A New Proof of Hilbert's Theorem on Homogeneous Functions *

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§1 Ordering of Products of Variables

The products P of n variables

$$x_1, x_2, \ldots, x_n$$

may be ordered such that for two products

$$P_1 = x_1^{h_1} x_2^{h_2} \cdots x_n^{h_n}, \qquad P_2 = x_1^{\kappa_1} x_2^{\kappa_2} \cdots x_n^{\kappa_n},$$

P_1 precedes P_2 if there exists an index σ for which

$$h_1 \leq \kappa_1, \quad h_2 \leq \kappa_2, \quad \ldots \quad h_{\sigma-1} \leq \kappa_{\sigma-1}, \quad h_\sigma < \kappa_\sigma.$$

P_1 is called *simpler* than P_2. We obtain products P, in which the exponent of x_σ is a given integer C, by multiplying x_σ^C by products consisting only of the remaining $n-1$ variables x.

§2 A Lemma

If each of the products given by

$$P = x_1^{h_1} x_2^{h_2} \cdots x_n^{h_n}, \qquad P_\varrho = x_1^{h_{\varrho,1}} x_2^{h_{\varrho,2}} \cdots x_n^{h_{\varrho,n}} \quad (\varrho = 1, 2, 3, \ldots)$$

has no other as a factor, then the number of them is finite.

Proof. I assume that the theorem holds for products of $n-1$ variables, hence that the number of products P_ϱ, for which the exponent of x_σ is the integer C, is finite.

Every P_ϱ has at least one exponent which satisfies the inequality $h_{\varrho,\sigma} < h_\sigma$. I assign to the product P_ϱ the first of these exponents h_σ and denote this exponent by $C_{\varrho,\sigma}$. The number of all possible $C_{\varrho,\sigma}$ is $h_1 + h_2 + \ldots + h_n$, thus finite. There are a finite number of P_ϱ corresponding to each $C_{\varrho,\sigma}$, so the number of P_σ is finite.

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* *Neuer Beweis des Hilbertschen Satzes über homogene Funktionen*, Nachr. der Königl. Ges. der Wiss. zu Göttingen **3** (1899), 240-242. Translation by Michael Abramson.

§3 Homogeneous Functions

Homogeneous functions f of $x_1, x_2, \ldots, x_n$ are polynomials of points P and can be written in the form

$$f = aP + \psi$$

where a is a nonzero constant. I order the products in f so that the products in ψ are simpler than the lead term P.

Given two functions

$$f_1 = a_1 P_1 + \psi_1, \qquad f_2 = a_2 P_2 + \psi_2,$$

if P_2 is a factor of P_1, meaning

$$P_1 = R P_2,$$

then the polynomial

$$f_1 - \frac{a_1}{a_2} f_2 R \quad = \quad \psi_1 - \frac{a_1}{a_2} R \psi_2$$

has a simpler lead term than f_1.

§4 The Theorem of Hilbert [1]

Let

$$F_1, F_2, \ldots$$

be homogeneous functions. I order them by their lead terms and, with a suitable choice of functions $A_1, A_2, \ldots, A_{\sigma-1}$, form those homogeneous polynomials

$$f_\varrho = A_1 F_1 + A_2 F_2 + \ldots + A_{\varrho-1} F_{\varrho-1} + F_\varrho$$

which have the simplest lead terms. This must be possible, since a sequence of products, in which every subsequent term is simpler than the preceding one, must terminate eventually.

Since by §3 these simplest lead terms are not factors of one another, the number of them is finite by §2. It corresponds to those special F_ϱ whose f_ϱ do not vanish.

Every given F is representable as polynomials of these special F, *i.e.* they are linear combinations of the special F, if we choose suitable functions in the variables $x_1, \ldots, x_n$ for coefficients in these linear expressions.

Munich, September 1899.

[1] D. Hilbert, *Über die Theorie der algebraischen Formen*, Math. Ann. **36** (1890), 474.