JOURNAL OF FORMALIZED MATHEMATICS
Volume 15, Released 2003, Published 2003
Inst. of Computer Science, Univ. of Białystok

Characterization and Existence of Gröbner Bases

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Summary. We continue the Mizar formalization of Gröbner bases following [8]. In this article we prove a number of characterizations of Gröbner bases among them that Gröbner bases are convergent rewriting systems. We also show the existence and uniqueness of reduced Gröbner bases.

MML Identifier: GROEB_1.

WWW: http://mizar.org/JFM/Vol15/groeb_1.html

The articles [24], [32], [34], [33], [10], [4], [18], [28], [11], [30], [12], [14], [5], [2], [31], [9], [7], [16], [17], [13], [21], [20], [25], [27], [19], [1], [6], [15], [23], [29], [26], [3], and [22] provide the notation and terminology for this paper.

1. Preliminaries

Let n be an ordinal number, let L be a right zeroed add-associative right complementable unital distributive non trivial double loop structure, and let p be a polynomial of n, L. Then $\{p\}$ is a non empty finite subset of Polynom-Ring(n,L).

We now state several propositions:

(1) Let n be an ordinal number, T be a connected term order of n, L be an add-associative right complementable right zeroed commutative associative well unital distributive field-like non trivial double loop structure, and f, p, g be polynomials of n, L. Suppose f reduces to g, p, T. Then there exists a monomial m of n, L such that $g = f - m * p$.

(2) Let n be an ordinal number, T be an admissible connected term order of n, L be an add-associative right complementable right zeroed commutative associative well unital distributive Abelian field-like non degenerated non empty double loop structure, and f, p, g be polynomials of n, L. Suppose f reduces to g, p, T. Then there exists a monomial m of n, L such that $g = f - m * p$ and $\mathrm{HT}(m*p,T) \notin \mathrm{Support}\, g$ and $\mathrm{HT}(m*p,T) \leq_T \mathrm{HT}(f,T)$.

(3) Let n be an ordinal number, T be a connected term order of n, L be an add-associative right complementable right zeroed commutative associative well unital distributive field-like non trivial double loop structure, f, g be polynomials of n, L, and P, Q be subsets of Polynom-Ring(n,L). If $P \subseteq Q$, then if f reduces to g, P, T, then f reduces to g, Q, T.

(4) Let n be an ordinal number, T be a connected term order of n, L be an add-associative right complementable right zeroed commutative associative well unital distributive field-like non trivial double loop structure, and P, Q be subsets of Polynom-Ring(n,L). If $P \subseteq Q$, then PolyRedRel$(P,T) \subseteq$ PolyRedRel(Q,T).

(5) Let n be an ordinal number, L be a right zeroed add-associative right complementable non empty double loop structure, and p be a polynomial of n, L. Then Support$(-p) =$ Support p.

(6) Let n be an ordinal number, T be a connected term order of n, L be a right zeroed add-associative right complementable unital distributive non trivial non empty double loop structure, and p be a polynomial of n, L. Then $\mathrm{HT}(-p,T)=\mathrm{HT}(p,T)$.

(7) Let n be an ordinal number, T be an admissible connected term order of n, L be a right zeroed add-associative right complementable unital distributive non trivial non empty double loop structure, and p, q be polynomials of n, L. Then $\mathrm{HT}(p-q,T)\leq_T \max_T(\mathrm{HT}(p,T),\mathrm{HT}(q,T))$.

(8) Let n be an ordinal number, T be an admissible connected term order of n, L be an add-associative right complementable right zeroed commutative associative well unital distributive field-like non trivial double loop structure, and p, q be polynomials of n, L. If $\mathrm{Support}\, q\subseteq \mathrm{Support}\, p$, then $q\leq_T p$.

(9) Let n be an ordinal number, T be a connected admissible term order of n, L be an add-associative right complementable right zeroed commutative associative well unital distributive field-like non degenerated non empty double loop structure, and f, p be non-zero polynomials of n, L. If f is reducible wrt p, T, then $\mathrm{HT}(p,T)\leq_T \mathrm{HT}(f,T)$.

