0$, we obtain $\hat{l}\hat{w}^{\leftarrow} + \hat{r}\hat{w}^{\rightarrow} = \begin{pmatrix} M\varphi & M\psi \\ N\varphi & N\psi \end{pmatrix} = \begin{pmatrix} 0 & m \\ n & 0 \end{pmatrix}$, where

$$m \equiv M\psi = (\lambda - \mu)^{-1}\left((\alpha + \beta\mu)(\gamma + \delta\lambda)\,\mathrm{e}^{-\mu} - (\alpha + \beta\lambda)(\gamma + \delta\mu)\,\mathrm{e}^{-\lambda}\right),$$
$$n \equiv N\varphi = (\lambda - \mu)^{-1}\left((\alpha + \beta\lambda)(\gamma + \delta\mu)\,\mathrm{e}^{\mu} - (\alpha + \beta\mu)(\gamma + \delta\lambda)\,\mathrm{e}^{\lambda}\right)$$

are used as abbreviations. Observing that $n = -m\,\mathrm{e}^{\lambda+\mu}$, we obtain the inverse $(\hat{l}\hat{w}^{\leftarrow} + \hat{r}\hat{w}^{\rightarrow})^{-1} = \begin{pmatrix} 0 & n^{-1} \\ m^{-1} & 0 \end{pmatrix} = -m^{-1}\begin{pmatrix} 0 & \mathrm{e}^{-\lambda-\mu} \\ -1 & 0 \end{pmatrix}$. According to Lemma 2, the nullspace projector is now

$$P = \lceil \hat{w}_1 \rceil (\hat{l}\hat{w}^{\leftarrow} + \hat{r}\hat{w}^{\rightarrow})^{-1}(L\hat{l} + R\hat{r})\hat{D}_2 = -m^{-1}\left(\lceil\varphi(x)\rceil \;\; \lceil\psi(x)\rceil\right)\begin{pmatrix} 0 & \mathrm{e}^{-\lambda-\mu} \\ -1 & 0 \end{pmatrix}\begin{pmatrix} M \\ N \end{pmatrix}$$
$$= m^{-1}(\lceil\psi(x)\rceil M - \mathrm{e}^{-\lambda-\mu}\lceil\varphi(x)\rceil N),$$

written as an analytic polynomial.

With these preparations, we can compute the *Green's operator* G as before via $G = (1 - P)\,T^{\blacklozenge}$. Substituting the nullspace projector and the right inverse, we obtain

$$m\,G = (m - \lceil\psi(x)\rceil M + \mathrm{e}^{-\lambda-\mu}\lceil\varphi(x)\rceil N)\lceil\mathrm{e}^{\lambda x}\rceil A \lceil\mathrm{e}^{(\mu-\lambda)x}\rceil A \lceil\mathrm{e}^{-\mu x}\rceil$$

as a preliminary answer.

For writing G in its *canonical form*, we normalize it by the **GreenEvaluator** after giving the necessary definitions and options:[12]

`In[20]:=` **Definition**["Abbreviations",
$M = \alpha L + \beta L D$
$N = \gamma R + \delta R D$]
$\varphi[x] = (\lambda - \mu)^{-1}((\alpha + \beta\lambda)\,\mathrm{e}^{\mu x} - (\alpha + \beta\mu)\,\mathrm{e}^{\lambda x})$
$\psi[x] = (\lambda - \mu)^{-1}((\gamma + \delta\lambda)\,\mathrm{e}^{\mu(x-1)} - (\gamma + \delta\mu)\,\mathrm{e}^{\lambda(x-1)})$]
$m = (\lambda - \mu)^{-1}((\alpha + \beta\mu)(\gamma + \delta\lambda)\mathrm{e}^{-\mu} - (\alpha + \beta\lambda)(\gamma + \delta\mu)\mathrm{e}^{-\lambda}$].

`In[21]:=` SetOptions[ReduceNoncommutativePolynomial,
FactorCoefficients → True].

`In[22]:=` Compute[$(m - \lceil\psi[x]\rceil M + \mathrm{e}^{-\lambda-\mu}\lceil\varphi[x]\rceil N)\,\lceil\mathrm{e}^{\lambda x}\rceil A \lceil\mathrm{e}^{(\mu-\lambda)x}\rceil A \lceil\mathrm{e}^{-\mu x}\rceil$,
using → **Definition**["Abbreviations"], by → GreenEvaluator].