2. Characterization of Gröbner Bases

The following propositions are true:

(10) Let n be a natural number, T be a connected admissible term order of n, L be an add-associative right complementable right zeroed commutative associative well unital distributive Abelian field-like non trivial double loop structure, and p be a polynomial of n, L. Then $\mathrm{PolyRedRel}(\{p\},T)$ is locally-confluent.

(11) Let n be a natural number, T be a connected admissible term order of n, L be an Abelian add-associative right complementable right zeroed commutative associative well unital distributive field-like non degenerated non empty double loop structure, and P be a subset of $\mathrm{Polynom\text{-}Ring}(n,L)$. Given a polynomial p of n, L such that $p\in P$ and P–ideal $=\{p\}$–ideal. Then $\mathrm{PolyRedRel}(P,T)$ is locally-confluent.

Let n be an ordinal number, let T be a connected term order of n, let L be a right zeroed add-associative right complementable unital distributive non trivial non empty double loop structure, and let P be a subset of $\mathrm{Polynom\text{-}Ring}(n,L)$. The functor $\mathrm{HT}(P,T)$ yielding a subset of $\mathrm{Bags}\, n$ is defined as follows:

(Def. 1) $\mathrm{HT}(P,T)=\{\mathrm{HT}(p,T);\, p$ ranges over polynomials of n, L: $p\in P \wedge p\neq 0_n L\}$.

Let n be an ordinal number and let S be a subset of $\mathrm{Bags}\, n$. The functor $\mathrm{multiples}(S)$ yields a subset of $\mathrm{Bags}\, n$ and is defined as follows:

(Def. 2) $\mathrm{multiples}(S)=\{b;\, b$ ranges over elements of $\mathrm{Bags}\, n : \bigvee_{b' : \mathrm{bag\ of}\ n} (b'\in S \wedge b' \mid b)\}$.

The following propositions are true:

(12) Let n be a natural number, T be a connected admissible term order of n, L be an add-associative right complementable right zeroed commutative associative well unital distributive Abelian field-like non degenerated non empty double loop structure, and P be a subset of $\mathrm{Polynom\text{-}Ring}(n,L)$. If $\mathrm{PolyRedRel}(P,T)$ is locally-confluent, then $\mathrm{PolyRedRel}(P,T)$ is confluent.

(13) Let n be an ordinal number, T be a connected term order of n, L be an add-associative right complementable right zeroed commutative associative well unital distributive field-like non trivial double loop structure, and P be a subset of $\mathrm{Polynom\text{-}Ring}(n,L)$. If $\mathrm{PolyRedRel}(P,T)$ is confluent, then $\mathrm{PolyRedRel}(P,T)$ has unique normal form property.

(14) Let n be a natural number, T be a connected admissible term order of n, L be an add-associative right complementable right zeroed commutative associative well unital distributive Abelian field-like non degenerated non empty double loop structure, and P be a subset of Polynom-Ring(n,L). Suppose PolyRedRel(P,T) has unique normal form property. Then PolyRedRel(P,T) has Church-Rosser property.

(15) Let n be a natural number, T be a connected admissible term order of n, L be an add-associative right complementable right zeroed commutative associative well unital distributive Abelian field-like non degenerated non empty double loop structure, and P be a non empty subset of Polynom-Ring(n,L). Suppose PolyRedRel(P,T) has Church-Rosser property. Let f be a polynomial of n, L. If $f \in P$–ideal, then PolyRedRel(P,T) reduces f to $0_n L$.

(16) Let n be an ordinal number, T be a connected term order of n, L be an add-associative right complementable right zeroed commutative associative well unital distributive field-like non trivial double loop structure, and P be a subset of Polynom-Ring(n,L). Suppose that for every polynomial f of n, L such that $f \in P$–ideal holds PolyRedRel(P,T) reduces f to $0_n L$. Let f be a non-zero polynomial of n, L. If $f \in P$–ideal, then f is reducible wrt P, T.

(17) Let n be a natural number, T be an admissible connected term order of n, L be an add-associative right complementable right zeroed commutative associative well unital distributive Abelian field-like non degenerated non empty double loop structure, and P be a subset of Polynom-Ring(n,L). Suppose that for every non-zero polynomial f of n, L such that $f \in P$–ideal holds f is reducible wrt P, T. Let f be a non-zero polynomial of n, L. If $f \in P$–ideal, then f is top reducible wrt P, T.

(18) Let n be an ordinal number, T be a connected term order of n, L be an add-associative right complementable right zeroed commutative associative well unital distributive field-like non trivial double loop structure, and P be a subset of Polynom-Ring(n,L). Suppose that for every non-zero polynomial f of n, L such that $f \in P$–ideal holds f is top reducible wrt P, T. Let b be a bag of n. If $b \in \mathrm{HT}(P\text{–ideal},T)$, then there exists a bag b' of n such that $b' \in \mathrm{HT}(P,T)$ and $b' \mid b$.

(19) Let n be an ordinal number, T be a connected term order of n, L be an add-associative right complementable right zeroed commutative associative well unital distributive field-like non trivial double loop structure, and P be a subset of Polynom-Ring(n,L). Suppose that for every bag b of n such that $b \in \mathrm{HT}(P\text{–ideal},T)$ there exists a bag b' of n such that $b' \in \mathrm{HT}(P,T)$ and $b' \mid b$. Then $\mathrm{HT}(P\text{–ideal},T) \subseteq \mathrm{multiples}(\mathrm{HT}(P,T))$.

(20) Let n be a natural number, T be a connected admissible term order of n, L be an Abelian add-associative right complementable right zeroed commutative associative well unital distributive field-like non degenerated non empty double loop structure, and P be a subset of Polynom-Ring(n,L). If $\mathrm{HT}(P\text{–ideal},T) \subseteq \mathrm{multiples}(\mathrm{HT}(P,T))$, then PolyRedRel$(P,T)$ is locally-confluent.

Let n be an ordinal number, let T be a connected term order of n, let L be an add-associative right complementable right zeroed commutative associative well unital distributive field-like non trivial double loop structure, and let G be a subset of Polynom-Ring(n,L). We say that G is a Groebner basis wrt T if and only if:

(Def. 3) PolyRedRel(G,T) is locally-confluent.

Let n be an ordinal number, let T be a connected term order of n, let L be an add-associative right complementable right zeroed commutative associative well unital distributive field-like non trivial double loop structure, and let G, I be subsets of Polynom-Ring(n,L). We say that G is a Groebner basis of I, T if and only if:

(Def. 4) G–ideal $= I$ and PolyRedRel(G,T) is locally-confluent.

The following propositions are true:

(21) Let n be a natural number, T be a connected admissible term order of n, L be an Abelian add-associative right complementable right zeroed commutative associative well unital distributive field-like non degenerated non empty double loop structure, and G, P be non empty subsets of Polynom-Ring(n,L). If G is a Groebner basis of P, T, then PolyRedRel(G,T) is a completion of PolyRedRel(P,T).

(22) Let n be a natural number, T be a connected admissible term order of n, L be an Abelian add-associative right complementable right zeroed commutative associative well unital distributive field-like non degenerated non empty double loop structure, p, q be elements of Polynom-Ring(n,L), and G be a non empty subset of Polynom-Ring(n,L). Suppose G is a Groebner basis wrt T. Then $p \equiv q(\mathrm{mod}\, G\text{–ideal})$ if and only if $\mathrm{nf}_{\mathrm{PolyRedRel}(G,T)}(p) = \mathrm{nf}_{\mathrm{PolyRedRel}(G,T)}(q)$.

(23) Let n be a natural number, T be a connected admissible term order of n, L be an add-associative right complementable right zeroed commutative associative well unital distributive Abelian field-like non degenerated non empty double loop structure, f be a polynomial of n, L, and P be a non empty subset of Polynom-Ring(n,L). Suppose P is a Groebner basis wrt T. Then $f \in P$–ideal if and only if PolyRedRel(P,T) reduces f to 0_nL.

(24) Let n be a natural number, T be a connected admissible term order of n, L be an add-associative right complementable right zeroed commutative associative well unital distributive Abelian field-like non degenerated non empty double loop structure, I be a subset of Polynom-Ring(n,L), and G be a non empty subset of Polynom-Ring(n,L). Suppose G is a Groebner basis of I, T. Let f be a polynomial of n, L. If $f \in I$, then PolyRedRel(G,T) reduces f to 0_nL.