12 Note that Theorema, just like Mathematica, uses square brackets rather than round parentheses for function application. Hence we have here e.g. $\lceil\varphi[x]\rceil$ instead of $\lceil\varphi(x)\rceil$ as before.

$$\texttt{Out[22]=}\quad (\alpha+\beta\mu)(\gamma+\delta\mu)(\lambda-\mu)^{-2}\,\mathrm{e}^{-\lambda}\,\lceil \mathrm{e}^{\lambda x}\rceil A\lceil \mathrm{e}^{-\mu x}\rceil$$
$$+\,(\alpha+\beta\mu)(\gamma+\delta\mu)(\lambda-\mu)^{-2}\,\mathrm{e}^{-\lambda}\,\lceil \mathrm{e}^{\lambda x}\rceil B\lceil \mathrm{e}^{-\mu x}\rceil$$
$$-\,(\alpha+\beta\lambda)(\gamma+\delta\mu)(\lambda-\mu)^{-2}\,\mathrm{e}^{-\lambda}\,\lceil \mathrm{e}^{\mu x}\rceil B\lceil \mathrm{e}^{-\mu x}\rceil$$
$$-\,(\alpha+\beta\lambda)(\gamma+\delta\mu)(\lambda-\mu)^{-2}\,\mathrm{e}^{-\lambda}\,\lceil \mathrm{e}^{\lambda x}\rceil A\lceil \mathrm{e}^{-\lambda x}\rceil$$
$$+\,(\alpha+\beta\lambda)(\gamma+\delta\lambda)(\lambda-\mu)^{-2}\,\mathrm{e}^{-\mu}\,\lceil \mathrm{e}^{\mu x}\rceil A\lceil \mathrm{e}^{-\lambda x}\rceil$$
$$+\,(\alpha+\beta\lambda)(\gamma+\delta\lambda)(\lambda-\mu)^{-2}\,\mathrm{e}^{-\mu}\,\lceil \mathrm{e}^{\mu x}\rceil B\lceil \mathrm{e}^{-\lambda x}\rceil$$
$$-\,(\alpha+\beta\mu)(\gamma+\delta\lambda)(\lambda-\mu)^{-2}\,\mathrm{e}^{-\mu}\,\lceil \mathrm{e}^{\mu x}\rceil A\lceil \mathrm{e}^{-\mu x}\rceil$$
$$-\,(\alpha+\beta\mu)(\gamma+\delta\lambda)(\lambda-\mu)^{-2}\,\mathrm{e}^{-\mu}\,\lceil \mathrm{e}^{\lambda x}\rceil B\lceil \mathrm{e}^{-\lambda x}\rceil.$$

From the above representation, one could extract the Green's function in the usual straightforward manner. For comparing our result with the literature, however, it is convenient to *factor* it as

$$\begin{aligned}
&(\lambda-\mu)^2\, m\, G\\
&\quad= \Big((\gamma+\delta\lambda)\,\mathrm{e}^{-\mu}\,\lceil \mathrm{e}^{\mu x}\rceil - (\gamma+\delta\mu)\,\mathrm{e}^{-\lambda}\,\lceil \mathrm{e}^{\lambda x}\rceil\Big)\ A\ \Big((\alpha+\beta\lambda)\,\lceil \mathrm{e}^{-\lambda x}\rceil\\
&\qquad -\ (\alpha+\beta\mu)\,\lceil \mathrm{e}^{-\mu x}\rceil\Big) + \Big((\alpha+\beta\lambda)\,\lceil \mathrm{e}^{\mu x}\rceil - (\alpha+\beta\mu)\,\lceil \mathrm{e}^{\lambda x}\rceil\Big)\\
&\qquad \times B\ \Big((\gamma+\delta\lambda)\,\mathrm{e}^{-\mu}\,\lceil \mathrm{e}^{-\lambda x}\rceil - (\gamma+\delta\mu)\,\mathrm{e}^{-\lambda}\,\lceil \mathrm{e}^{-\mu x}\rceil\Big)\\
&\quad= \lceil(\gamma+\delta\lambda)\,\mathrm{e}^{\mu(x-1)} - (\gamma+\delta\mu)\,\mathrm{e}^{\lambda(x-1)}\rceil\ A\ \Big\lceil \mathrm{e}^{-(\lambda+\mu)x}\,\Big((\alpha+\beta\lambda)\,\mathrm{e}^{\mu x}\\
&\qquad -\ (\alpha+\beta\mu)\,\mathrm{e}^{\lambda x}\Big)\Big\rceil + \lceil(\alpha+\beta\lambda)\,\mathrm{e}^{\mu x} - (\alpha+\beta\mu)\,\mathrm{e}^{\lambda x}\rceil\\
&\qquad \times B\ \Big\lceil \mathrm{e}^{-(\lambda+\mu)x}\,\Big((\gamma+\delta\lambda)\,\mathrm{e}^{\mu(x-1)} - (\gamma+\delta\mu)\,\mathrm{e}^{\lambda(x-1)}\Big)\Big\rceil\\
&\quad= \lceil(\lambda-\mu)\,\psi(x)\rceil\ A\ \lceil \mathrm{e}^{-(\lambda+\mu)x}\,(\lambda-\mu)\,\varphi(x)\rceil + \lceil(\lambda-\mu)\,\varphi(x)\rceil\\
&\qquad \times B\ \lceil \mathrm{e}^{-(\lambda+\mu)x}\,(\lambda-\mu)\,\psi(x)\rceil,
\end{aligned}$$

which yields immediately the Green's operator

$$G = \lceil\psi(x)\rceil\ A\ \lceil m^{-1}\,\mathrm{e}^{-(\lambda+\mu)x}\,\varphi(x)\rceil + \lceil\varphi(x)\rceil\ B\ \lceil m^{-1}\,\mathrm{e}^{-(\lambda+\mu)x}\,\psi(x)\rceil$$

in a condensed representation.