(25) Let n be an ordinal number, T be a connected term order of n, L be an add-associative right complementable right zeroed commutative associative well unital distributive field-like non trivial double loop structure, and G, I be subsets of Polynom-Ring(n,L). Suppose that for every polynomial f of n, L such that $f \in I$ holds PolyRedRel(G,T) reduces f to 0_nL. Let f be a non-zero polynomial of n, L. If $f \in I$, then f is reducible wrt G, T.

(26) Let n be a natural number, T be an admissible connected term order of n, L be an add-associative right complementable right zeroed commutative associative well unital distributive Abelian field-like non degenerated non empty double loop structure, I be an add closed left ideal subset of Polynom-Ring(n,L), and G be a subset of Polynom-Ring(n,L). Suppose $G \subseteq I$. Suppose that for every non-zero polynomial f of n, L such that $f \in I$ holds f is reducible wrt G, T. Let f be a non-zero polynomial of n, L. If $f \in I$, then f is top reducible wrt G, T.

(27) Let n be an ordinal number, T be a connected term order of n, L be an add-associative right complementable right zeroed commutative associative well unital distributive field-like non trivial double loop structure, and G, I be subsets of Polynom-Ring(n,L). Suppose that for every non-zero polynomial f of n, L such that $f \in I$ holds f is top reducible wrt G, T. Let b be a bag of n. If $b \in \mathrm{HT}(I,T)$, then there exists a bag b' of n such that $b' \in \mathrm{HT}(G,T)$ and $b' \mid b$.

(28) Let n be an ordinal number, T be a connected term order of n, L be an add-associative right complementable right zeroed commutative associative well unital distributive field-like non trivial double loop structure, and G, I be subsets of Polynom-Ring(n,L). Suppose that for every bag b of n such that $b \in \mathrm{HT}(I,T)$ there exists a bag b' of n such that $b' \in \mathrm{HT}(G,T)$ and $b' \mid b$. Then $\mathrm{HT}(I,T) \subseteq \mathrm{multiples}(\mathrm{HT}(G,T))$.

(29) Let n be a natural number, T be a connected admissible term order of n, L be an Abelian add-associative right complementable right zeroed commutative associative well unital distributive field-like non degenerated non empty double loop structure, I be an add closed left ideal non empty subset of Polynom-Ring(n,L), and G be a non empty subset of Polynom-Ring(n,L). If $G \subseteq I$, then if $\mathrm{HT}(I,T) \subseteq \mathrm{multiples}(\mathrm{HT}(G,T))$, then G is a Groebner basis of I, T.

3. EXISTENCE OF GRÖBNER BASES

The following four propositions are true:

(30) Let n be a natural number, T be a connected admissible term order of n, and L be an add-associative right complementable right zeroed commutative associative well unital distributive Abelian field-like non trivial double loop structure. Then $\{0_n L\}$ is a Groebner basis of $\{0_n L\}$, T.

(31) Let n be a natural number, T be a connected admissible term order of n, L be an add-associative right complementable right zeroed commutative associative well unital distributive Abelian field-like non trivial double loop structure, and p be a polynomial of n, L. Then $\{p\}$ is a Groebner basis of $\{p\}$–ideal, T.

(32) Let T be an admissible connected term order of $\emptyset$, L be an add-associative right complementable right zeroed commutative associative well unital distributive Abelian field-like non degenerated non empty double loop structure, I be an add closed left ideal non empty subset of Polynom-Ring$(\emptyset, L)$, and P be a non empty subset of Polynom-Ring$(\emptyset, L)$. If $P \subseteq I$ and $P \neq \{0_\emptyset L\}$, then P is a Groebner basis of I, T.

(33) Let n be a non empty ordinal number, T be an admissible connected term order of n, and L be an add-associative right complementable right zeroed commutative associative well unital distributive field-like non degenerated non empty double loop structure. Then there exists a subset P of Polynom-Ring(n, L) such that P is not a Groebner basis wrt T.

Let n be an ordinal number. The functor DivOrder(n) yields an order in Bags n and is defined by:

(Def. 5) For all bags b_1, b_2 of n holds $\langle b_1, b_2\rangle \in$ DivOrder(n) iff $b_1 \mid b_2$.

Let n be a natural number. Observe that $\langle$Bags n, DivOrder$(n)\rangle$ is Dickson.

The following three propositions are true:

(34) For every natural number n and for every subset N of $\langle$Bags n, DivOrder$(n)\rangle$ holds there exists a finite subset of Bags n which is Dickson basis of N, $\langle$Bags n, DivOrder$(n)\rangle$.

(35) Let n be a natural number, T be a connected admissible term order of n, L be an Abelian add-associative right complementable right zeroed commutative associative well unital distributive field-like non degenerated non empty double loop structure, and I be an add closed left ideal non empty subset of Polynom-Ring(n, L). Then there exists a finite subset of Polynom-Ring(n, L) which is a Groebner basis of I, T.

(36) Let n be a natural number, T be a connected admissible term order of n, L be an Abelian add-associative right complementable right zeroed commutative associative well unital distributive field-like non degenerated non empty double loop structure, and I be an add closed left ideal non empty subset of Polynom-Ring(n, L). Suppose $I \neq \{0_n L\}$. Then there exists a finite subset G of Polynom-Ring(n, L) such that G is a Groebner basis of I, T and $0_n L \notin G$.

Let n be an ordinal number, let T be a connected term order of n, let L be a non empty multiplicative loop with zero structure, and let p be a polynomial of n, L. We say that p is monic wrt T if and only if:

(Def. 6) $\mathrm{HC}(p, T) = \mathbf{1}_L$.

Let n be an ordinal number, let T be a connected term order of n, let L be a right zeroed add-associative right complementable commutative associative well unital distributive field-like non trivial non empty double loop structure, and let P be a subset of Polynom-Ring(n, L). We say that P is reduced wrt T if and only if:

(Def. 7) For every polynomial p of n, L such that $p \in P$ holds p is monic wrt T and irreducible wrt $P \setminus \{p\}$, T.

We introduce P is autoreduced wrt T as a synonym of P is reduced wrt T.

One can prove the following propositions:

(37) Let n be an ordinal number, T be an admissible connected term order of n, L be an add-associative right complementable right zeroed commutative associative well unital distributive Abelian field-like non degenerated non empty double loop structure, I be an add closed left ideal subset of Polynom-Ring(n,L), m be a monomial of n, L, and f, g be polynomials of n, L. Suppose $f \in I$ and $g \in I$ and $\mathrm{HM}(f,T) = m$ and $\mathrm{HM}(g,T) = m$. Suppose that

(i) it is not true that there exists a polynomial p of n, L such that $p \in I$ and $p <_T f$ and $\mathrm{HM}(p,T) = m$, and

(ii) it is not true that there exists a polynomial p of n, L such that $p \in I$ and $p <_T g$ and $\mathrm{HM}(p,T) = m$.

Then $f = g$.

(38) Let n be a natural number, T be a connected admissible term order of n, L be an Abelian add-associative right complementable right zeroed commutative associative well unital distributive field-like non degenerated non empty double loop structure, I be an add closed left ideal non empty subset of Polynom-Ring(n,L), G be a subset of Polynom-Ring(n,L), p be a polynomial of n, L, and q be a non-zero polynomial of n, L. Suppose $p \in G$ and $q \in G$ and $p \neq q$ and $\mathrm{HT}(q,T) \mid \mathrm{HT}(p,T)$. If G is a Groebner basis of I, T, then $G \setminus \{p\}$ is a Groebner basis of I, T.

(39) Let n be a natural number, T be a connected admissible term order of n, L be an Abelian add-associative right complementable right zeroed commutative associative well unital distributive field-like non degenerated non empty double loop structure, and I be an add closed left ideal non empty subset of Polynom-Ring(n,L). If $I \neq \{0_n L\}$, then there exists a finite subset G of Polynom-Ring(n,L) which is a Groebner basis of I, T and reduced wrt T.

(40) Let n be a natural number, T be a connected admissible term order of n, L be an Abelian add-associative right complementable right zeroed commutative associative well unital distributive field-like non degenerated non empty double loop structure, I be an add closed left ideal non empty subset of Polynom-Ring(n,L), and G_1, G_2 be non empty finite subsets of Polynom-Ring(n,L). Suppose G_1 is a Groebner basis of I, T and reduced wrt T and G_2 is a Groebner basis of I, T and reduced wrt T. Then $G_1 = G_2$.

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Received June 11, 2003

Published January 2, 2004