An easy computation shows that $m\,\mathrm{e}^{(\lambda+\mu)x}$ is just the Wronskian $W(x) \equiv \det \hat{w}(x) = \varphi(x)\,\psi'(x) - \varphi'(x)\,\psi(x)$, hence we obtain the *Green's function* in the form

$$g(x,\xi) = \begin{cases} \psi(x)\,\varphi(\xi)\,W(\xi)^{-1} & \text{if}\quad 0 \le \xi \le x \le 1,\\ \varphi(x)\,\psi(\xi)\,W(\xi)^{-1} & \text{if}\quad 0 \le x \le \xi \le 1, \end{cases}$$

in accordance with Stakgold (1979, p. 195).

4. Conclusion

We have presented a new algorithm for solving linear two-point BVPs symbolically. Unlike the usual methods that translate the operator problem into a functional setting, our approach represents the abstract quotient structure encoding the relevant operators (differentiation, integration, boundary evaluation, functional multiplication) with their essential properties (Leibniz equality, fundamental theorem of calculus, etc) canonically in an ismorphic algorithmic domain: the quotient ring $\mathfrak{An}(\mathfrak{F})/\mathfrak{Gr}(\mathfrak{F})$ may be considered as an abstract condensate capturing the *algebraic characteristics of the operators involved*, whereas the isomorphic system $\mathfrak{Gr}_0(\mathfrak{F})$ makes this structure available to computations via the reduction induced by the Green's system.

At this point it is natural to ask ourselves how far the method presented in this article could be generalized. Let us first look at some straightforward *generalizations*; most of these have also been discussed in Rosenkranz et al. (2003a).

- We can investigate *systems of differential equations* (together with their boundary conditions) instead of a single one. In the linear case, the resulting theory is very similar to scalar BVPs, using a Green's matrix instead of a Green's function; see e.g. p. 249 in Kamke (1983). Our method should be extensible to this case in a fairly simple manner. In the worst case, we have to recede to our original approach in Rosenkranz et al. (2003a) via Gröbner bases and adapt them to work for vectors of polynomials rather than single ones. Essentially this amounts to computing Gröbner bases in modules, a routine task for commutative polynomials—see e.g. Becker and Weispfenning (1993, pp. 485ff)—that may be expected to carry over to the noncommutative case.
- It is certainly a much greater challenge to move from ordinary to *partial differential equations*. In principle, the algebraization embodied in our approach extends in a straightforward way, e.g. introducing the partial differential operators D_x and D_y instead of D and analogous operators for integration. Certain concepts and results from Riquier–Janet theory and Lie analysis may come in handy here. Of course, one will have to adapt the treatment of boundary values. Besides this, the analog of right inversion will in general be far more complex for partial differential operators—maybe somewhat similar to the elimination techniques used by the holonomic paradigm (Zeilberger, 1990).
- One of the most difficult generalizations is probably the step towards *nonlinear* BVPs. The reason is that our algebraic model does not lend itself easily to describe nonlinear differential operators, and the systematic approach seems to lead to general rewriting (still modulo the polynomial congruence), with substitution in addition to replacement. Maybe this could be handled by a suitable combination of Gröbner bases and the Knuth-Bendix algorithm; see Bachmair and Ganzinger (1994) and Marché (1996).
- It can also be expected to treat certain *integro-differential equations* by our approach. In fact, the Green's polynomials provide a uniform way for expressing integral as well as differential equations—and their mixtures.

Beyond these rather direct generalizations of the problem considered in this article, we believe that our approach has a certain intrinsic interest not directly tied to BVPs of any kind. The essence of our method can be described as solving problems at the operator level by polynomial methods. This could be a new research paradigm applicable to various problems of a field that might be called *symbolic functional analysis*. Up to now, symbolic methods have conquered the following two "main floors" (cum grano salis): (1) numbers $\to$ computer algebra, e.g. solving a system of polynomial equations; (2) functions $\to$ computer analysis, e.g. solving a differential equation. Naturally, the third floor would be:

(3) operators → symbolic functional analysis, e.g. solving BVPs. We have described these ideas in more detail in Rosenkranz et al. (2003b). Let us just mention two further examples of “problems on the third floor”:

- Certain problems in *potential theory* have a flavor very similar to that of BVPs for PDEs, at least when seen from the symbolic viewpoint. It is therefore natural to ask to what extent one could transfer some ideas from BVPs to the potential setting. In particular, one would like to formulate an algebraic set-up that allows us to express the operator induced by the potential function (analogous to the Green’s operator induced by the Green’s function).
- The field of *inverse problems* (Engl et al., 1996) opens a whole arena of possible applications of symbolic functional analysis. Even though one cannot usually expect algebraic solutions for such problems, the polynomial approach will certainly uncover a great deal about the solution manifold. In particular, it may be possible to transform a given problem into a different one that has more profitable properties.

Pondering such examples, we do hope that it will be possible to develop fruitful ideas along these lines in the near future.

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